M.K. HOME TUITION

Mathematics Revision Guides
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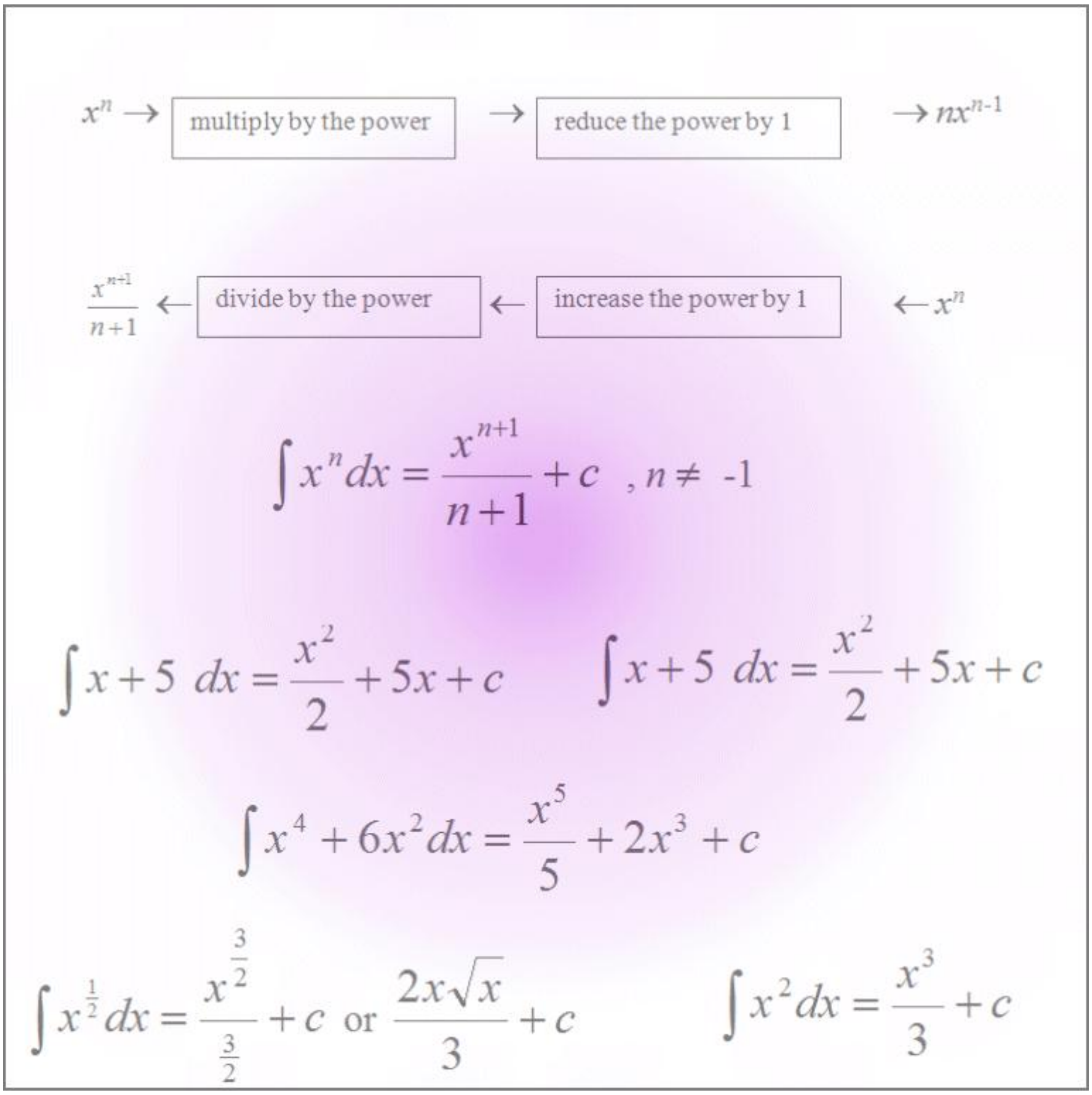
INTRODUCTION TO INTEGRATION



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INTRODUCTION TO INTEGRATION

Differentiation is the process of finding out the gradient of a function and obtaining a derived function. The reverse process is **integration**, where we are given a gradient function and have to obtain the original function from which it was derived.

The steps of differentiating an expression can be thought of diagrammatically as:

x^n → [multiply by the power] → [reduce the power by 1] → nx^{n-1}

This suggests that the reverse process is given by

$\dfrac{x^{n+1}}{n+1}$ ← [divide by the power] ← [increase the power by 1] ← x^n

This is almost correct, but there is an important fact to bear in mind.

If we differentiate x^2 we obtain the derived function of $2x$, but we can differentiate $x^2 + 1$, $x^2 + 1000$, $x^2 - 123.45$ and so on and still obtain $2x$. This is because the derivative of a constant is zero, and this must be taken into account when carrying out the reverse process, i.e. by integrating.

The result of integrating a power of x, x^n, is therefore denoted by

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$$

The elongated S is the symbol for integration, the '*dx*' indicates that the integration is performed with respect to x, and the c is the constant of integration (or arbitrary constant).

Note also that this expression cannot be used for $n = -1$ (division by zero !)

Constant multiples, sums and differences are handled in the same way as in differentiation:

$\int af(x)dx = a\int f(x)dx$ where a is a constant.

$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Examples (1): Integrate with respect to x: a) x^2 ; b) $x + 5$; c) $8x^3$; d) $x^4 + 6x^2$; e) $\sqrt{x}$; f) $\dfrac{1}{x^2}$

a) $\int x^2 dx = \dfrac{x^3}{3} + c$ b) $\int x + 5 \ dx = \dfrac{x^2}{2} + 5x + c$ c) $\int 8x^3 dx = 2x^4 + c$

d) $\int x^4 + 6x^2 dx = \dfrac{x^5}{5} + 2x^3 + c$

e) $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, so $\int x^{\frac{1}{2}} dx = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$ or $\dfrac{2x\sqrt{x}}{3} + c$

f) $\dfrac{1}{x^2}$ is the same as x^{-2}, so $\int x^{-2} dx = \dfrac{x^{-1}}{-1} + c$ or $\dfrac{-1}{x} + c$

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Sometimes there may be enough information to determine the value of the constant c.

Example (2): Find the equation of the curve whose gradient function is $4x$, and which passes through the point (3, 11).

Here $\frac{dy}{dx} = 4x$, and so $y = \int 4x\,dx$, or $y = 2x^2 + c$.

Substituting $x = 3$, $2 \times 3^2 + c = 11$, and therefore $c = 11 - 18$, or -7.

The equation of the curve is therefore $\boldsymbol{y = 2x^2 - 7}$**.**

Example (3): The gradient of a curve is equal to 3 at the point (1, -1) and 9 at the point (4, 17). Given that the curve is a quadratic, find its equation.

The gradient function of a quadratic is a linear function , thus here $\frac{dy}{dx} = mx + c$.

When $x = 1$, $mx + c = 3$; when $x = 4$, $mx + c = 9$. Substituting we have $m + c = 3$ and $4m + c = 9$, hence $m = 2$ and $c = 1$.

The gradient function is therefore $\frac{dy}{dx} = 2x + 1$.

Integrating , $y = \int 2x + 1\,dx$, or $y = x^2 + x + c$.

Substituting $x = 1$ and $y = -1$ gives $1 + 1 + c = -1 \Rightarrow c = -3$.

$\therefore$ The equation of the curve is $y = x^2 + x - 3$.

Example (4): The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 2x - 8$.
We are also told that the curve passes through the point (1, 4).

i) Find the equation of the curve.
ii) Show that the curve touches the x-axis at point A and cuts it at another point B. State the coordinates of A and B.
iii) The curve cuts the y-axis at point C. Find the gradient of the tangent at C and also the x-coordinate of point D such that the tangents at C and D are parallel.

i) If $\frac{dy}{dx} = 3x^2 - 2x - 8$, then $y = \int 3x^2 - 2x - 8\,dx = x^3 - x^2 - 8x + c$.

When $x = 1$, $x^3 - x^2 - 8x + c = 4 \Rightarrow 1 - 1 - 8 + c = 4 \Rightarrow c = 12$.

$\therefore$ The equation of the curve is $y = x^3 - x^2 - 8x + 12$.

ii) The curve touches the x-axis at point A, signifying that its equation has a repeated factor.

Using the Factor Theorem, we can substitute certain values of x into the equation.
When $x = 2$, $y = 8 - 4 - 16 + 12 = 0$, therefore $(x - 2)$ is a factor of $x^3 - x^2 - 8x + 12$.

Division of the cubic (not shown here) by $x - 2$ gives a quotient of $x^2 + x - 6$, which again factorises by inspection to $(x-2)(x+3)$.

$\therefore \quad x^3 - x^2 - 8x + 12 = (x - 2)^2(x+3)$

The coordinates of A are therefore (2, 0) since the curve is a tangent to the axis at a repeated root; the coordinates of B are (-3, 0).

iii) Substituting $x = 0$ into the equation of the curve gives the coordinates of C as (0, 12).
The gradient of the tangent at C is -8, by substituting $x = 0$ into the derivative function $3x^2 - 2x - 8$.

Since parallel lines have the same gradient, the gradient of the tangent at D must also satisfy
$3x^2 - 2x - 8 = -8$, or $3x^2 - 2x = 0$, or $x(3x-2) = 0$.

We have already 'used' $x = 0$, so the x-coordinate of point D is therefore $\frac{2}{3}$.