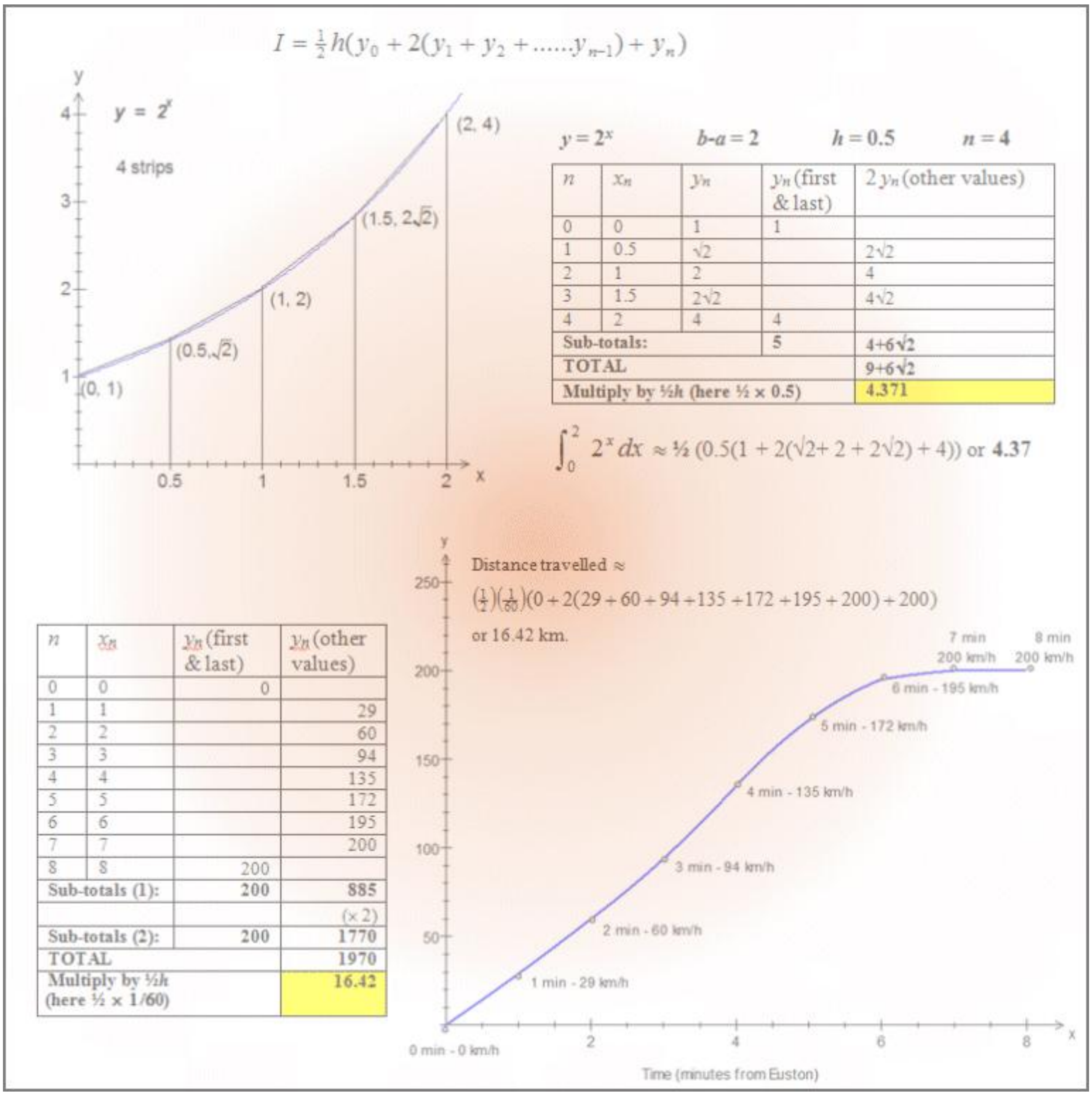
# M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C2** Edexcel: **C2** OCR: **C2** OCR MEI: **C2**

# NUMERICAL INTEGRATION - TRAPEZIUM RULE




| $n$ | $x_n$ | $y_n$ | $y_n$ (first & last) | $2y_n$ (other values) |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | |
| 1 | 0.5 | √2 | | 2√2 |
| 2 | 1 | 2 | | 4 |
| 3 | 1.5 | 2√2 | | 4√2 |
| 4 | 2 | 4 | 4 | |
| Sub-totals: | | | 5 | 4+6√2 |
| TOTAL | | | | 9+6√2 |
| Multiply by ½h (here ½ x 0.5) | | | | 4.371 |

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 0 | |
| 1 | 1 | | 29 |
| 2 | 2 | | 60 |
| 3 | 3 | | 94 |
| 4 | 4 | | 135 |
| 5 | 5 | | 172 |
| 6 | 6 | | 195 |
| 7 | 7 | | 200 |
| 8 | 8 | 200 | |
| Sub-totals (1): | | 200 | 885 |
| | | | (× 2) |
| Sub-totals (2): | | 200 | 1770 |
| TOTAL | | | 1970 |
| Multiply by ½h (here ½ x 1/60) | | | 16.42 |

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## NUMERICAL INTEGRATION.

Often, a function may prove difficult or impossible to integrate analytically. Definite integrals can however be obtained by numerical approximation.

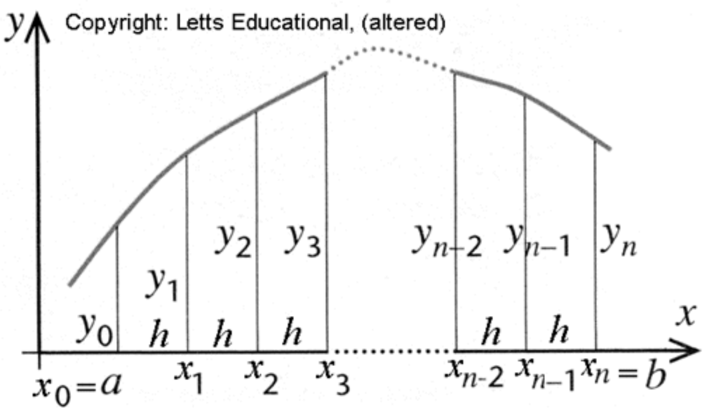
In the diagram below, the value of $I = \int_a^b f(x)dx$ represents the area under the graph below of $y = f(x)$ between the points $x = a$ and $x = b$.

This area can be approximated using the **trapezium rule.** This divides the area into a number of strips of equal width, treats each strip as a trapezium, and adds together the areas of all the trapezia.

The diagram shows that the width of each strip, and hence the height of each trapezium (turned on its side) has a value $h$. Because $x$ takes values from $a$ to $b$, it follows that $h = \frac{b-a}{n}$ where $n$ is the number of strips.

Moving from left to right, the first strip has an area of $\frac{1}{2}h(y_0 + y_1)$ , the second one an area of $\frac{1}{2}h(y_1 + y_2)$, the third $\frac{1}{2}h(y_2 + y_3)$ and so on until the last strip whose area is $\frac{1}{2}h(y_{n-1} + y_n)$ .

Note how each $y$-value is counted double except the first and the last.
Taking out $\frac{1}{2}h$ as a factor, the total area $I$ can be given by the trapezium rule as
$I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + ......y_{n-1}) + y_n)$.

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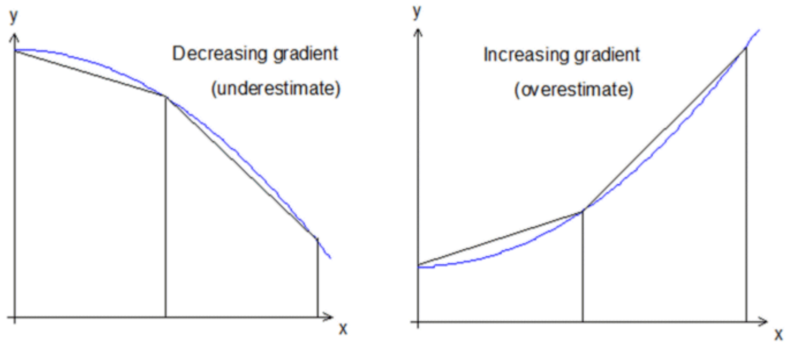
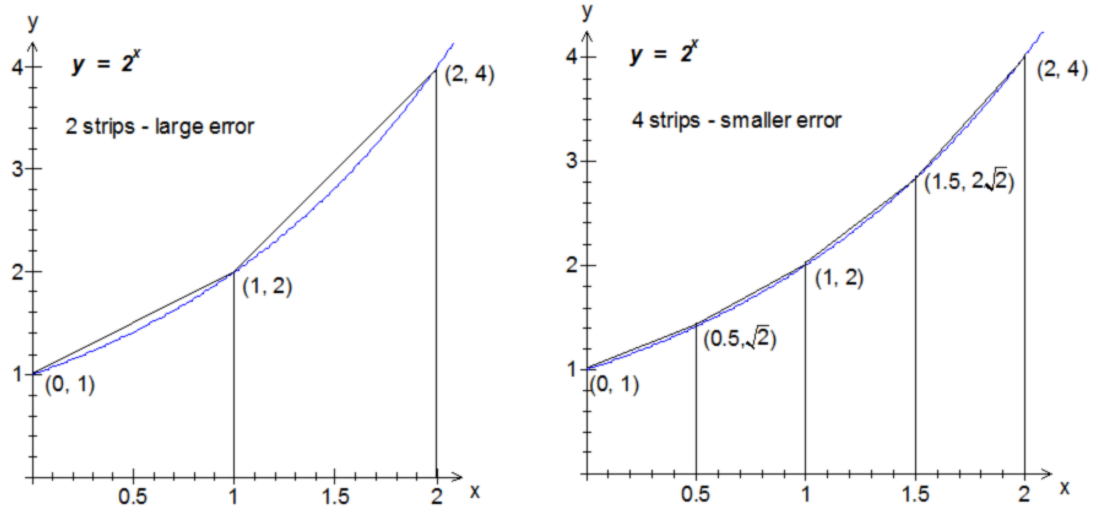
The trapezium rule usually requires quite a large number of strips to give a fairly good approximation to the area under a curve. If the gradient of the function to be integrated is decreasing, then the computed integral will be an underestimate; if the gradient is increasing, the computed value will be an overestimate.




In the left-hand diagram, the curve lies *above* the upper edges of the trapezia, so the error in using the trapezium rule is one of underestimation.

In the right-hand diagram, the reverse occurs. The curve lies *below* the upper edges of the trapezia, so the error is one of overestimation.

**Example (1)**: Use the trapezium rule with 4 strips to estimate the value of $\int_0^2 2^x \, dx$. Is the estimate going to be too large or too small ?




As can be seen in the above diagram, there is a large error in the example on the left, which is the result of applying the trapezium rule with only two strips. Because the upper edges of the trapezia are above the curve, the error will take the form of an over-estimate. The example on the right shows the working with four strips. The error is less pronounced here, so using the trapezium rule gives an area estimate of

$$I = \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots\ldots y_{n-1}) + y_n)$$

where the number of strips $n$ = 4, the interval $b - a$ = 2, the width of a single strip $h = \frac{b-a}{n}$ = 0.5, and the values of $y$ (ordinates) are 1, 1.414, 2, 2.828 and 4 to 3 decimal places.
(In surd form, they are 1, √2, 2, 2√2 and 4).

The estimated area under the curve is thus ½ (0.5(1 + 2( 1.414 + 2 + 2.828) + 4)) or **4.37 square units**.
(Or in surd form: ½ (0.5(1 + 2(√2+ 2 + 2√2) + 4)) or 4.37 square units).

The true value for the integral is 4.328 square units, suggesting an error of about 1% in the estimate.

Compare this with the result for two strips ($h$ = 1), which is ½ (1 + 2( 2) + 4) or **4.5 square units –** giving an error of about 4%.

It is best to tabulate the values for ease of checking; also, work to one more decimal place than is required for the final answer, or use exact forms. The first and last ordinates ($y$-values) are added in one column, and the others doubled and added together in another.

On the other hand, where there are fewer strips, it might be more convenient to write the calculation out in linear form, as in the earlier paragraphs.

For homework examples, such tables can be created rapidly using a spreadsheet program. Examination questions will usually be limited to 4 strips due to time issues.

In the longer form of the table, the $y$-values (other than the first and last) are shown doubled in the right-hand column and then added together.

$y = 2^x$ $b-a = 2$ $h = 0.5$ $n = 4$

Here, we have used exact surd forms for the working.

| $n$ | $x_n$ | $y_n$ | $y_n$ (first & last) | 2 $y_n$ (other values) |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | |
| 1 | 0.5 | √2 | | 2√2 |
| 2 | 1 | 2 | | 4 |
| 3 | 1.5 | 2√2 | | 4√2 |
| 4 | 2 | 4 | 4 | |
| **Sub-totals:** | | | **5** | **4+6√2** |
| **TOTAL** | | | | **9+6√2** |
| **Multiply by ½*h* (here ½ × 0.5)** | | | | **4.371** |

In the shorter form of the table, the $y$-values (other than the first and last) are shown undoubled in the right-hand column, and the result only doubled after the $y$-values are added together. This involves fewer calculator clicks, and might be handier if there are more than about 4 strips.

$y = 2^x$ $b-a = 2$ $h = 0.5$ $n = 4$

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 1 | |
| 1 | 0.5 | | √2 |
| 2 | 1 | | 2 |
| 3 | 1.5 | | 2√2 |
| 4 | 2 | 4 | |
| **Sub-totals (1):** | | **5** | **2+3√2** |
| | | | (× 2) |
| **Sub-totals (2):** | | **5** | **4+6√2** |
| **TOTAL** | | | **9+6√2** |
| **Multiply by ½*h* (here ½ × 0.5)** | | | **4.371** |

In each form of the table, the values of $n$ and the $x$-values are included for completeness; only the $y$-values are used in the calculation.

We could have also written the sum out in linear form:

$\int_0^2 2^x \, dx \approx \frac{1}{2}(0.5(1 + 2(\sqrt{2} + 2 + 2\sqrt{2}) + 4))$ or **4.37** .

**Example (2):** Use the trapezium rule with 8 strips to estimate $\int_0^2 2^x\,dx$ to two decimal places.

The previous result was about 1% in error and was not correct to one decimal place. Increasing the number of strips to 8 would reduce the error considerably.

We will again use the rule $I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots\ldots y_{n-1}) + y_n)$

but now there are 8 strips ($n$ = 8), the interval $b - a$ = 2, and so the width of a strip, $h$, is now 0.25.

Using $I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + \ldots\ldots y_{n-1}) + y_n)$, the results are:

***y* = 2*^x^*** ***b-a* = 2** ***h* = 0.25** ***n* = 8**

(Short form of table – other *y*-values doubled *after* summing in ‘Sub-totals (1)’)

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 1 | |
| 1 | 0.25 | | 1.189 |
| 2 | 0.5 | | 1.414 |
| 3 | 0.75 | | 1.682 |
| 4 | 1 | | 2 |
| 5 | 1.25 | | 2.378 |
| 6 | 1.5 | | 2.828 |
| 7 | 1.75 | | 3.364 |
| 8 | 2 | 4 | |
| **Sub-totals (1):** | | **5** | **14.855** |
| | | | (× 2) |
| **Sub-totals (2):** | | **5** | **29.711** |
| **TOTAL** | | | **34.711** |
| **Multiply by ½*h* (here ½ × 0.5)** | | | **4.339** |

The value of $\int_0^2 2^x\,dx$ is therefore estimated at 4.34 to two decimal places .

The true value for the integral is 4.3281 square units, and therefore the error has been reduced to just one unit in the second decimal place, or about 0.2%.

We have used decimal approximations here using 3 decimal places but again we could have used exact calculator values.

We could have also written the sum out in linear form:

$\int_0^2 2^x\,dx$ ≈ ½ (0.25(1 + 2(1.189 + 1.414 + 1.682 + 2 + 2.378 + 2.828 + 3.364) + 4)) or **4.34**.

**Example (3):** Use the trapezium rule with 5 strips to estimate the value of $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ to three decimal places.

The number of strips $n$ = 5, the interval is $b – a$ = 0.5, and so the width of a single strip, $h$, is 0.1.
Using $I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + ......y_{n-1}) + y_n)$, the results are:

$y = \frac{1}{\sqrt{1-x^2}}$ ***b*-*a* = 0.5** ***h* = 0.1** ***n* = 5**

(Short form of table – other *y*-values doubled *after* summing in 'Sub-totals (1)')

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 1 | |
| 1 | 0.1 | | 1.0050 |
| 2 | 0.2 | | 1.0206 |
| 3 | 0.3 | | 1.0483 |
| 4 | 0.4 | | 1.0911 |
| 5 | 0.5 | 1.1547 | |
| **Sub-totals (1):** | | **2.1547** | **4.1650** |
| | | | (× 2) |
| **Sub-totals (2):** | | **2.1547** | **8.3301** |
| **TOTAL** | | | **10.4848** |
| **Multiply by ½*h* (here ½ × 0.1)** | | | **0.5242** |

The value of $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ is approximately 0.524 to three decimal places.

(The true value is $\frac{\pi}{6}$ or 0.5236 to four decimal places, suggesting a smaller relative error in this example).

We could have also written the sum out in linear form:
$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ ≈ ½ (0.1(1 + 2(1.0050 + 1.0206 + 1.0483 + 1.0911) + 1.1547)) or **0.524**.

**Example (4):** Use the trapezium rule with 4 strips to estimate the value of $\int_0^4 x\sqrt{(x^2+9)}\,dx$ to one decimal place.

The number of strips $n$ = 4, the interval is $b - a$ = 4, and so the width of a single strip, $h$, is 1.

$y = x\sqrt{(x^2+9)}$ **$b$-$a$ = 4** **$h$ = 1** **$n$ = 4**

(Short form of table – other $y$-values doubled *after* summing in 'Sub-totals (1)')

Also, we have used exact surds when writing down the $y$-values.

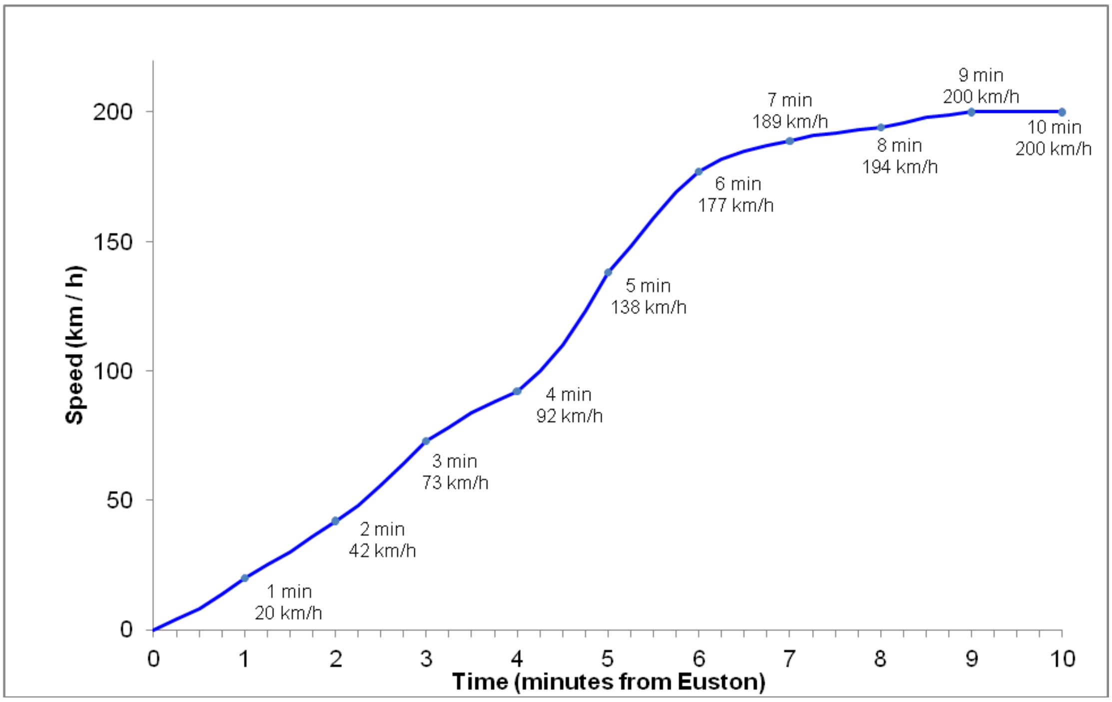
| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 0 | |
| 1 | 1 | | √10 |
| 2 | 2 | | 2√13 |
| 3 | 3 | | 3√18 |
| 4 | 4 | 20 | |
| **Sub-totals (1):** | | **20** | **23.10** |
| | | | (× 2) |
| **Sub-totals (2):** | | **20** | **46.20** |
| **TOTAL** | | | **66.20** |
| **Multiply by ½*h* (here ½ × 1)** | | | **33.1** |

Linear form of working:

$\int_0^4 x\sqrt{(x^2+9)}\,dx$ ≈ ½ (1(0 + 2(√10 + 2√13 + 3√18) + 20)) or **33.1**.

The previous examples were purely mathematical in nature, but the method can equally be applied in real-life situations. Two such examples follow overleaf.

**Example (5):** The travel graph below shows the speed (in km/h) of a train leaving London's Euston station, over a ten-minute time interval. (The area under the curve represents the distance travelled.)




Use the trapezium rule with 10 strips to find the distance covered by the train during this 10-minute interval, to the nearest 0.1 km. Remember to divide $h$ by 60 due to the use of km/h as the unit of speed.

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 0 | |
| 1 | 1 | | 20 |
| 2 | 2 | | 42 |
| 3 | 3 | | 73 |
| 4 | 4 | | 92 |
| 5 | 5 | | 138 |
| 6 | 6 | | 177 |
| 7 | 7 | | 189 |
| 8 | 8 | | 194 |
| 9 | 9 | | 200 |
| 10 | 10 | 200 | |
| **Sub-totals (1):** | | **200** | **1125** |
| | | | (× 2) |
| **Sub-totals (2):** | | **200** | **2250** |
| **TOTAL** | | | **2450** |
| **Multiply by ½*h* (here ½ × 1/60)** | | | **20.83** |

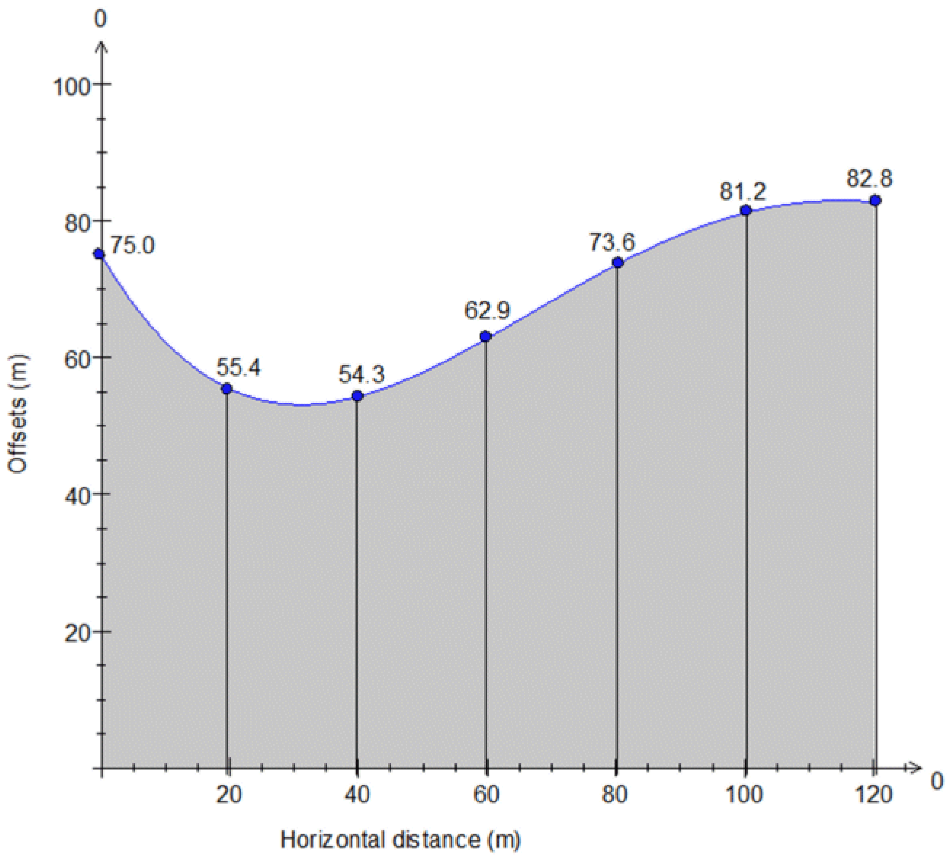
Linear form:

Distance travelled $\approx \left(\frac{1}{2}\right)\left(\frac{1}{60}\right)(0 + 2(20 + 42 + 73 + 92 + 138 + 177 + 189 + 194 + 200) + 200)$ km
= 20.83 km.

∴ The train has covered **20.8 km** in 10 minutes, using the trapezium rule with 10 strips.

**Example (6):** Estimate the area of the plot of land below, divided into 20m-wide strips. The perpendicular offset distances from the baseline are also shown here.

Use the trapezium rule with 6 strips to estimate the area in square metres, to 3 significant figures.




Here, $h$ = 20 and the $y$-values are the offsets.

| $n$ | $x_n$ | $y_n$ (first & last) | $y_n$ (other values) |
|---|---|---|---|
| 0 | 0 | 75.0 | |
| 1 | 20 | | 55.4 |
| 2 | 40 | | 54.3 |
| 3 | 60 | | 62.9 |
| 4 | 80 | | 73.6 |
| 5 | 100 | | 81.2 |
| 6 | 120 | 82.8 | |
| **Sub-totals (1):** | | **157.8** | **327.4** |
| | | | (× 2) |
| **Sub-totals (2):** | | **157.8** | **654.8** |
| **TOTAL** | | | **812.6** |
| **Multiply by ½*h* (here ½ × 20)** | | | **8126** |

Linear form:
Area $\approx \left(\frac{1}{2}\right)(20)(75.0 + 2(55.4 + 54.3 + 62.9 + 73.6 + 81.2) + 82.8)$ or 8126 $m^2$.

$\therefore$ Estimated area of the plot = **8130 $m^2$** to 3 s.f., using the trapezium rule with 6 strips.