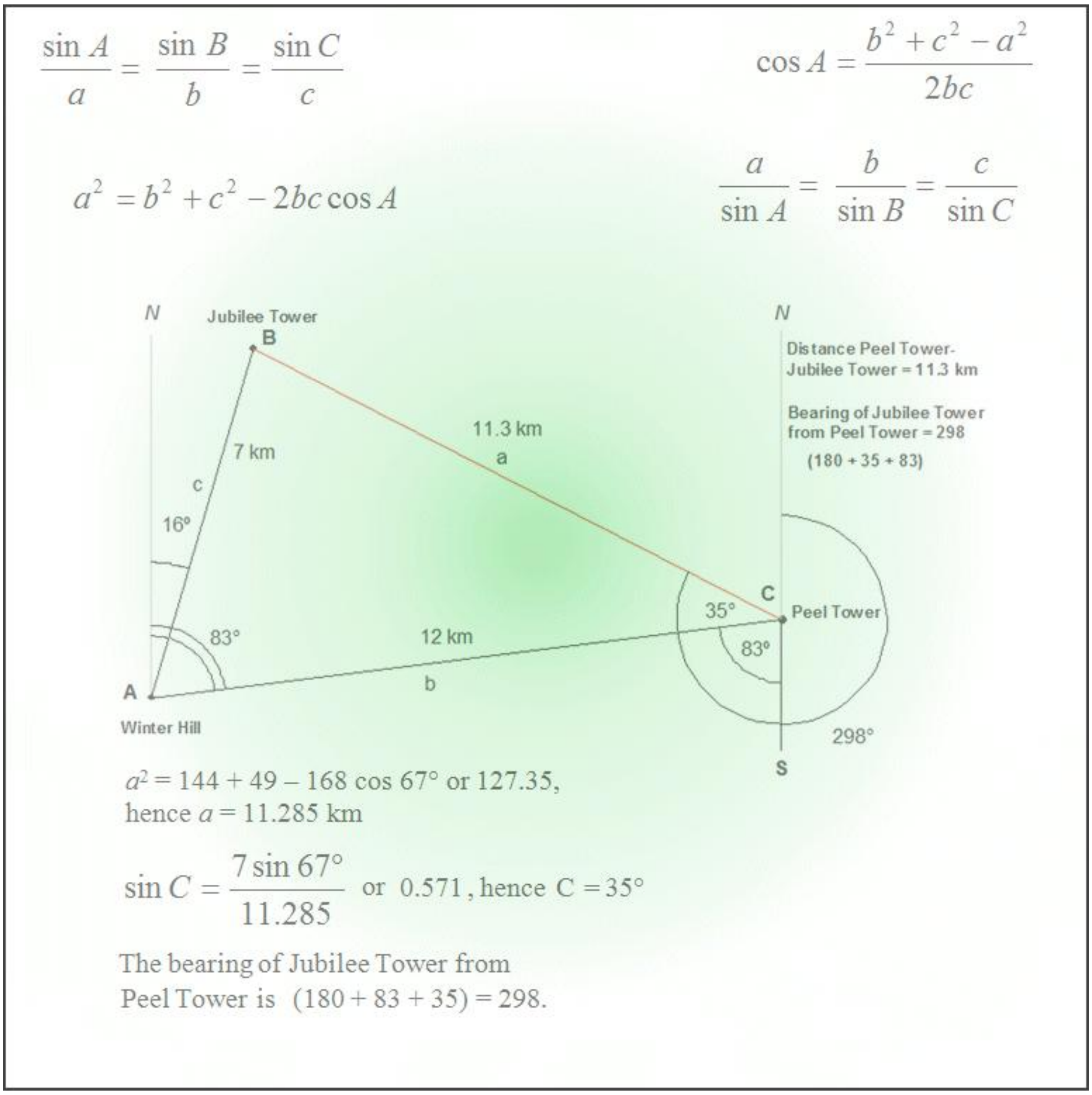
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C2** Edexcel: **C2** OCR: **C2** OCR MEI: **C2**

SOLVING TRIANGLES USING THE SINE AND COSINE RULES



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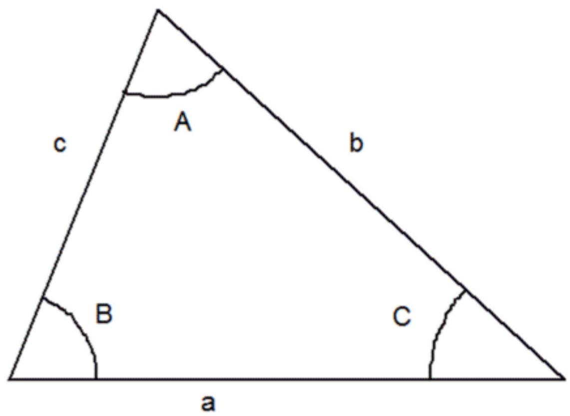
SOLUTION OF GENERAL TRIANGLES - THE SINE AND COSINE RULES.

The sine and cosine rules are used for finding missing sides or angles for **any** triangle, and not just for right-angled examples.

All lettered sides are opposite the corresponding lettered angles.

Area of a triangle.

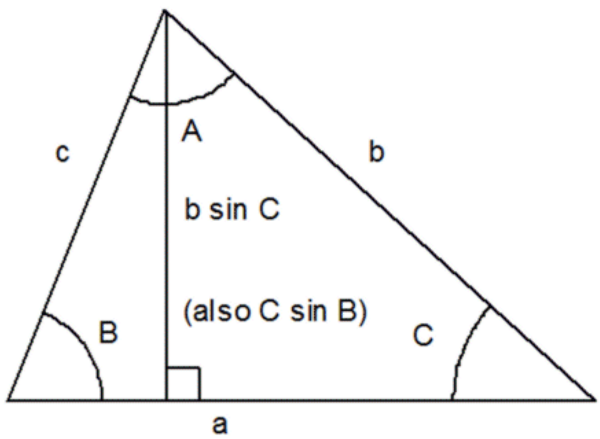


One formula for finding the area of a triangle is ½ (base) × (height). This can be adapted as follows:

By drawing a perpendicular from A, its length can be deduced by realising that it is opposite angle C, and that the hypotenuse is of length *b*. The length of the perpendicular, and thus the height of the triangle, is *b* sin C.

The area of the triangle is therefore ½*ab* sin C. Since any side can be used as the base, the formula can be juggled about as ½*ac* sin B or ½*bc* sin A.

This formula applies to both acute- and obtuse-angled triangles.



The sine rule.

The sides and angles of a triangle are related by this important formula:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The formula is normally used in the rearranged forms

$\sin A = \dfrac{a \sin B}{b}$ when finding an unknown angle, or $a = \dfrac{b \sin A}{\sin B}$ when finding an unknown side.
(In the cases shown above, angle *B* and the opposite side *b* are known, but not both angle *A* and side *a*.

(The corresponding letter-pairs are interchangeable, thus $\sin B = \dfrac{B \sin C}{c}$ and $c = \dfrac{a \sin C}{\sin A}$ are examples of other equally valid forms.)

Note that an equation of the form sin *A* = *x* has *two* solutions in the range 0° to 180°. Thus 30° is not the only angle with a sine of 0.5 - 150° is another one.

Any angle *A* will have the same sine as (180°-*A*). This is important when solving certain cases. However, this ambiguous case is not generally covered in the syllabus.

The cosine rule.

This is another formula relating the sides and angles of a triangle, slightly harder to apply than the sine rule.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

It is used in this form when finding an unknown side *a* where sides *b* and *c* are known, along with the included angle *A*.

If we are given three sides but need to find an unknown angle, then the other form is used:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The sides containing the angle *A* are added and the side opposite angle *A* is subtracted to give the top line, whilst twice the product of the sides containing the angle gives the bottom line.

This formula can also be rotated between different sides and angles: thus

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

all have the same effect.

The formulae for missing angles can be similarly rotated:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Which rules should we use ?

This depends on the information given.

i) Given **two angles and one side** - use the **sine rule**. (If the side is not opposite one of the angles, you can work out the third angle simply by subtracting from 180°).

ii) Given **two sides and an angle opposite one of them** - use the **sine rule**. Care is needed here, as some cases can give rise to two possible solutions, although this ambiguous case is not generally covered in the syllabus.

iii) Given **two sides and the included angle** - use the **cosine rule** to find the third side and then continue with the **three sides and one angle** case below.

iv) Given **three sides and no angles** - use the **cosine rule** to find the angle opposite the longest side, followed by the sine rule for either of the others. The third angle can be found by subtraction.

We find the angle opposite the longest side first, as it will be the largest angle. That angle might be either obtuse or acute, but the other two could only be acute. There would therefore be no danger of ambiguity when applying the sine rule.

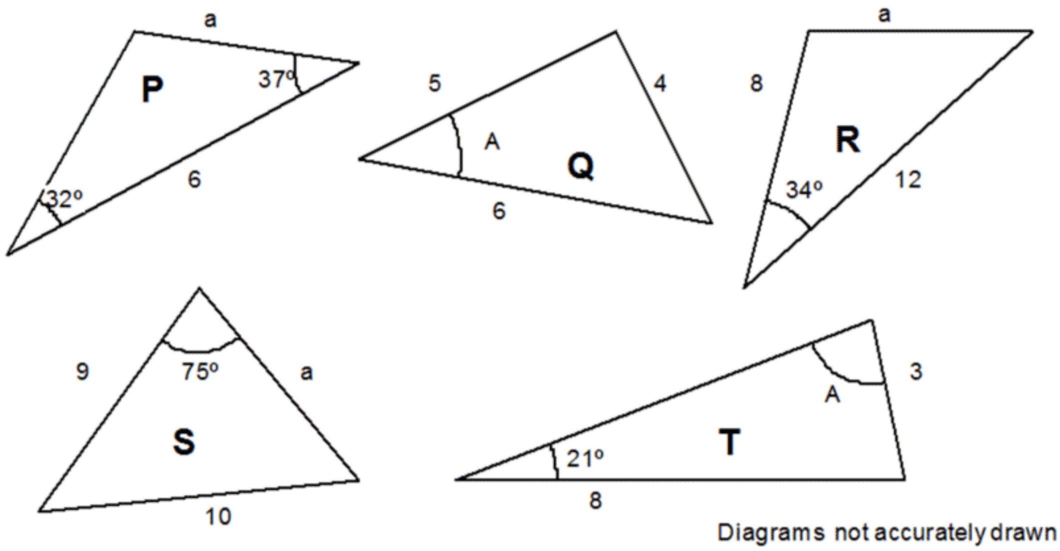
v) Given **three sides and one angle** - either rule can be used.

If the known angle is opposite the longest side (it could be obtuse), then apply the **sine rule** to find the angle opposite either of the remaining sides.

If the known angle is opposite one of the other sides, then either
a) apply the **sine rule** to find the angle opposite the shorter of the remaining sides (must be acute), and then find the third angle by subtraction, or
b) apply the **cosine rule** to find the angle opposite the longer of the remaining sides (might be obtuse), and then find the third angle by subtraction.

The sine rule is easier to use, but the cosine rule will never give ambiguous results.

Example (1): Find the angles marked A and the sides marked *a* in the triangles below. Assume in this example that triangle **T** is **acute-angled.**



<u>Triangle **P**.</u>

Two angles and a side are known. The known side is not opposite either of the known angles, but the opposite angle (call it *B*) can easily be worked out by subtracting the other two angles from 180°. This makes *B* = 111° and *b* = 6 units. We will also label the 32° angle *A* as it is opposite side *a*.

We therefore use the sine rule in the form $a = \dfrac{b\sin A}{\sin B} \Rightarrow a = \dfrac{6\sin 32°}{\sin 111°}$, or 3.41 units to 2 d.p.

<u>Triangle **Q**.</u>

All three sides are known here but we are required to find angle *A*. Labelling side *a* as the opposite side (length 4 units), we will call the side of length 5 side *b* and the side of length 6 side *c*.

This time we use the cosine rule in the form $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Substituting for *a*, *b* and *c* gives

$\cos A = \dfrac{25 + 36 - 16}{60} \Rightarrow$ *A* = 41.4° to 1 d.p.

<u>Triangle **R**.</u>

Here we have two sides plus the included angle given. Label the angle of 34° as *A*, the side of length 8 as *b*, and the side of length 12 as *c*.

We must therefore substitute the values of *A*, *b* and *c* into the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

This gives $a^2 = 64 + 144 - 192\cos 34° \Rightarrow a = \sqrt{208 - 159.2}$ or 6.99 units to 2 d.p.

<u>Triangle **S**.</u>

Here we have two sides given, plus an angle *not* included. Label the angle opposite *a* as *A*, the 75° angle as *B*, the side of length 10 as *b*, the side of length 9 as *c*, and the angle opposite *c* as *C*. To find *a* we need to apply the sine rule twice.

First we find angle *C* using $\sin C = \dfrac{c \sin B}{b} \Rightarrow \sin C = \dfrac{9 \sin 75°}{10}$

The value of sin *C* is 0.8693 to 4 dp, but it must be remembered that there are two possible solutions to this. One value of *C* is 60.4°, but the angle of (180°-60.4°) or 119.6° also has the same sine.

The obtuse angle of 119.6° can be rejected however, because one angle of the triangle is 75° and the other two cannot add up to more than 105°. Angle *C* is therefore 60.4°.

To find side *a*, we must find angle *A.* The angle can be worked out as 180 - (75 + 60.4) degrees, or 44.6°.

Then we use the sine rule again: $a = \dfrac{b \sin A}{\sin B} \Rightarrow a = \dfrac{10 \sin 44.6°}{\sin 75°}$, giving *a* = 7.27 units to 2 d.p.

<u>Triangle **T**.</u>

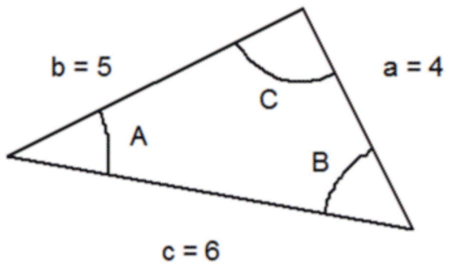
Again we have two sides given, plus an angle *not* included. We use the sine rule again , this time to find angle *A*. Label the side of length 8 as *a*, the angle of 21° as *B*, and the side of length 3 as *b*.

Applying the sine formula in the form $\sin A = \dfrac{a \sin B}{b}$ we get $\sin A = \dfrac{8 \sin 21°}{3}$,

or sin *A* = 0.9556 to 4 d.p.

We are told that the triangle is acute-angled, so angle *A* = 72.9°.

Example (2): Solve triangle **Q** from example 1 by finding all three missing angles – also calculate its area.



After labelling as above, the first step would be to find angle *C*, opposite the longest side**.** This uses the cosine formula.

We use the form
$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.
Substituting for a, b and c gives

$\cos C = \dfrac{16 + 25 - 36}{40} \Rightarrow A = 82.8°$ to 1 d.p.
(keep more accuracy, 82.82°, for future working)
We now have enough information to work out the area of the triangle, as we have found the included angle *C*.

The area of the triangle is thus ½*ab* sin *C*, or 10 sin 82.8° = 9.85 sq.units.

To find the other two angles, we use the sine rule to find one of them and then subtract the sum of the other two angles from 180° to find the third.

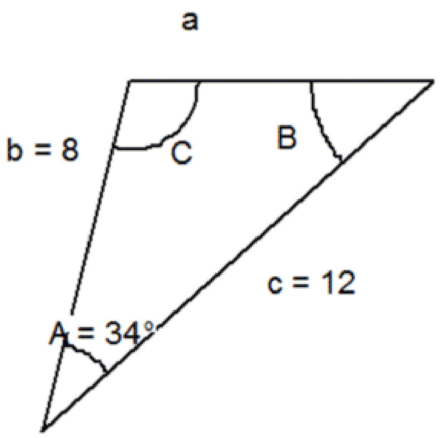
The reason for using the longest side first is to prevent ambiguous results when using the sine rule. No triangle can have more than one obtuse angle, and the longest side is always opposite the largest angle. The cosine rule would take care of the obtuse angle if there was one, leaving no possibility of confusion when using the sine rule to work out the other two. In fact, angle *C* is acute in this case, so we have an acute-angled triangle.

We can choose either remaining side to work out the other angles - here we'll find *B* first using the sine rule.
$\sin B = \dfrac{b \sin C}{c} \Rightarrow \sin B = \dfrac{5 \sin 82.82°}{6} \Rightarrow \sin B = 0.8268.$

This gives *B* = 55.8° to 1 d.p. (only the acute angle is valid here)

To find *C*, we subtract the sum of *A* and *B* from 180°, hence *C* = 41.4° to 1 d.p.

Example (3): Solve triangle **R** from example 1 by finding its area, the two missing angles and the missing side.



We can work out the area at once as ½bc sin A. This gives 48 sin 34° or , 26.84 sq.units.

Then, we find the missing side a by substituting the values of A, b and c into the cosine formula
$$a^2 = b^2 + c^2 - 2bc\cos A$$

This gives $a^2 = 64 + 144 - 192\cos 34°$ or 48.8 $\Rightarrow$ $a = \sqrt{208 - 159.2}$ or 6.99 units to 2 d.p.
(Keep greater accuracy for future calculation - 6.987).

After finding a , the next step is to find one of the two missing angles. Both methods are shown here for illustrative purposes - choose the one you're happier with.

Using Cosine Rule.
Choose angle C as the next angle**,** since it is opposite the longer side, here c. (This will take care of a potential obtuse angle solution.)

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \cos C = \frac{48.8 + 64 - 144}{2 \times 6.987 \times 8} \Rightarrow \cos C = -0.2791.$$

This gives C = 106.2° to 1 d.p. (note that obtuse angles have a negative cosine).

Angle B can be found simply by subtracting the sum of A and C from 180°. It is thus 180 - (34 + 106.2)° or 39.8°.

Using Sine Rule.
We have one known angle, A, of 34°, so we know that one of the remaining ones must be acute since all triangles have at least two acute angles. We therefore use the sine rule to find the angle opposite the shorter of the remaining sides, namely side b.

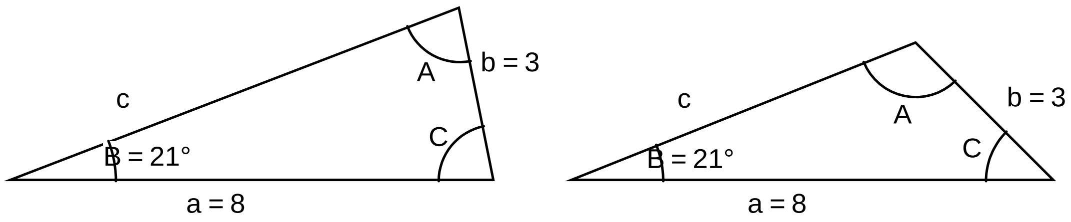
Applying the sine formula in the form $\sin B = \dfrac{b\sin A}{a}$ we get $\sin B = \dfrac{8\sin 34°}{6.987}$ $\Rightarrow$ sin B = 0.6403 to 4 d.p. $\Rightarrow$ B = 39.8°

Angle C can be found by subtraction, being equal to (180 - (34 +39.8))°, or 106.2°.

The ambiguous case. (Rare in practice)

Example (4): Solve triangle **T** from example 1 completely by finding the two missing angles and the missing side, and find the area. This time, triangle **T** is not necessarily acute-angled.

This question illustrates the ambiguity of solutions when an angle *not* included and two sides are given. There are *two* triangles possible, as the diagram below shows.



The first step is to find angle A using the sine formula in the form $\sin A = \dfrac{a \sin B}{b}$.

We get $\sin A = \dfrac{8 \sin 21°}{3}$ $\Rightarrow$ $\sin A = 0.9556$ to 4 d.p.
This gives *two* possible values for A, 72.9° or 107.1°.

The obtuse angle is valid this time, as angle B is equal to 21°. Angles A and B would add up to 128.1°, well short of 180°.

The two possible angle solutions given the data above are thus:

Acute-angled solution: A = 72.9°, B = 21°, C = 86.1°
Obtuse-angled solution: A = 107.1°, B = 21°, C = 51.9°

Because both triangles are of different shape, the lengths of side c and the areas will be different in each case.

Using the sine rule for the acute-angled solution we have

$c = \dfrac{b \sin C}{\sin B}$ or $c = \dfrac{3 \sin 86.1°}{\sin 21°}$ $\Rightarrow$ c = 8.35 units to 2 d.p.

The area of the acute-angled triangle is ½ab sin C or 12 sin 86.1° sq.units = 11.97 sq.units.

For the obtuse-angled solution we would have

$c = \dfrac{3 \sin 51.9°}{\sin 21°}$ $\Rightarrow$ c = 6.59 units to 2 d.p.

The area of the obtuse-angled triangle is ½ab sin C or 12 sin 51.9° sq.units = 9.44 sq.units.

Real-life applications of Sine and Cosine Rules.

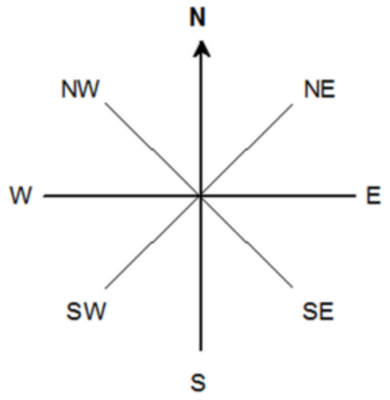
The sine and cosine rules can be used to solve real-life trigonometry problems.

Bearings (revision)

A **bearing** of a point **B** from point **A** is its compass direction generally quoted to the nearest degree, and stated as a number from 000° (North) to 359°.
(In practice, leading zeros are included when quoting bearings.)

Bearings are measured **clockwise** from the **northline**.

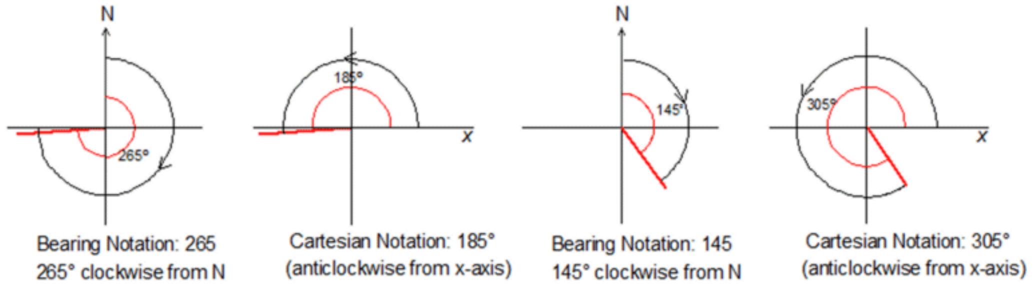
Example(1): Express the eight points of the compass shown in the diagram as bearings from north.



N – 000°;	NE – 045°;	**E – 090°;**	SE – 135°
S – 180°;	SW – 225°;	**W – 270°;**	NW – 315°

<u>Difference between bearing and Cartesian angular notation.</u>

Bearings are measured clockwise from north, but Cartesian angles are measured anticlockwise from the *x*-axis. This is why diagrams are useful to avoid confusion between the two.



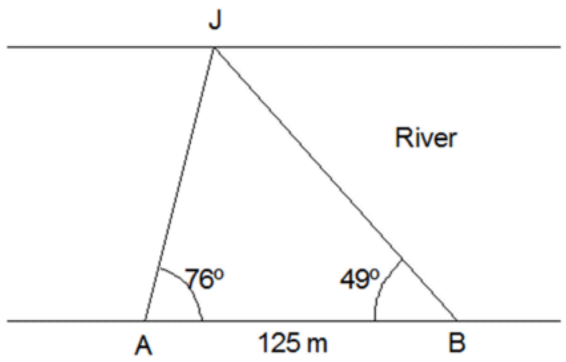
In Cartesian form, a bearing of 265° corresponds to 185°; a bearing of 145° corresponds to 305°.

Example(5):

A and B are points 125 metres apart on the same side of a straight river with parallel banks. The points A and B make angles of 76° and 49° respectively with a jetty J.

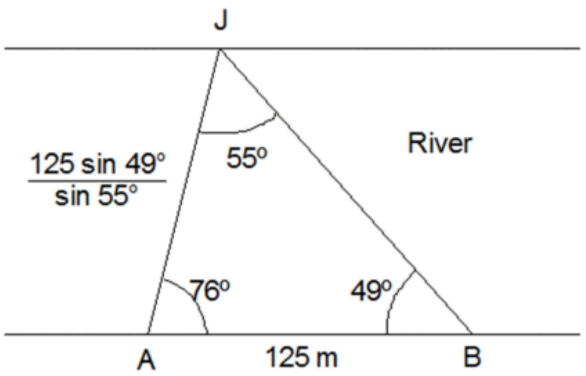
Calculate the width of the river to the nearest metre.

(Copyright OCR 2004, MEI Mathematics Practice Paper C2-A, 2004, Q. 6, altered)



The first step is to find ∠AJB, which works out as 55° (sum of angles), and from there we use the sine rule to find the length AJ.

AJ = $\dfrac{125\sin 49°}{\sin 55°}$ = 115.17m to 2 d.p.

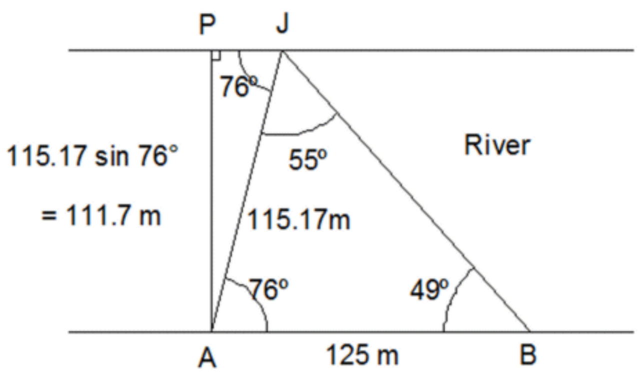


(We could equally well have chosen to find the length of BJ.

BJ = $\dfrac{125\sin 76°}{\sin 55°}$ = 148.06m to 2 d.p.)

Finally we draw a perpendicular from A at the point P.

Side AJ is the hypotenuse of the triangle APJ, and therefore the length AP, and hence the width of the river, is 115.17 sin 76° m, or 111.7m, or **112m** to the nearest metre.

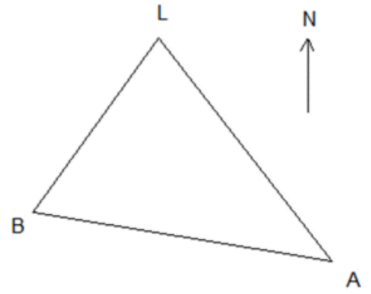


(Had we used side BJ, the width of the river would be 148.06 sin 49° or again 112m.)

Either way, the width of the river could have been calculated in one step as

$\dfrac{125\sin 76°\sin 49°}{\sin 55°}$ = 112m.

Example (6): A ship's captain measures the bearing of a lighthouse L and finds that it is 322° at 13:45, when his position is at A on the diagram.



At 14:30 he is at point B, and takes another reading of the lighthouse's bearing and finds it to be 036°. During this time the ship's course is 279°.

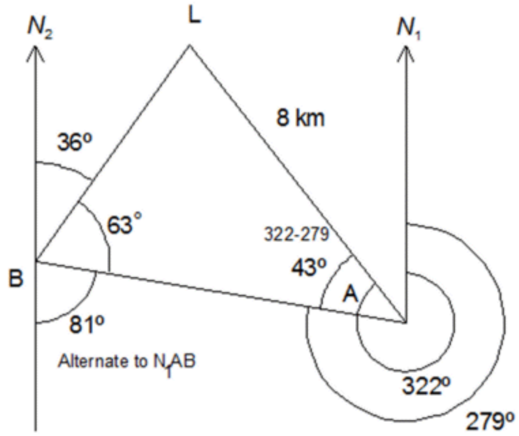
i) Write down the size of the angles LAB and LBA.

ii) The captain reckons that point A is 8 km from L. Assuming that LA is exactly 8 km from A, show that LB is 6.12 km correct to 2 d.p, and find AB, thus calculating the ship's speed.

iii) The actual speed of the ship is 12.5 km/h. Given that the bearings and the ship's course are all correct, calculate the true distance LA.

i) The first step is to draw in northlines at N_1 and N_2 to points A and B, and then use the known angles to determine the unknown ones. (Remember bearings are measured clockwise from north.)

∠LAB = $\angle N_1AL - \angle N_1AB$ = 322°-279° = **43°.**
(using the reflex angles)

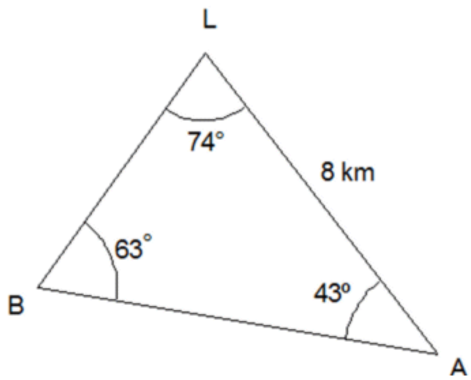


We also have $\angle N_2BL$ = 36°, and because the non-reflex value of $\angle N_1AB$ = 360° - 279° = 81°, then by alternate angles, the angle between the line AB and the south is also 81°. Hence **∠LBA = 63°** (angle in a straight line = 180°).

ii) By sine rule, LB = $\dfrac{8\sin 43°}{\sin 63°}$ = 6.12 km (2 d.p.)

To find AB we deduce that ∠BLA = 74° (angle sum of triangle), and then we can apply either the sine or the cosine rule.

By sine rule, AB = $\dfrac{8\sin 74°}{\sin 63°}$ = 8.63 km (2 d.p.)_



By the cosine rule,

$(AB)^2 = (LA)^2 + (LB)^2 - 2.LA.LB \cos 74°$

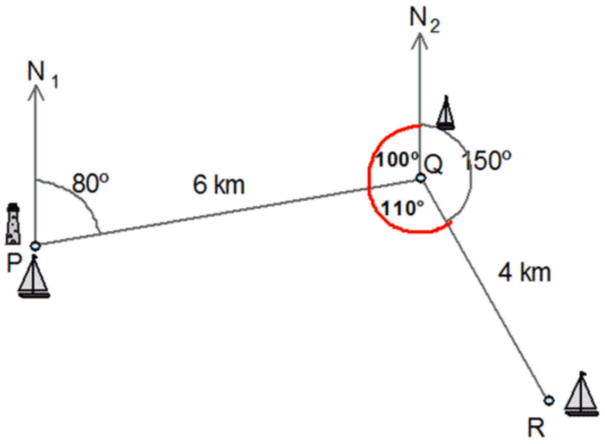
$= 8^2 + 6.12^2 - 97.8 \cos 74° =$
$64 + 37.5 - 27.0 = 74.5 \Rightarrow AB = 8.63$ km to 2 d.p.

Because 45 minutes, or 0.75 hours, elapse between the ship's positions at points A and B, the speed of the ship is $\dfrac{8.63}{0.75}$ or 11.52 km/h.

Example (7): A yachtsman passes a lighthouse at point **P** and sails for 6 km on a bearing of 080° until he reaches point **Q**. He then changes direction to sail for 4 km on a bearing of 150°.

Work out the yachtsman's distance and bearing from the lighthouse at point **R**, after the second stage of his sailing.

(Although this is an accurate diagram, only a sketch is required).

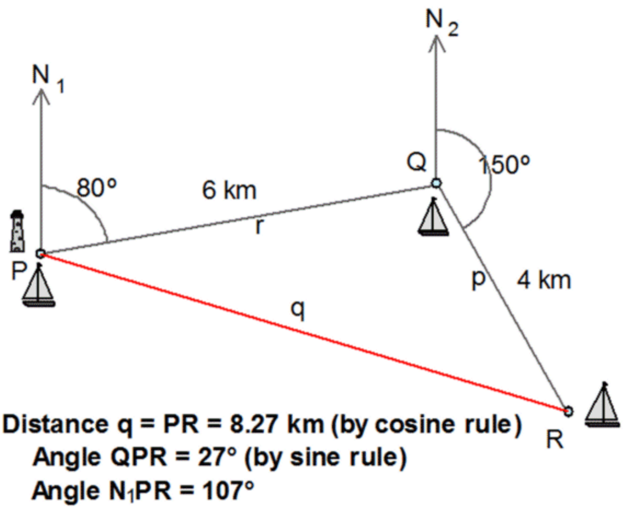


We can find $\angle$ PQR by realising that $\angle$ N_1PQ and $\angle$ PQN_2 are supplementary, i.e. their sum is 180°. Hence add $\angle$ PQN_2 = (180 – 80) = 100°.

Because angles at a point add to 360°, $\angle$ PQR = 360 – (100 + 150) = 110°

We can now find the distance PR as being the third side of triangle PQR – we have two sides (4 km and 6 km) and the included angle of 110°.

We label each side as opposite the angles, and use the cosine rule to find side q (PR) first :



$q^2 = r^2 + p^2 - 2pr \cos Q$

This gives $q^2 = 36 + 16 - 48 \cos 110°$ or 68.41, and hence q = 8.27 km (Keep higher accuracy for future calculation - 8.271).

We can then use the sine rule to find angle QPR (P) and hence the yacht's final bearing from northline N_1.

Applying the sine rule, we have

$$\frac{\sin \angle QPR}{4} = \frac{\sin 110°}{8.271} \Rightarrow$$

$$\sin \angle QPR = \frac{4 \sin 110°}{8.271}$$

or sin $\angle$QPR = 0.4545 to 4 d.p. Angle QPR is hence 27°

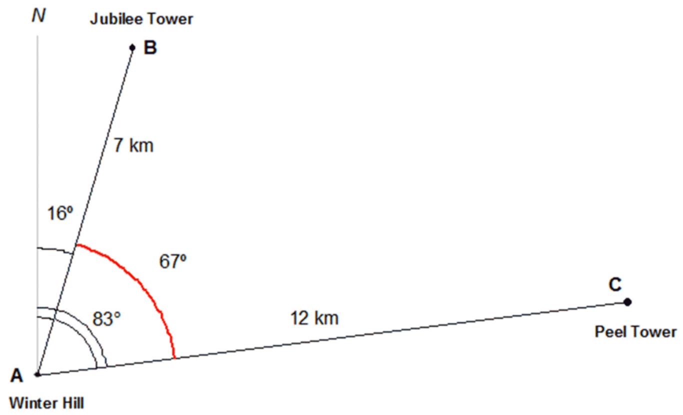
The yacht's final bearing from northline N_1 is (80 + 27) °, or 107°, and its distance from the lighthouse at P is 8.27 km.

Example (8): Peel Tower is 12 km from Winter Hill, on a bearing of 083°, whereas Jubilee Tower is 7 km from Winter Hill, on a bearing of 016°.

Find the distance and bearing of Jubilee Tower from Peel Tower.

First, we sketch the positions of the northline and the three landmarks in question.

We also label sides opposite corresponding angles with lower-case letters.
Note that angle A = (83 – 16) = 67°.



Firstly, we find the length of the side *a* of the triangle, and to do so, we use the cosine rule.

$a^2 = b^2 + c^2 - 2bc \cos A$.

Substituting,

$a^2 = 144 + 49 - 168 \cos 67° = 127.4$,
and hence the distance between Jubilee Tower and Peel Tower is $a = 11.29$ km (Keep higher accuracy for future calculations).

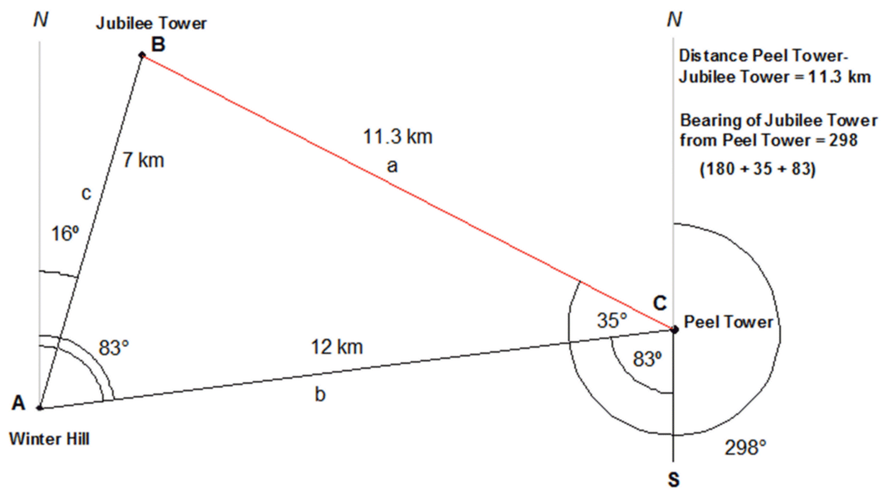
To find the bearing of Jubilee Tower from Peel Tower, we draw a southern continuation of the northline at S and use alternate angles to find ∠ACS = 83°

The bearing required is therefore (180 + 83)° + angle C (to be determined).

Angle C can be found by the sine rule:

$$\frac{\sin C}{7} = \frac{\sin 67°}{11.29} \Rightarrow \sin C = \frac{7 \sin 67°}{11.29} \Rightarrow \sin C = 0.571 \text{ to 3 d.p.}$$

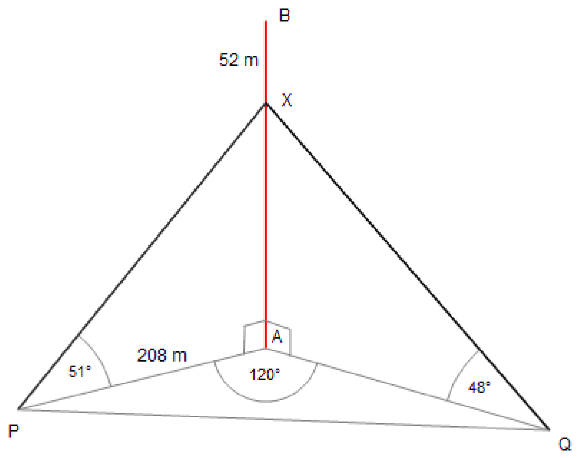
Angle C is hence 35°, so the bearing of Jubilee Tower from Peel Tower is (180 + 83 + 35)° = 298°.



∴ **Jubilee Tower is 11.3 km from Peel Tower on a bearing of 298.**

The next example is in three dimensions.

Example (9): A TV mast *AB* is anchored at *X* by two cables *XP* and *XQ*.
The ground angle at *PAQ* = 120°.
The distance *PA* from the foot of the mast *A* to the ground anchor at *P* = 208 m.
The angle of elevation of *X* from the ground anchor at *P* = 51°.
The angle of elevation of *X* from the ground anchor at *Q* = 48°.
The height *XB* from the anchor at *X* to the top of the mast at *B* = 52 m.

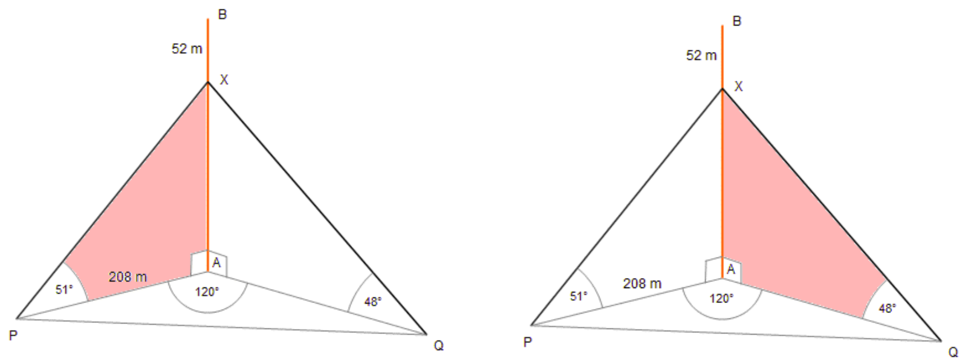


i) Find the length of the cable *PX*.

ii) Find the height *AX*, and hence the total height of the mast, *AB*.

iii) Using the results from ii), find the length of the cable *QX*.

iv) Find the ground distance *PQ* between the cable anchors, and hence the angle *PXQ* between the cables at *X*.

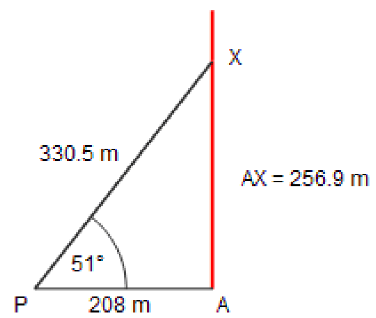


Firstly we spot the two right-angled triangles *PAX* and *QAX*.

i) $PX = \dfrac{208}{\cos 51°} = 330.5$ m.

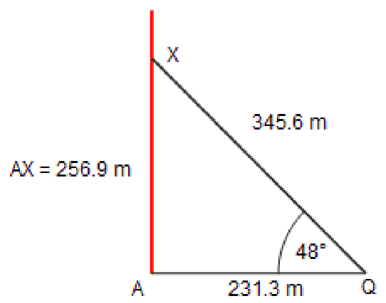
ii) $AX = 208 \tan 51° = 256.9$ m.

Hence the total height of the mast *AB* = 52 + 256.9 = **308.9 m**.



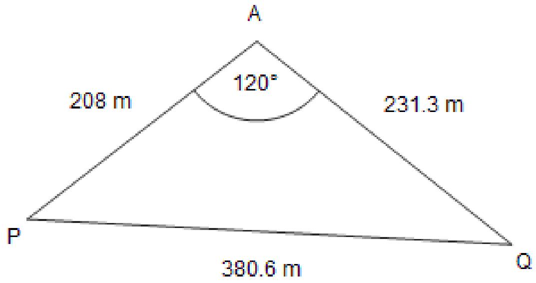
iii) $QX = \dfrac{256.9}{\sin 48°} = 345.6$ m.

iv) $AQ = \dfrac{256.9}{\tan 48°} = 231.3$ m.



Unlike triangles *PAX* and *QAX*, *PAQ* is not right-angled, so we need to use the cosine rule to find the ground distance *PQ* between the anchors.

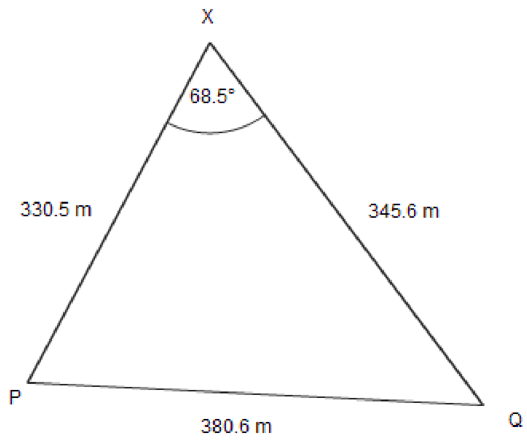
$(PQ)^2 = 208^2 + 231.3^2 - 2(208)(231.3)\cos 120°$
and hence *PQ* = 380.6 m.



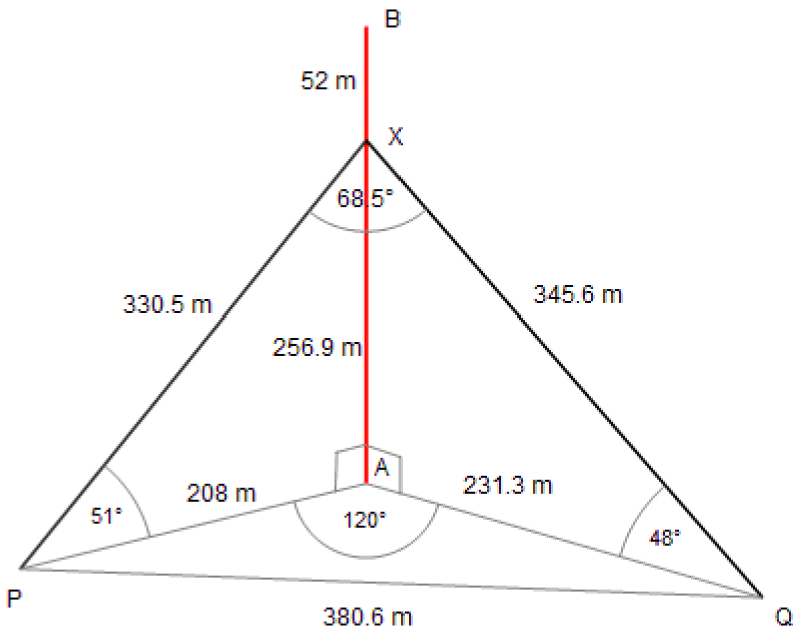
We apply the cosine rule again to find the angle *PXQ* between the cables at *X* :

$$\cos PXQ = \frac{330.5^2 + 345.6^2 - 380.6^2}{2 \times 330.5 \times 345.6}$$

and therefore *PXQ* = 68.5°.



Completed diagram:



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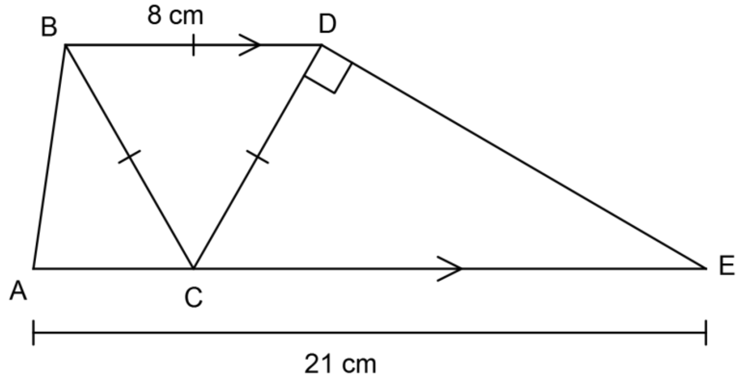
Example (10): (Non-calculator)

ABDE is a trapezium whose base length *AE* is 21 cm, and additionally *BC* = *CD* = *BD* = 8 cm.
In addition, angle *CDE* = 90°.

Calculate the perimeter of the trapezium, giving your result in the form $a + b\sqrt{c}$ where *a*,*b* and *c* are integers.



From the given data, the triangle *BCD* is equilateral, so $\angle CBD = \angle CDB = 60°$, and because *AE* and *BD* are parallel, angles *ACB* and *DCE* equal 60° by alternate angles.

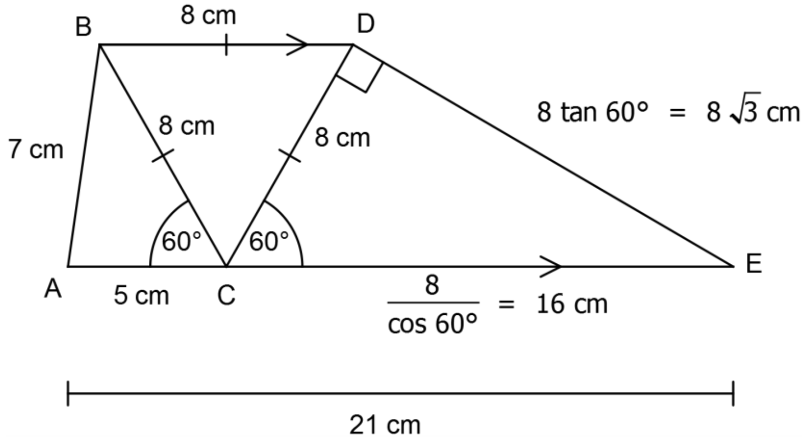
Triangle *CDE* is right-angled, so $DE = 8 \tan 60° = 8\sqrt{3}$ cm, and $CE = \dfrac{8}{\cos 60°} = 16$ cm.

(Note that cos 60° = ½ and tan 60° = √3).

By subtraction, *AC* = (21 – 16) cm = 5 cm, which leaves us with side *AB*.
The length of that side can be worked out using the cosine rule.

$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC) \cos 60°$

This gives $(AB)^2 = 25 + 64 - 80 \cos 60°$ or 89 – 40, or 49. Hence *AB* = 7cm.



∴ The perimeter of the trapezium is $(8 + 7 + 21 + 8\sqrt{3})$ cm, or $\mathbf{36 + 8\sqrt{3}}$ **cm**.