M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

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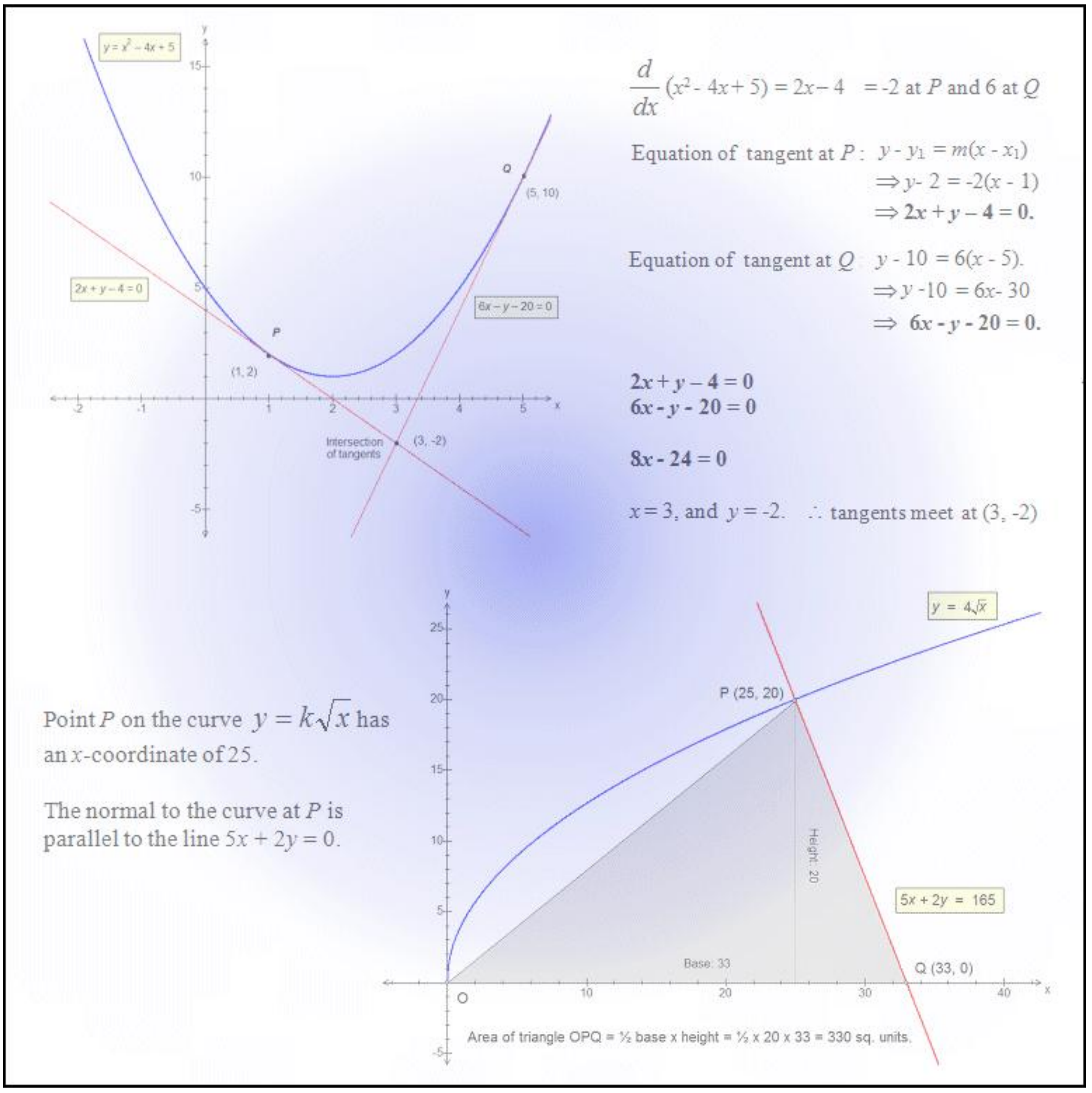
TANGENTS AND NORMALS



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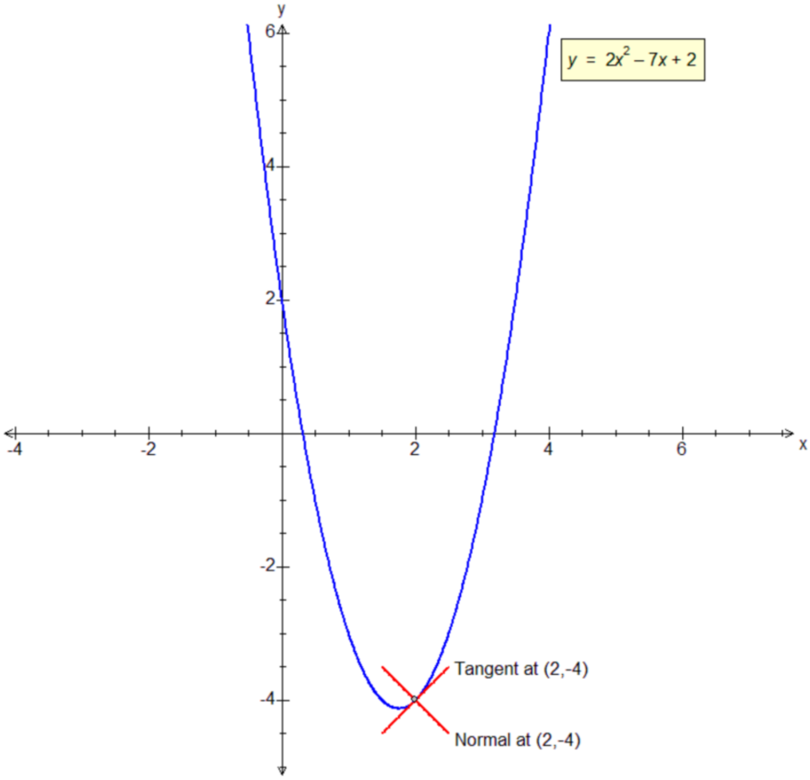
Tangents and Normals.

Differentiation helps to find the gradient of the **tangent** to a curve, but we can use ideas learnt in the section of straight lines to find the actual equation of the tangent at a given point. The **normal** at the same point is perpendicular to the tangent, therefore the product of their gradients is –1.



Example (1): Find the gradients, and hence the equations, of the tangent and the normal to the curve $2x^2 - 7x + 2$ at the point (2, -4).

The tangent and normal have been shown here for reference. (Note that the axes of the graph must be shown to uniform scales, or the normal and tangent might not appear at right angles)

The derivative of $2x^2 - 7x + 2$ is $4x – 7$; substituting for $x = 2$ gives a gradient of 1.

The equation of the tangent at (2, -4) is therefore $y - y_1 = m(x - x_1)$
$\Rightarrow y + 4 = 1(x - 2)$.

In gradient-intercept form ('$mx + c$'):
$y + 4 = x – 2$
$\Rightarrow y = x - 6$ $\therefore$ equation of tangent is $\boldsymbol{y = x - 6}$.

In '$ax + by + c = 0$' form :
$y + 4 = 1(x – 2) \Rightarrow y + 4 = x – 2$
$\Rightarrow x – 2 - y - 4 = 0 \Rightarrow x – y - 6 = 0$
$\therefore$ equation of tangent is $\boldsymbol{x – y - 6 = 0}$.

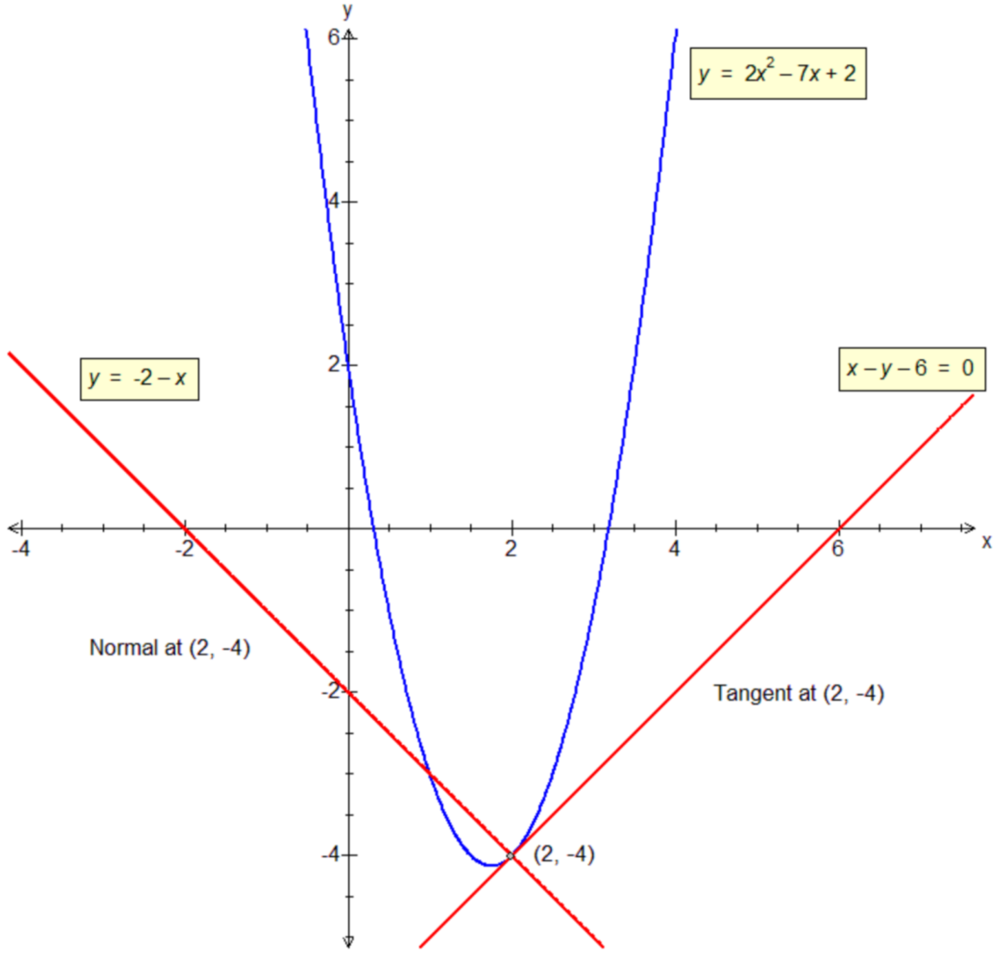
The product of the gradients of the tangent and the normal must be –1, and therefore the normal to the curve must have a gradient of –1.

The equation of the normal at (2, -4) is therefore $y + 4 = -1(x - 2)$.

In gradient-intercept form (‘$mx + c$’):
$y + 4 = -1(x - 2) \Rightarrow y + 4 = 2 - x \Rightarrow y = -2 - x$ $\therefore$ equation of normal is $\boldsymbol{y = -2 - x}$.

In ’$ax + by + c = 0$’ form :
$y + 4 = -1(x - 2) \Rightarrow y + 4 = 2 - x$
$\Rightarrow y + 4 - 2 + x = 0$
$\Rightarrow x + y + 2 = 0$ $\therefore$ equation of normal is $\boldsymbol{x + y + 2 = 0}$.

See diagram below.



Example (2): Find the equations of the tangent and normal to the curve $x^3 - 4x^2 + 2x$ at the point (2, -4). Give the equations in gradient-intercept ('$mx + c$') form, and hence show that the tangent passes through the origin.

The derivative of $x^3 - 4x^2 + 2x$ is $3x^2 - 8x + 2$, and thus its value at (2, -4) is $(3 \times 2^2) - (8 \times 2) + 2$, or -2 . The tangent to the curve therefore has a gradient of –2.

The equation of the tangent will be therefore $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y + 4 = -2(x - 2)$.

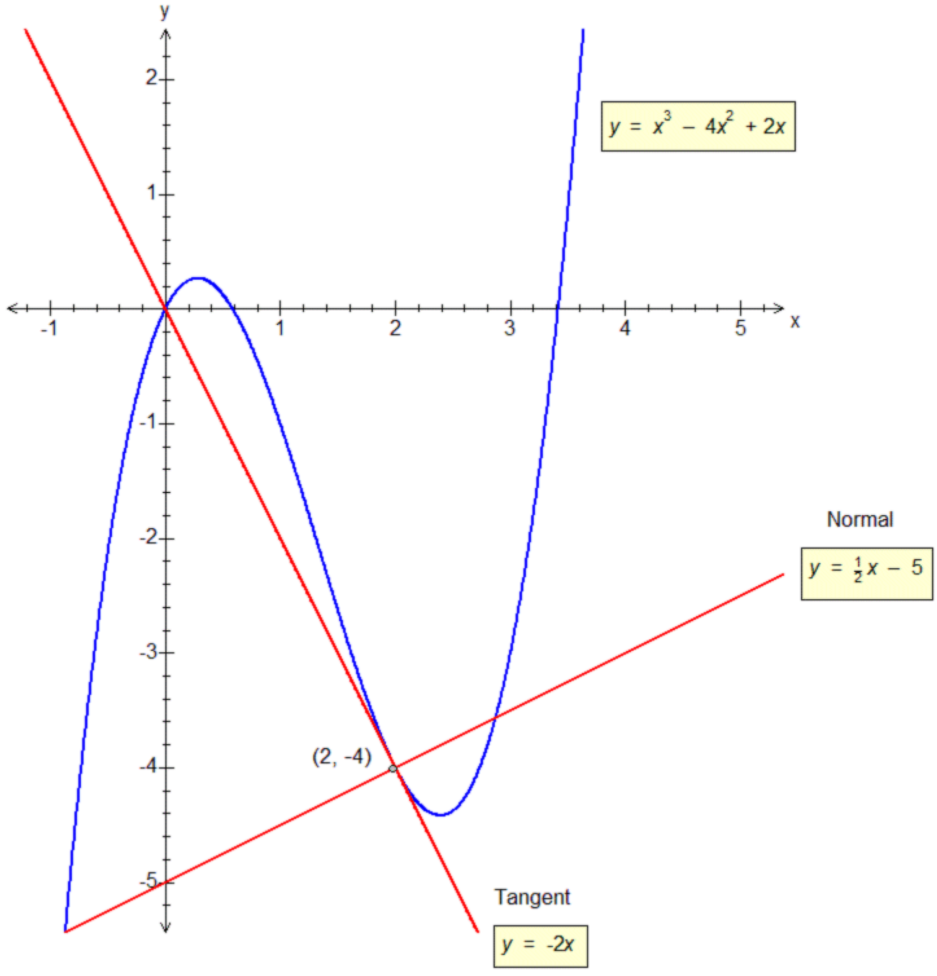
$\Rightarrow$ $y + 4 = -2x + 4$

$\Rightarrow$ $y = -2x$ $\therefore$ equation of tangent is $\mathbf{y = -2x}$. This line has a y-intercept at the origin.

The gradient of the normal must be $\frac{1}{2}$ (product of the gradients of tangent and normal must be –1).

The equation of the normal is therefore $y + 4 = \frac{1}{2}(x - 2) \Rightarrow y + 4 = \frac{1}{2}x - 1$

$\Rightarrow$ $y = \frac{1}{2}x - 5$ $\therefore$ equation of normal is $\mathbf{y = \frac{1}{2}x - 5}$.



Example (3): Find the equations of the tangent and normal to the curve $\frac{4}{x^2}$ at (4, $\frac{1}{4}$).
Give the result in '$ax + by + c = 0$' form.

The tangent has a gradient of $\frac{-8}{x^3}$, so when $x = 4$, its gradient is $-\frac{1}{8}$,
and its equation is $y - y_1 = m(x - x_1)$ or $y - \frac{1}{4} = -\frac{1}{8}(x-4) \Rightarrow 8y - 2 = -(x-4)$
$\Rightarrow 8y - 2 + x - 4 = 0$
$\Rightarrow x + 8y - 6 = 0 \quad \therefore$ equation of tangent is $\mathbf{x + 8y - 6 = 0.}$

The gradient of the normal at (4, $\frac{1}{4}$) is thus 8 by the rule of the product of gradients.
Its equation is therefore $y - \frac{1}{4} = 8(x - 4) \Rightarrow 4y - 1 = 32(x-4) \Rightarrow 32x - 128 - 4y + 1 = 0$
$\Rightarrow 32x - 4y - 127 = 0 \quad \therefore$ equation of normal is $\mathbf{32x - 4y - 127 = 0.}$

Example (4): A curve has equation $y = x^3 - 10x^2 + 12x + 40$.

i) Find the equation of the tangent to the curve when $x = 5$, in '$ax + by + c = 0$' form.
ii) Find the x-coordinate of the point on the curve where the tangent is parallel to the one at $x = 5$.

i) If $y = x^3 - 10x^2 + 12x + 40$, then $\frac{dy}{dx} = 3x^2 - 20x + 12$.

When $x = 5$, $y = 125 - 250 + 60 + 40 = -25$, so the curve passes through (5, -25).

Also, when $x = 5$, $\frac{dy}{dx} = 75 - 100 + 12 = -13$.

$\therefore$ the equation of the tangent at (5, -25) is $y + 25 = -13(x - 5) \Rightarrow y + 25 = -13x + 65$
$\Rightarrow \mathbf{13x + y - 40 = 0.}$

ii) Since the gradient of the tangent in part i) is equal to -13, the gradient of the required parallel tangent will also be -13.

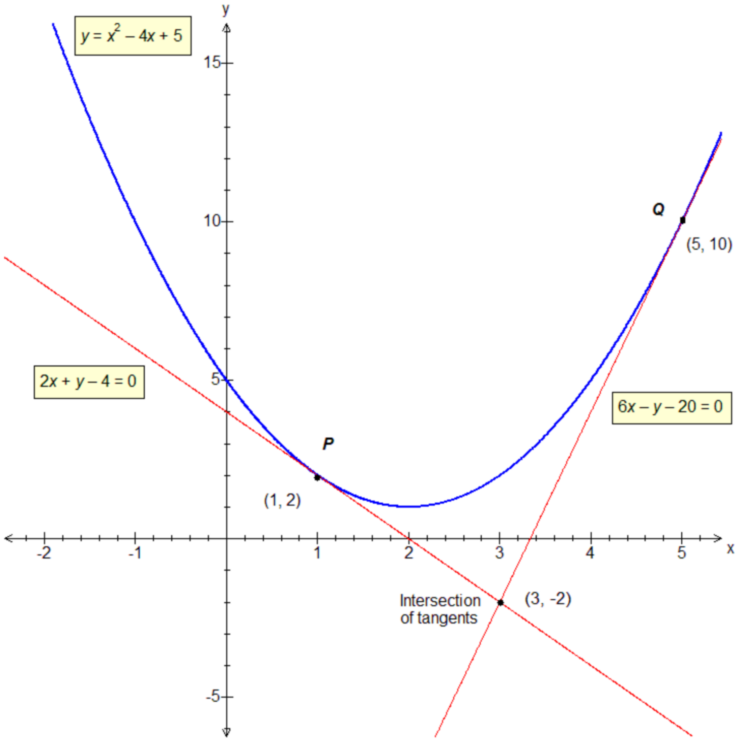
Hence $\frac{dy}{dx} = 3x^2 - 20x + 12 = -13$.

Rearranging as $3x^2 - 20x + 25 = 0$, the resulting quadratic derivative factorises out as $(3x - 5)(x - 5)$.
The solutions of $(3x - 5)(x - 5) = 0$ are $x = 5$ (corresponding to the result from i)) and $x = \frac{5}{3}$,

$\therefore$ the x-coordinate of the point where the curve and the parallel tangent meet is $\frac{5}{3}$.

Example (5): Two tangents are drawn to the curve $x^2 - 4x + 5$ at the points P (1, 2) and Q (5, 10). Give the coordinates of their point of intersection.

The derivative of $x^2 - 4x + 5$ is $2x - 4$, and thus its values at P and Q are –2 and 6 respectively.



The equation of the tangent at P is

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 2 = -2(x - 1)$

$\Rightarrow y - 2 = -2x + 2$

$\Rightarrow y - 2 + 2x - 2 = 0.$

$\Rightarrow$ **2x + y – 4 = 0.**

Similarly the equation of the tangent at Q is

$y - 10 = 6(x - 5).$

$\Rightarrow y - 10 = 6x - 30$

$\Rightarrow 6x - 30 - y + 10 = 0.$

$\Rightarrow$ **6x - y - 20 = 0.**

We now have a pair of simultaneous linear equations which can be solved by elimination:

2x + y – 4 = 0 *A*
6x - y - 20 = 0 *B*

8x - 24 = 0 *A+B*

This gives $x = 3$, and substituting into either equation gives $y = -2$.

$\therefore$ the two tangents meet at (3, -2).

Example (6): The equation of a curve is given by $y = \frac{x^3}{3} - 16x$.

i) Find $\frac{dy}{dx}$ and hence the coordinates of the stationary points on the curve $y = \frac{x^3}{3} - 16x$.

ii) Distinguish between the maximum and minimum points obtained in part i).
iii) Given that the line 20x - y – 144 = 0 is the equation of the tangent to the curve at the point (p, q), find the values of p and q.

i) If $y = \frac{x^3}{3} - 16x$, then $\frac{dy}{dx} = x^2 - 16 \Rightarrow \frac{dy}{dx} = (x+4)(x-4)$.

The x-coordinates of the stationary points are 4 and -4.

$\therefore$ The two stationary points are $\left(4, \frac{-128}{3}\right)$ and $\left(-4, \frac{128}{3}\right)$ after substituting in $y = \frac{x^3}{3} - 16x$.

ii) The second derivative, $\frac{d^2y}{dx^2} = 2x$.

At $\left(4, \frac{-128}{3}\right)$, $\frac{d^2y}{dx^2} = 8$ (i.e. > 0), hence $\left(4, \frac{-128}{3}\right)$ is a local minimum.

On the other hand, at $\left(-4, \frac{128}{3}\right)$, $\frac{d^2y}{dx^2} = -8$ (i.e. < 0), hence $\left(-4, \frac{128}{3}\right)$ is a local maximum.

iii) We can find the gradient of the line 20x - y – 144 = 0 $\Rightarrow$ 20x = y + 144 $\Rightarrow$ y = 20x – 144.
$\therefore$ the gradient of the line 20x - y – 144 = 0 is 20.

(Or we could have used the fact that a line with equation $ax + by + c = 0$ has a gradient of $-\frac{a}{b}$).

Next, we must find the points on the curve where the gradient is also 20, i.e. we solve
$x^2 - 16 = 20 \Rightarrow x^2 - 36 = 0 \Rightarrow (x + 6)(x - 6) = 0$.

When x = 6, $\frac{x^3}{3} - 16x$ = 72 - 96 = -24; similarly when x = -6, $\frac{x^3}{3} - 16x$ = -72 + 96 = 24.
The two possible values for (p, q) are (6, -24) or (-6, 24).

We then substitute each pair of values into the expression 20x - y – 144; the correct pair should give a result of 0.

At (6, -24), 20x – y –144 = 0; at (-6, 24), 20x – y –144 = -288.

$\therefore$ the line 20x - y – 144 = 0 is a tangent to the curve $y = \frac{x^3}{3} - 16x$ at the point (6, -24).

Example (7): The point *P* on the curve $y = k\sqrt{x}$ has an *x*-coordinate of 25. The normal to the curve at *P* is parallel to the line 5*x* + 2*y* = 0.

i) Find the value of *k* and hence the coordinates of *P*.
ii) The normal to the curve meets the *x*-axis at the point *Q*. Calculate the area of the triangle *OPQ* where *O* is the origin.

i) The gradient of the normal to the curve $y = k\sqrt{x}$ when *x* = 25 can be found by rearranging 5*x* + 2*y* = 0 as 2*y* = -5*x* and $y = -\frac{5}{2}x$. $\therefore$ the gradient of the normal is $-\frac{5}{2}$ when *x* = 25.

(Or we could have used the fact that a line with equation *ax* + *by* + *c* = 0 has a gradient of $-\frac{a}{b}$).

Since a tangent and normal to a curve at a given point are perpendicular, the gradient of the tangent to the curve when *x* = 25 is $\frac{2}{5}$.

Differentiating the function $y = k\sqrt{x}$ gives $\frac{dy}{dx} = \frac{k}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}k\left(x^{-\frac{1}{2}}\right)$.

The gradient of the tangent at *x* = 25 is $\frac{2}{5}$ i.e. $\frac{k}{2\sqrt{x}} = \frac{2}{5}$.

We then rearrange to find *k* ; $\frac{k}{2\sqrt{25}} = \frac{2}{5} \Rightarrow \frac{k}{10} = \frac{2}{5} \Rightarrow k = 4$.

Hence $y = 4\sqrt{x}$ and the coordinates of point *P* are (25, 20).

ii) We know that the normal at *P* is parallel to the line 5*x* + 2*y* = 0, and also that it passes through the point (25, 20). Substituting for (*x*, *y*) gives the equation of the normal as 5*x* + 2*y* = 165.

The *x*-coordinate of the *x*-intercept at *Q* satisfies 5*x* + 2*y* = 165 for *y* = 0, so 5*x* = 165 and *x* = 33. The coordinates of *Q* are therefore (33,0).

See diagram for the calculation of the area of triangle *OPQ*.