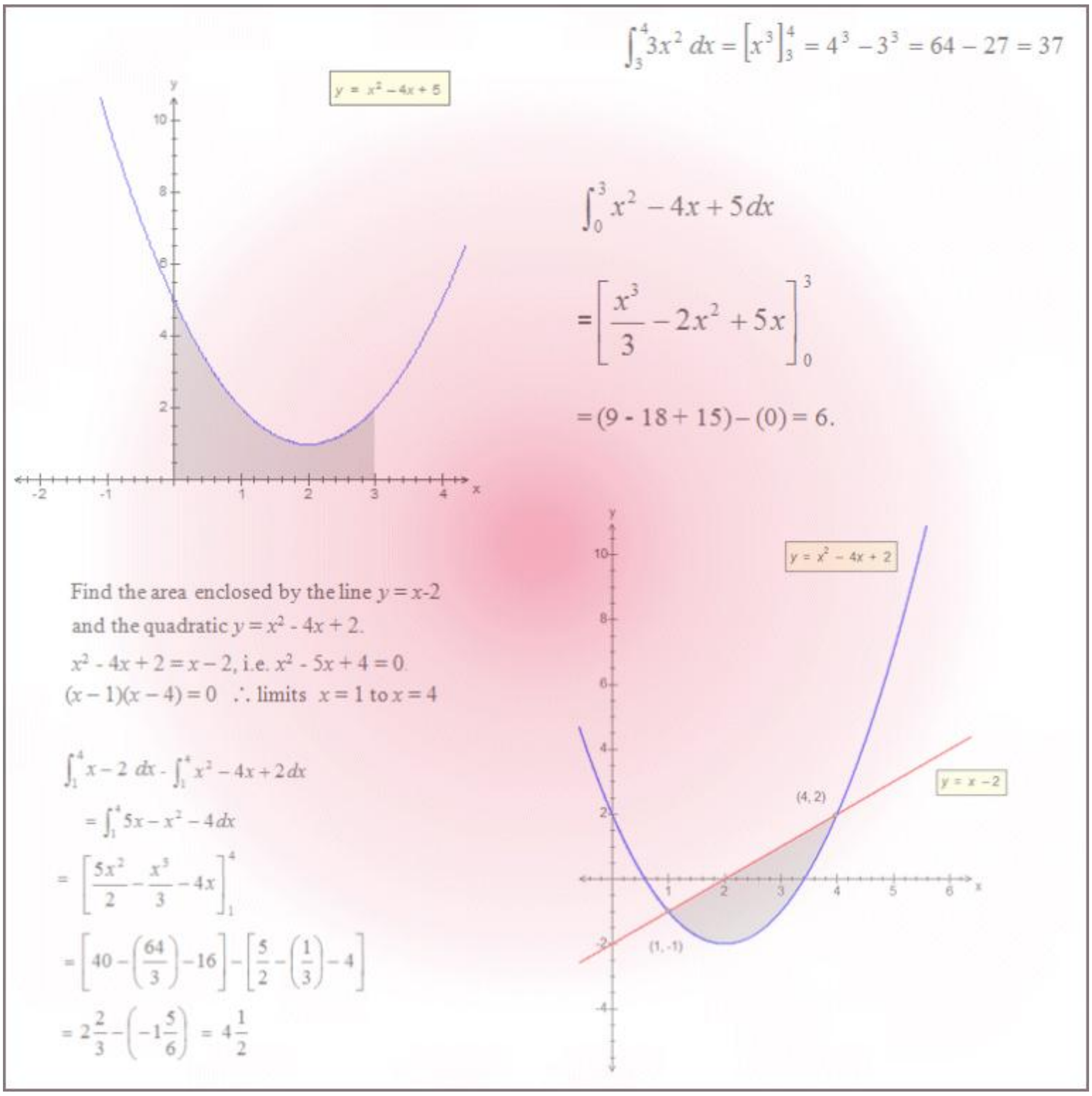
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C2** Edexcel: **C2** OCR: **C1** OCR MEI: **C2**

DEFINITE INTEGRALS - AREA UNDER A CURVE



Version : 4.1 Date: 11-01-2015

Introduction to definite integrals.

We have come across integration earlier, such as $\int x^2 dx = \frac{x^3}{3} + c$.

Because any constant term becomes zero when differentiated, then all integrals of this type will need to include an arbitrary constant *c* (unless other details are stated to enable us to find a unique solution).

These integrals are therefore termed **indefinite integrals** due to the need to include this constant. In addition, indefinite integrals give a **function** as a result.

An integral of the form $\int_a^b f(x)dx$ is a **definite integral** and it returns a **numerical** result.

$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a)$$

where $f(x) = g'(x)$.

To evaluate a definite integral, we first substitute the upper limit and the lower limit in turn, and then subtract the value for the lower limit from the value for the upper one.

Example (1): Find the value of $\int_3^4 3x^2\, dx.$.

$$\int_3^4 3x^2\, dx = \left[x^3\right]_3^4 = 4^3 - 3^3 = 64 - 27 = 37.$$

Note that definite integrals do not include an arbitrary constant, because it would be cancelled out by the subtraction:

$$\int_3^4 3x^2\, dx = \left[x^3 + c\right]_3^4 = (4^3 + c) - (3^3 + c) = 64 - 27 = 37.$$

(OCR only) There may be a need to tackle a definite integral with an infinite upper limit, such as this example:

Example (1a): Find the value of $\int_1^\infty \frac{1}{x^2}\, dx.$.

$$\int_1^\infty \frac{1}{x^2}\, dx = \left[\frac{-1}{x}\right]_1^\infty = \lim_{x\to\infty}\left(-\frac{1}{x}\right) - \left(-\frac{1}{1}\right) = 0 + 1 = 1.$$

Area under a curve.

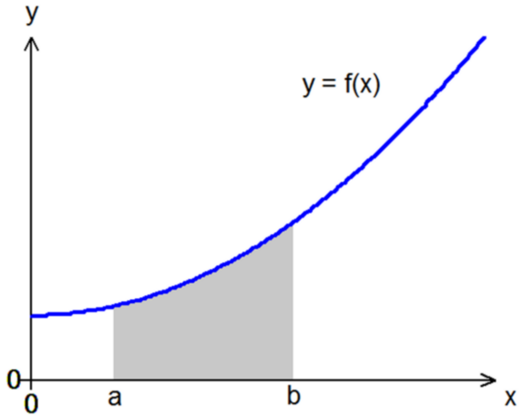
The area enclosed by the curve $y = f(x)$, the x-axis and the lines

$x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx.$$

For areas *below* the x-axis, the definite integral gives a *negative* value.

Also beware of cases where the curve is partly above the x-axis and partly below it.



Example (2): Find the area under the curve $y = x^2 - 4x + 5$ between $x = 0$ and $x = 3$.

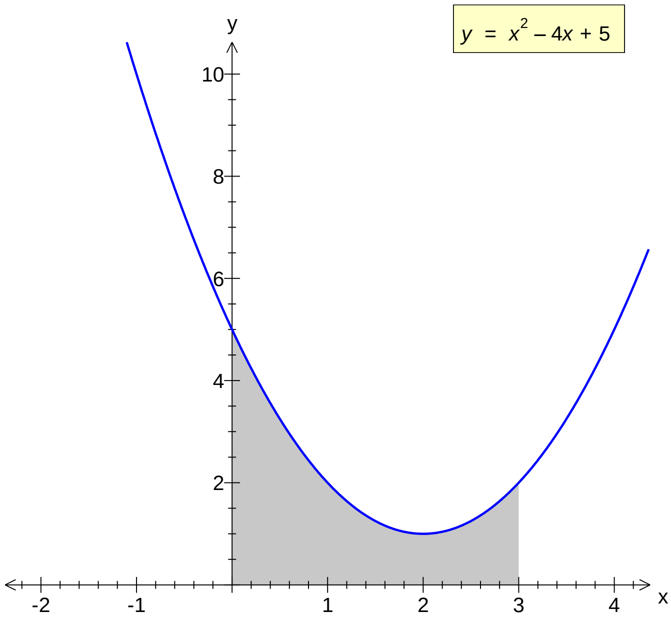
The area is obtained by

$$\int_0^3 x^2 - 4x + 5\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 5x\right]_0^3$$

= (9 - 18 + 15) – (0) = 6.

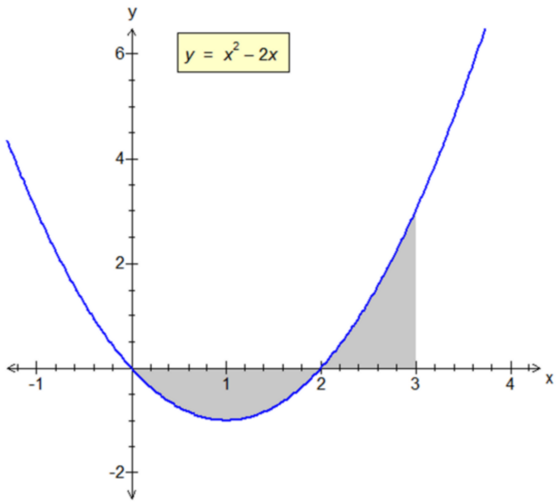
The area under the curve is 6 sq.units.



Example (3): Find the area under the curve $y = x^2 - 2x$ between $x = 0$ and $x = 3$, and explain the result.

The area under the curve is given as $\int_0^3 x^2 - 2x\,dx$ which works out as

$$\left[\frac{x^3}{3} - x^2\right]_0^3 = (9-9) - (0-0) = 0.$$



The graph is below the x-axis when x is between 0 and 2, and above the x-axis when x is between 2 and 3.

Those two parts are equal in area, but because they are on different sides of the x-axis, the algebraic area cancels out to zero.

Example (4): Find the area under the curve $y = x^2 - 2x$ between a) $x = 0$ and $x = 2$, and b) $x = 2$ and $x = 3$, and thus find the total area under the curve. .

From $x = 0$ to 2, the area is $\int_0^2 x^2 - 2x\,dx$

$$= \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - (0-0) = -\left(\frac{4}{3}\right).$$

(This result is negative since the area is below the x-axis.)

From $x = 2$ to 3, the area is $\int_2^3 x^2 - 2x\,dx$

$$= \left[\frac{x^3}{3} - x^2\right]_2^3 = (9-9) - \left(\frac{8}{3} - 4\right) = \left(\frac{4}{3}\right).$$

(This result is positive since the area is above the x-axis.)

Both areas are numerically equal to $\frac{4}{3}$ square units, and so the total area under the curve is double that, or $2\frac{2}{3}$ square units.

Sometimes we might be asked to find the area between a line (or curve) and the y-axis.

In such cases, if y is defined as a function of x, then we re-express x as a function of y and integrate with respect to y.

Example (5): Find the area between the curve $y = x^2 - 2$ (for positive x) and the y – axis from $y = 2$ to $y = 7$.

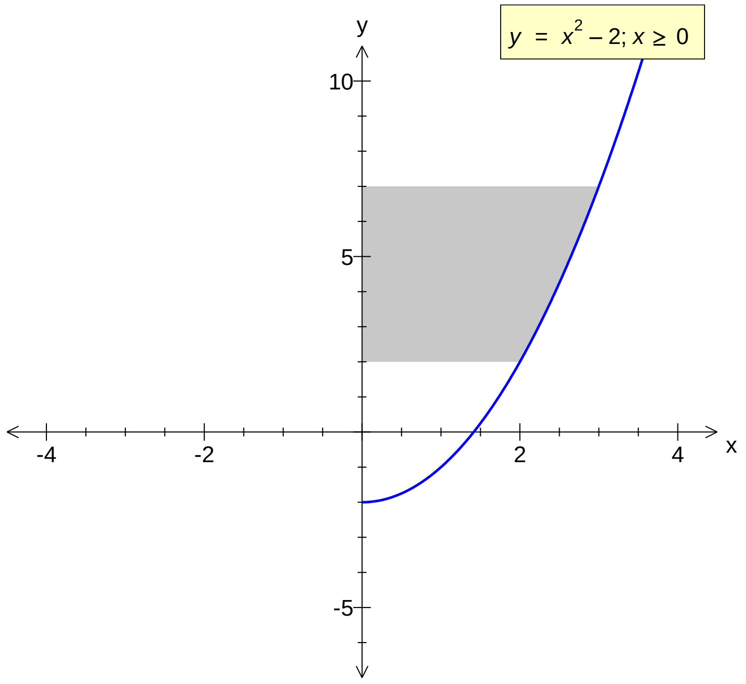
Use the result $\int \sqrt{(x+a)}\,dx = \frac{2}{3}(x+a)\sqrt{(x+a)} + c$.

We re-express $y = x^2 - 2$ as a function of y:

$y = x^2 - 2$

$\Rightarrow\ x^2 = y + 2$

$\Rightarrow\ x = \sqrt{(y + 2)}$



Using the result above we have $\int_2^7 \sqrt{(y+2)}\,dy = \left[\frac{2}{3}(y+2)\sqrt{(y+2)}\right]_2^7$

$= \frac{2}{3}(27-8) = 12\frac{2}{3}$.

$\therefore$ the shaded area is $12\frac{2}{3}$ square units.

To find the area between a line and a curve, a method is to find the areas under the line and the curve separately, and then subtract to find the required area.

Example (6): Find the area enclosed by the line $y = 2x$ and the quadratic $y = x^2 - 4x + 5$.

The first step is to find the limits of integration, which would be the values of x where the line and the curve cross.

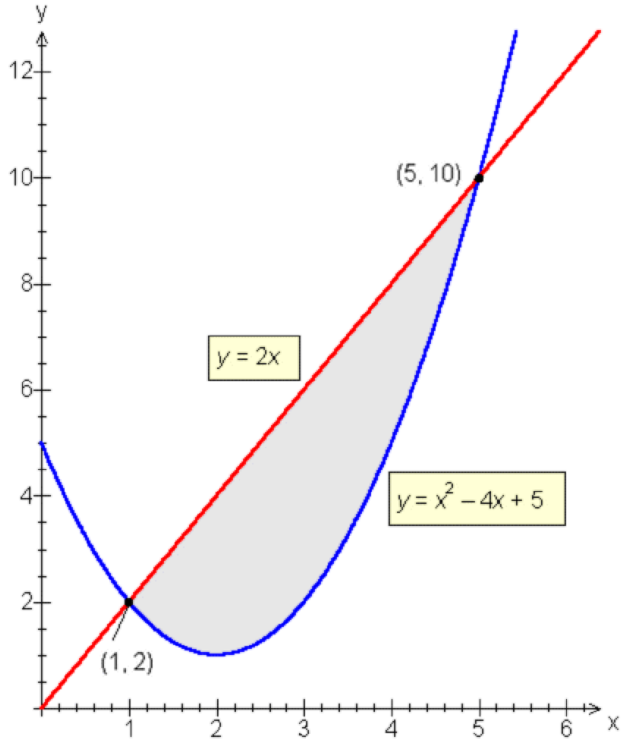
At the intersections,
$x^2 - 4x + 5 = 2x \Rightarrow x^2 - 6x + 5 = 0$.

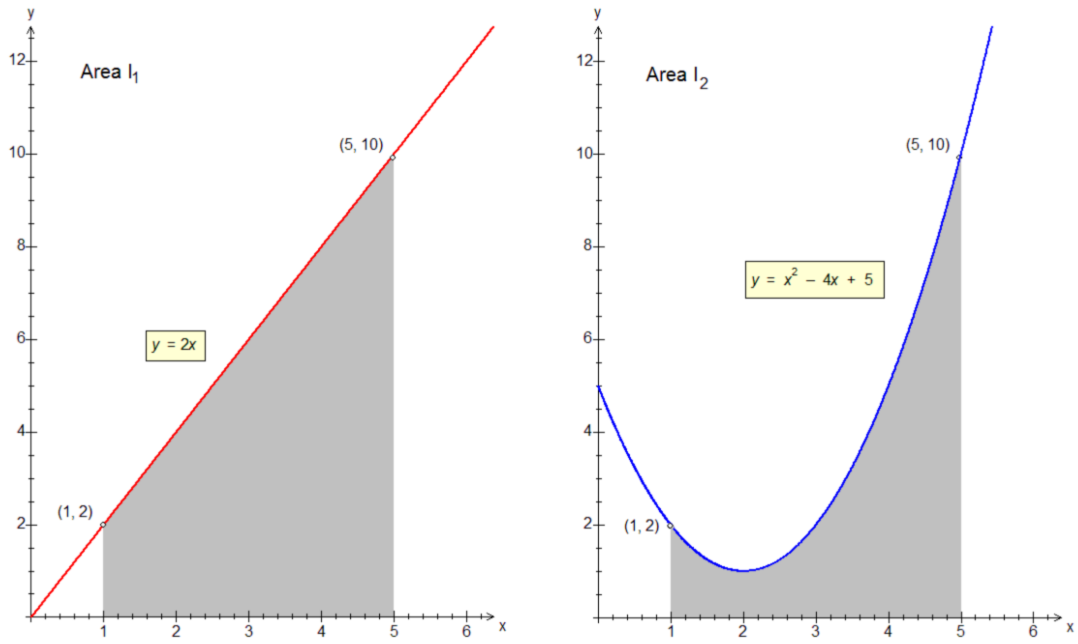
Factorising the resulting quadratic gives $(x - 1)(x - 5) = 0$, so the limits of the required integral would be $x = 1$ to $x = 5$.

The shaded area can be obtained by firstly taking the area under the line $y = 2x$ (call it I_1), and secondly taking the area under the quadratic $y = x^2 - 4x + 5$ (call it I_2).

Note that the line $y = 2x$ is the upper bound of the area to be integrated, and the quadratic $y = x^2 - 4x + 5$ is the lower bound.

The shaded area will be $I_1 - I_2$.





The area under the line, I_1, is therefore $\int_1^5 2x\,dx$, or $\left[x^2\right]_1^5$, = 25 – 1 , or 24 square units.

The area under the parabola, I_2, is $\int_1^5 x^2 - 4x + 5\,dx \;=\; \left[\frac{x^3}{3} - 2x^2 + 5x\right]_1^5$

$= \left[\left(\frac{125}{3}\right) - 50 + 25\right] - \left[\left(\frac{1}{3}\right) - 2 + 5\right]$

$= 16\frac{2}{3} - 3\frac{1}{3}$, or $13\frac{1}{3}$ square units.

The shaded area is therefore $24 - 13\frac{1}{3}$, or $10\frac{2}{3}$ square units.

Alternative method (preferred).

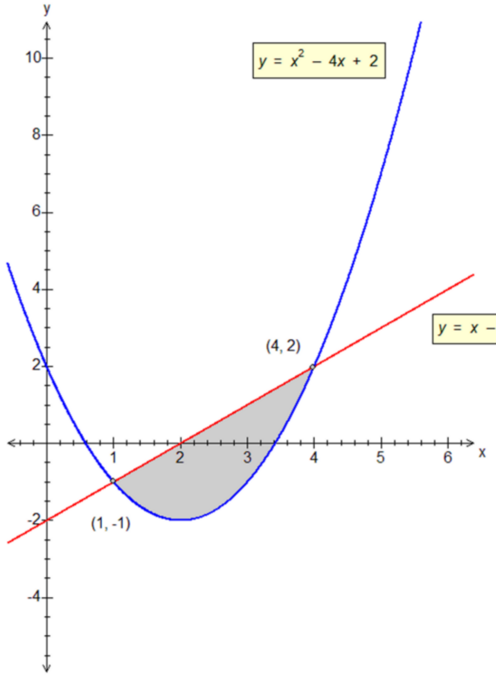
The method above was shown for convenience of illustration. The area under the curve could have been subtracted from that under the line and the whole working performed in one integration.

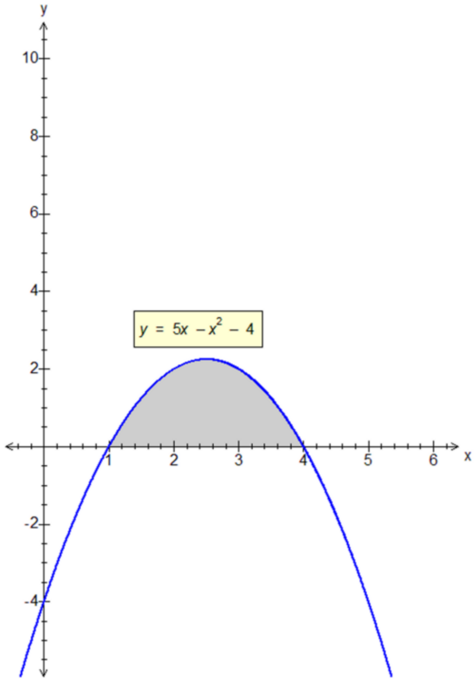
The resulting integral would be $\int_1^5 2x\,dx$ - $\int_1^5 x^2 - 4x + 5\,dx \;=\; \int_1^5 6x - x^2 - 5\,dx$

evaluating to $\left[3x^2 - \frac{x^3}{3} - 5x\right]_1^5 = \left[75 - \left(\frac{125}{3}\right) - 25\right] - \left[3 - \left(\frac{1}{3}\right) - 5\right]$

$= 8\frac{1}{3} - \left(-2\frac{1}{3}\right)$, or $10\frac{2}{3}$ square units.

Example (7): Find the area enclosed by the line $y = x-2$ and the quadratic $y = x^2 - 4x + 2$.





The line and the curve cross when $x^2 - 4x + 2 = x - 2$, i.e. $x^2 - 5x + 4 = 0$.
Factorising gives $(x - 1)(x - 4) = 0$, so the limits of the required integral would be $x = 1$ to $x = 4$.

The graph on the left shows that $y = x-2$ is the upper function of the two, and so we need to work out the integral as $\int_1^4 x - 2\ dx$ - $\int_1^4 x^2 - 4x + 2\,dx$.

Notice how the area to be integrated appears to be partly above the x–axis and partly below it, as on the graph on the left. This does not present a problem as in Example (3) because we are integrating the *difference* between the two functions, whose graph is on the right. This difference is positive throughout the interval as can be seen by the shaded area.

We can either work out the integrals separately and subtract the lower one from the upper one, or we can subtract before integrating. The second method is quicker, as it involves finding *one* integral rather than *two*, and so we will use this shorter method.

The resulting integral is $\int_1^4 x - 2\ dx$ - $\int_1^4 x^2 - 4x + 2\,dx = \int_1^4 5x - x^2 - 4\,dx$

which evaluates to $\left[\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right]_1^4 = \left[40 - \left(\frac{64}{3}\right) - 16\right] - \left[\frac{5}{2} - \left(\frac{1}{3}\right) - 4\right]$

$= 2\frac{2}{3} - \left(-1\frac{5}{6}\right)$, or $4\frac{1}{2}$ square units.

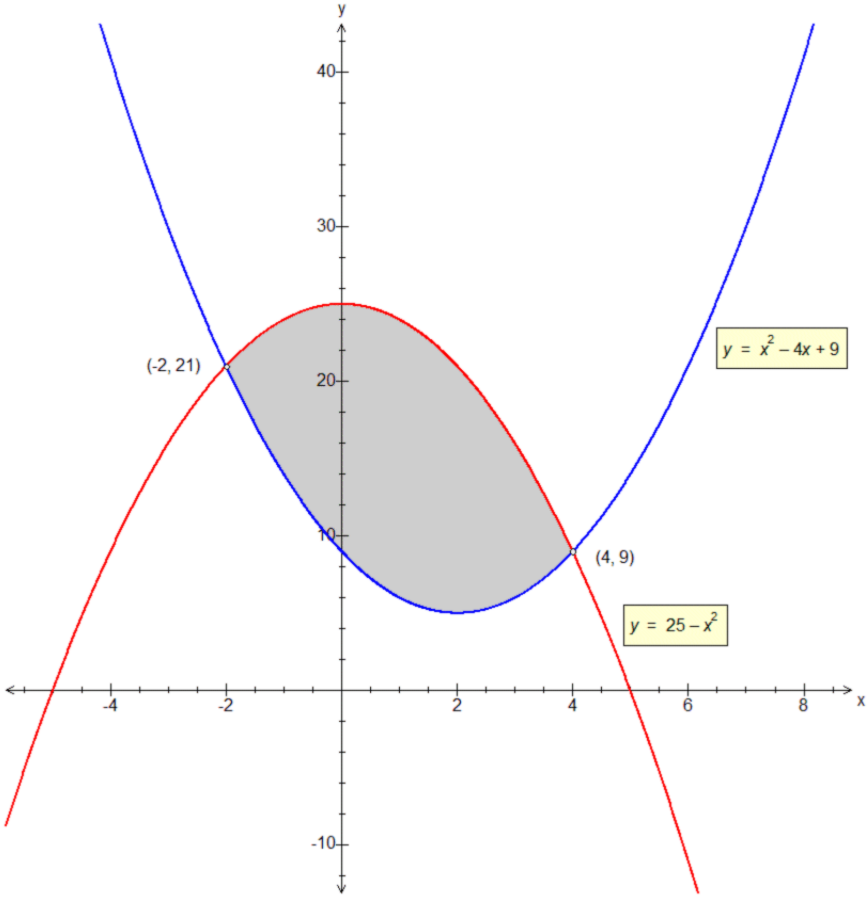
Example (8): Find the area enclosed by the quadratics $y = x^2 - 4x + 9$ and $y = 25 - x^2$.

The limits of the required integrand are the values of x where the two quadratic graphs intersect.

We must solve the equation
$x^2 - 4x + 9 = 25 - x^2$,
i.e. $2x^2 - 4x - 16 = 0$.

Taking out the common factor of 2 and factorising
$x^2 - 2x - 8 = 0$ gives
$(x + 2)(x – 4) = 0$.

$\therefore$ the required integrand would have limits $x = -2$ to $x = 4$.



The shaded area can be obtained by taking the area under the upper curve $y = 25 - x^2$, followed by taking the area under the lower curve $y = x^2 - 4x + 9$ and finally subtracting the second value from the first.

Therefore, the required area is given by

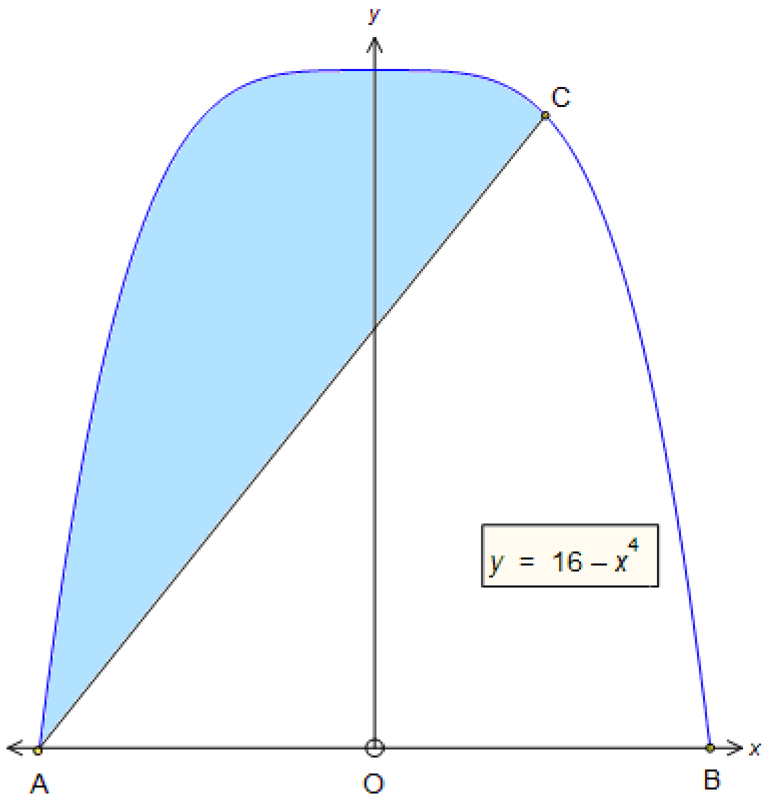
$$\int_{-2}^{4} 25 - x^2 \, dx - \int_{-2}^{4} x^2 - 4x + 9 \, dx = \int_{-2}^{4} 16 + 4x - 2x^2 \, dx$$

$$= \left[16x + 2x^2 - \frac{2x^3}{3} \right]_{-2}^{4} = \left[\left(64 + 32 - \frac{128}{3} \right) - \left(-32 + 8 + \frac{16}{3} \right) \right]$$

$$= \left[\left(96 - \frac{128}{3} \right) - \left(-24 + \frac{16}{3} \right) \right] = 120 - \frac{144}{3} = 72 .$$

Example (9) (Omnibus).

The points A, B and C lie on the curve $y = 16\text{-}x^4$.



i) Find the y-coordinate of point C, given that the x-coordinate of C is 1.

ii) Find the x-coordinates of points A and B.

iii) Find $\int_{-2}^{1} (16 - x^4)\,dx$.

iv) Hence calculate the area of the shaded region bounded by the curve and the line AC.

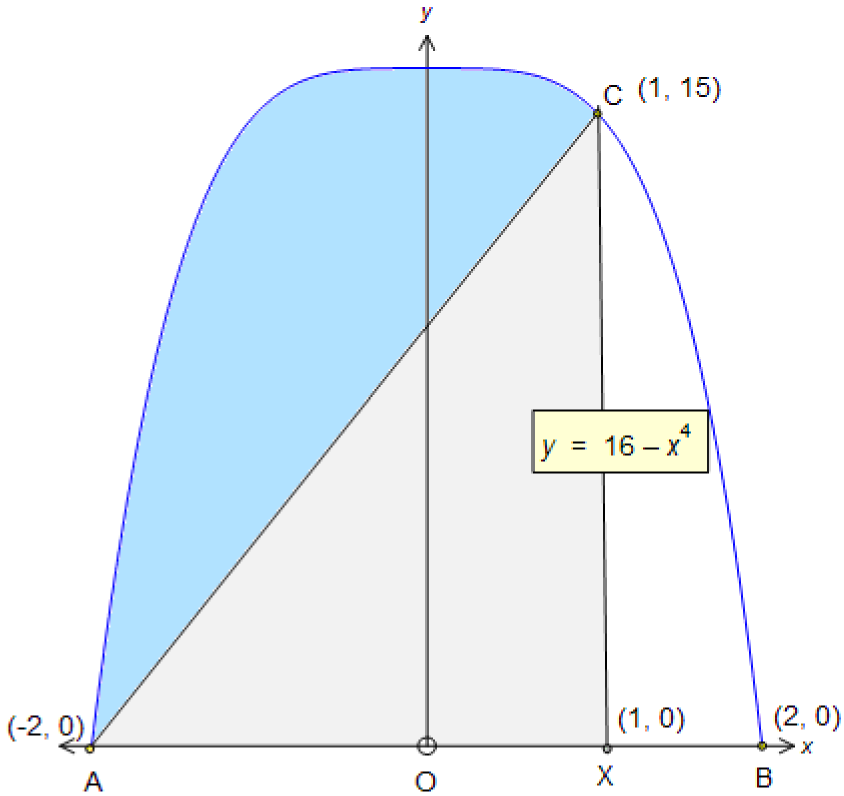
(Copyright AQA, GCE Mathematics Paper C1, May 2008, Q.5, altered)

i) The y-coordinate of point C is $16\text{-}1^4$, namely 15.

ii) The solutions of $16 - x^4 = 0$ are those of $x^4 = 16$, i.e. $x = 2$ and $x = -2$.
Therefore points A and B are (-2, 0) and (2, 0).

iii) $\int_{-2}^{1} (16 - x^4)\,dx = \left[16x - \frac{x^5}{5}\right]_{-2}^{1} = \left[\left(16 - \frac{1}{5}\right) - \left(-32 - \frac{32}{5}\right)\right]$

$= 15\frac{4}{5} + 25\frac{3}{5} \quad = \quad 41\frac{2}{5}$.

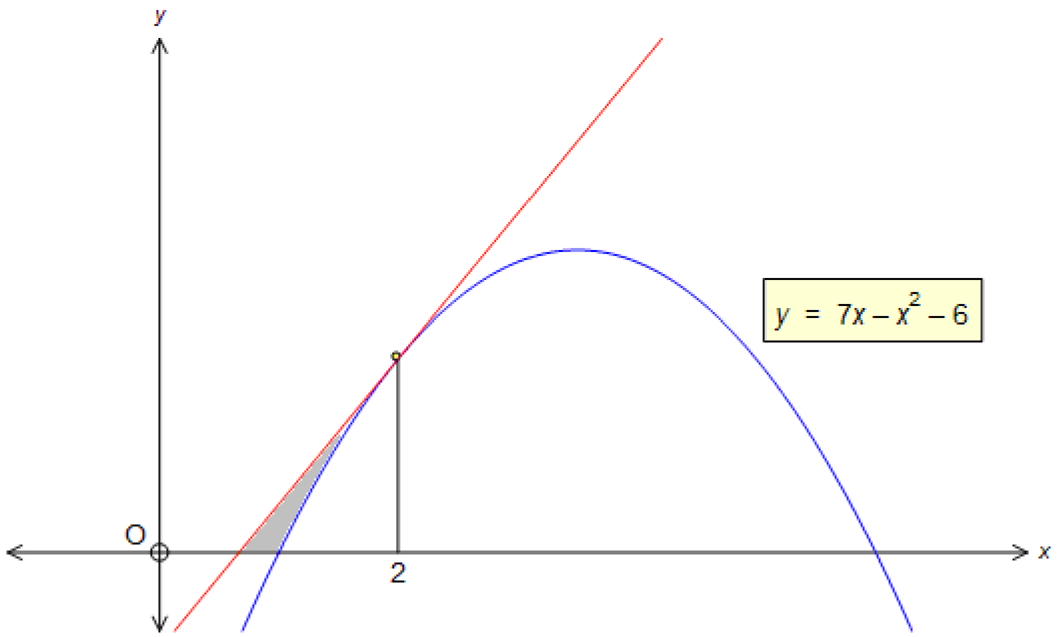


iv) The area of the shaded region can most easily be calculated by subtracting the area of the triangle AXC from the integral result of part iii). (This is easier than finding the equation of the line AC, which only creates extra work).

Since the base of this triangle is 3 units and the height is 15 units, its area is ½ × 3 × 15 = 22½ sq. units

The shaded area is therefore $41\frac{2}{5} - 22\frac{1}{2} \quad = \quad 18\frac{9}{10}$ square units.

Example (10) (Omnibus). The graph of $y = 7x - x^2 - 6$ is sketched below.



i) Find $\frac{dy}{dx}$ and hence find the equation of the tangent to the curve at the point on the curve where $x = 2$.
Show that this tangent crosses the x-axis where $x = \frac{2}{3}$.

ii) Show that the curve crosses the x-axis where $x = 1$ and find the x-coordinate of the other point of intersection.

iii) Find $\int_1^2 (7x - x^2 - 6)\,dx$.
Hence find the area of the shaded region bounded by the curve, the tangent and the x-axis.

i) $\frac{dy}{dx} = 7 - 2x$, so when $x = 2$, the gradient of the tangent is $(7 - 4) = 3$ at that point, whose y-coordinate is $(14 - 4 - 6)$ or 4. The curve therefore passes through the point (2, 4).

The equation of this tangent is $y - 4 = 3(x - 2) \Rightarrow y = 3x - 2$.
The tangent crosses the x-axis when $3x - 2 = 0$, i.e. when $x = \frac{2}{3}$.

ii) When $x = 1$, $y = 7x - x^2 - 6 \Rightarrow y = 7 - 1 - 6 = 0$, so the curve crosses the x-axis when $x = 1$.

Since, by the Factor Theorem, $(x - 1)$ is a factor of $7x - x^2 - 6$, inspection of the quadratic and constant terms reveals $(6 - x)$ as the other factor.

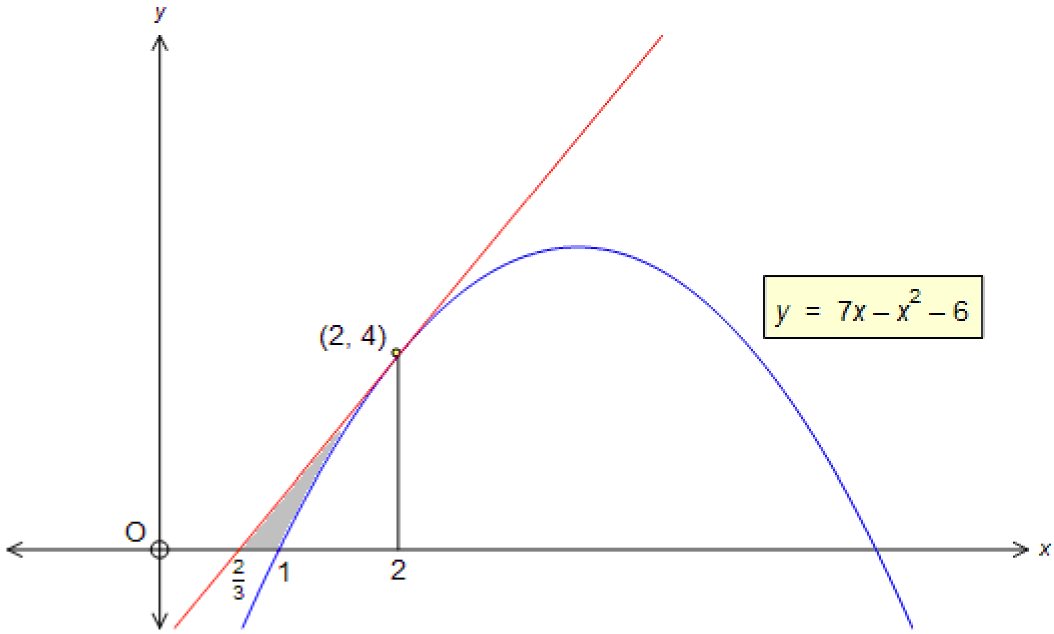
The other point of intersection of the curve and the x-axis is therefore (6, 0).

iii) $\int_1^2 (7x - x^2 - 6)\,dx = \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^2 = \left[\left(14 - \frac{8}{3} - 12\right) - \left(\frac{7}{2} - \frac{1}{3} - 6\right)\right]$

$= \left[\left(-\frac{2}{3}\right) - \left(-2\frac{5}{6}\right)\right] = 2\frac{1}{6}$.

To find the area of the shaded region, we need to subtract the integral just found from the area of the triangle bounded by the points ($\frac{2}{3}$, 0), (2, 0) and (2, 4).

The base of this triangle is $\frac{4}{3}$ units and its height 4 units, so its area is $\frac{1}{2} \times \frac{4}{3} \times 4 = 2\frac{2}{3}$ square units.



The area of the shaded region is therefore $2\frac{2}{3} - 2\frac{1}{6} = \frac{1}{2}$ square unit.

Mechanics examples.

Recall the following:

Velocity is the rate of change of displacement with respect to time, *t*, in other words, $v = \frac{ds}{dt}$.

Acceleration is the rate of change of velocity with respect to time, so $a = \frac{dv}{dt}$ or $a = \frac{d^2s}{dt^2}$,

To obtain velocity from displacement, or to obtain acceleration from velocity, we differentiated.

We can also carry out the processes in reverse as follows:

$s = \int v\,dt$ To find displacement from velocity, we integrate.

$v = \int a\,dt$ To find velocity from acceleration, we integrate.

Also, the area under a velocity / time graph is equal to the total displacement.

Example (11): A high-performance car is being driven along a long straight track for 80 seconds.

Its velocity is modelled by $v = 14t - 0.75t^2$. where t is the time in seconds and $0 \leq t \leq 8$.

i) Find the velocity of the car after 8 seconds.
ii) Find an expression for s , the distance travelled, given that at $t = 0$, $s = 0$.
iii) Find the total distance travelled by the car after 8 seconds according to this model.

After the car has been driven for 8 seconds, the velocity is modelled by
$v = 32\sqrt[3]{t},\ 8 < t \leq 64$.

iv) Explain why the previous model, $v = 14t - 0.75t^2$, cannot be used for $t > 20$.
v) Find the velocity of the car after 27 seconds.
vi) Find the **total** distance travelled by the car after 27 seconds.

vii) The car reaches its maximum velocity after 64 seconds, and assuming that this velocity remains constant, find the total distance travelled after 80 seconds.

i) When $t = 8$, $v = (14 \times 8) - 0.75(8^2) = 112 - 48 =$ **64 m/s**.

ii) We integrate to find the distance travelled by the car:

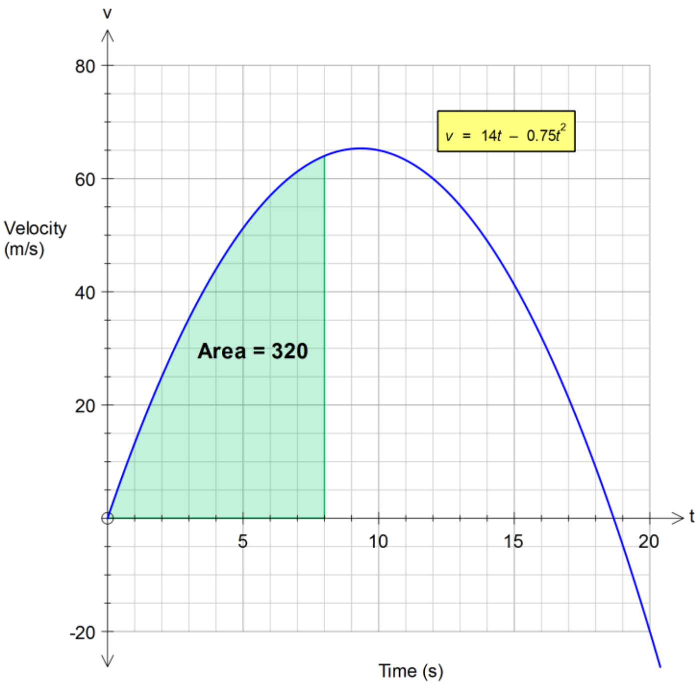
$$\int 14t - 0.75t^2\, dt = 7t^2 - 0.25t^3 + c$$

Since $s = 0$ when $t = 0$, $c = 0$, so $s = 7t^2 - 0.25t^3$.

iii) The total distance travelled by the car in 8 seconds is

$$\int_0^8 14t - 0.75t^2 dt = \left[7t^2 - 0.25t^3\right]_0^8 = 448 - 128 = 320 \text{ metres.}$$

This distance also corresponds to the area under the curve below.



iv) Substituting $t = 20$ into the formula $v = 14t - 0.75t^2$ would give a result of $v = 280 - 300$ or -20 m/s, which implies the car is being driven in reverse ! (See diagram above)

v) After 27 seconds, $v = 32\sqrt[3]{27} = 32 \times 3 = 96$ m/s.

vi) We need to evaluate $\int_8^{27} 32\sqrt[3]{t}\ dt$ here;

$$\int_8^{27} 32\sqrt[3]{t}\ dt = \int_8^{27} 32t^{\frac{1}{3}}\ dt = \left[\frac{32t^{\frac{4}{3}}}{\frac{4}{3}}\right]_8^{27} = \left[24t^{\frac{4}{3}}\right]_8^{27} = 24(81-16) = 1560.$$

The car has travelled 1560 metres from the end of the 8th second to the end of the 27th , but we must add the 320 metres travelled in the first 8 seconds, giving a total distance of **1880 metres**.

iii) To find the distance covered between the 8th second and the 64th, we evaluate

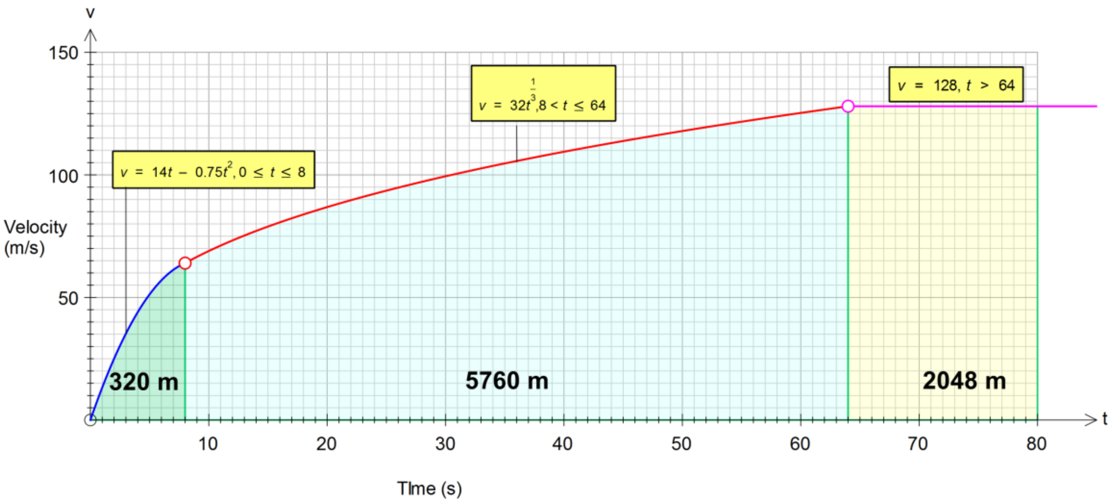
$$\int_8^{64} 32t^{\frac{1}{3}}\ dt = \left[24t^{\frac{4}{3}}\right]_8^{64} = 24(256-16) = 5760.$$

After 64 seconds, the velocity is $v = 32\sqrt[3]{64} = 32 \times 4 = 128$ m/s. (that is 286 mph !)

From 64s to 80s is another 16s, so the car will be travelling a further 16 × 128 metres, or 2048 metres.

More formally, $\int_{64}^{80} 128\ dt = [128t]_{64}^{80} = 10240 - 8192 = 2048.$

The total distance covered is therefore 320 + 5760 + 2048 = 8128 m (or about 5 miles !)
See diagram below.



Example (12): A particle moves in a straight line which passes through the fixed point *O*.

Its acceleration, in cm/s^2, is given by $a = 12 - 6t$ where t is the time in seconds and $0 \le t \le 8$.

i) Given that the initial velocity of the particle is 15 cm/s, find v in terms of t.

ii) Find the maximum value of v, and the time at which this maximum occurs.

iii) By factorising the expression for v, show that the particle has a maximum displacement at $t = 5$.

iv) Using the expression for v obtained in i), find $\int_0^5 v\,dt$ and $\int_5^8 v\,dt$.

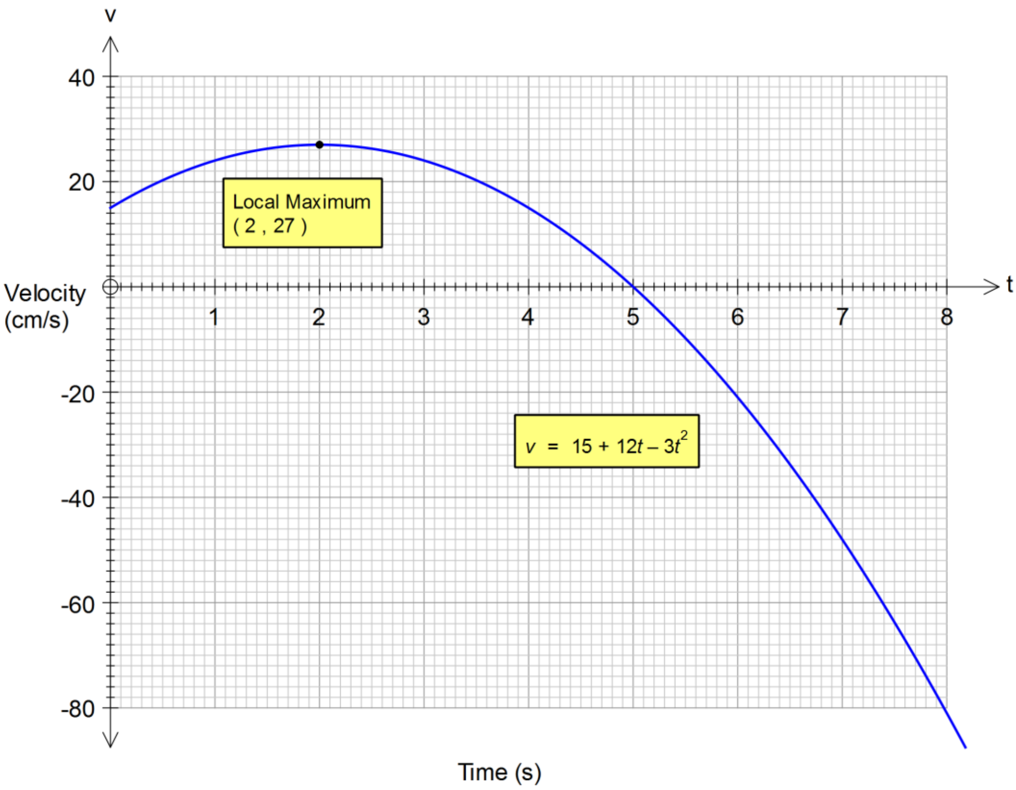
What do those results tell us about the displacement of the particle
a) after 5 seconds and b) after 8 seconds ?

i) We integrate a to find the velocity; $v = \int a\,dt = \int 12 - 6t\,dt = 12t - 3t^2 + c.$

Given $v = 15$ when $t = 0$, we have $c = 15$, so $\mathbf{v = 15 + 12t - 3t^2}$.

ii) Since $a = 0$ when v takes a maximum value, we solve $12 - 6t = 0$ giving $t = 2$.
Substituting $t = 2$, we have $v = 15 + 24 - 12 = 27$.

∴ The particle attains its maximum velocity of 27 cm/s after 2 seconds.



iii) Maximum displacement implies $v = 0$, so we solve the equation $15 + 12t - 3t^2 = 0$.

$15 + 12t - 3t^2 = 0 \quad \Rightarrow \quad 5 + 4t - t^2 = 0$ (dividing by 3) $\Rightarrow (5 - t)(1 + t) = 0$ (factorising)

Given the condition $0 \le t \le 8$, the only applicable solution is $t = 5$.

iv) Since $v = 15 + 12t - 3t^2$, we can integrate v to obtain the displacement s.

$$\int_0^5 15 + 12t - 3t^2 \; dt = \left[15t + 6t^2 - t^3\right]_0^5 = 75 + 150 - 125 = 100 .$$

$\therefore$ The particle moves 100 cm in the positive direction in the first five seconds, and from the result in iii), that value of 100 cm is the maximum displacement.

$$\int_5^8 15 + 12t - 3t^2 \; dt = \left[15t + 6t^2 - t^3\right]_5^8 = (120 + 384 - 512) - (75 + 150 - 125) = -108$$

$\therefore$ The particle then moves 108 cm in the negative direction between the fifth and the eighth seconds. As a result, is has a net negative displacement of 8 cm.

As an aside, we could have shown that result by evaluating
$$\int_0^8 15 + 12t - 3t^2 \; dt = \left[15t + 6t^2 - t^3\right]_0^8 = 120 + 384 - 512 = -8 .$$

See diagram below.