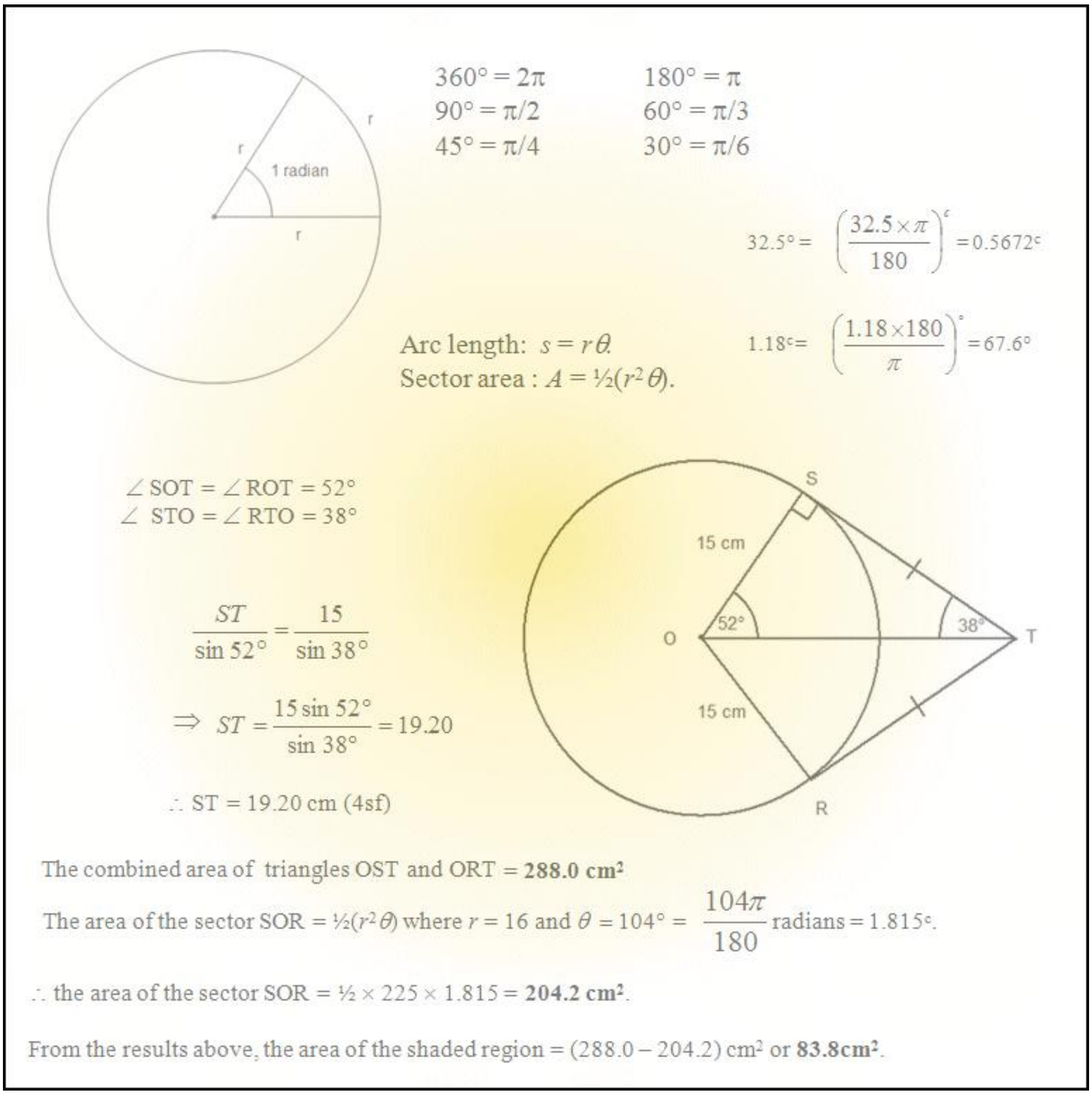
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C2** Edexcel: **C2** OCR: **C2** OCR MEI: **C2**

RADIAN MEASURE OF ANGLES



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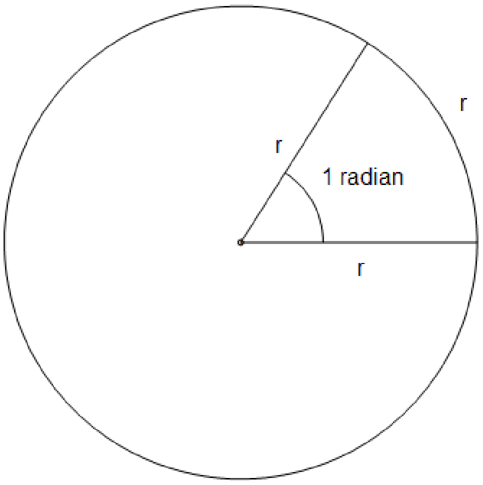
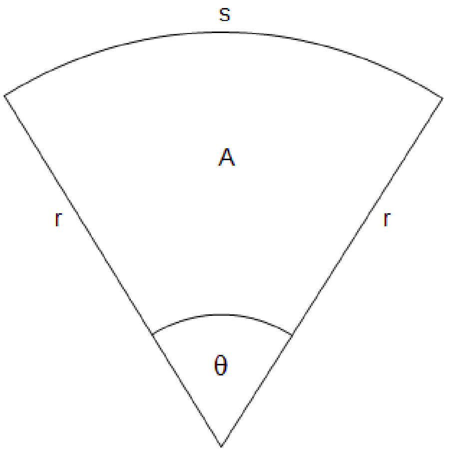
Radian measure.

Although angles are usually measured in degrees, sometimes it is more convenient to use what is known as **radian** measure, especially in calculus.

A radian is the angle subtended by the arc of a circle, where the arc's length is equal to the circle's radius.





Because the formula for the circumference is C = 2πr, it follows that 360° is equal to 2π radians.
1 radian, abbreviated to 1 rad or 1^c, is therefore (180/π)°, or about 57.3°.
The radian symbol is not normally written when the angle is given in terms of π.

Familiar angles in radian measure:

360° = 2π	180° = π
90° = π/2	60° = π/3
45° = π/4	30° = π/6

To convert degrees into radians, multiply by π and divide by 180.

To convert radians into degrees, multiply by 180 and divide by π.

Arc length and sector area.

When an angle θ is given in radians (see figure on upper right), the sector area and arc length are given by the following formulae:

The length of the arc is $s = r\theta$.
The area of the sector is $A = ½(r^2\theta)$.

Example (1): Convert 32.5° to radians, and 1.18 radians to degrees.

$$32.5° = \left(\frac{32.5\times\pi}{180}\right)^c = 0.5672^c \text{ to 4 decimal places.}$$

$$1.18^c = \left(\frac{1.18\times180}{\pi}\right)^\circ = 67.6° \text{ to 1 decimal place.}$$

Example (2): Find the arc length and area of a) a 28° sector of a circle whose radius is 12cm; a 45° sector of a circle whose radius is 16cm . Leave the answers in part (b) in terms of π.

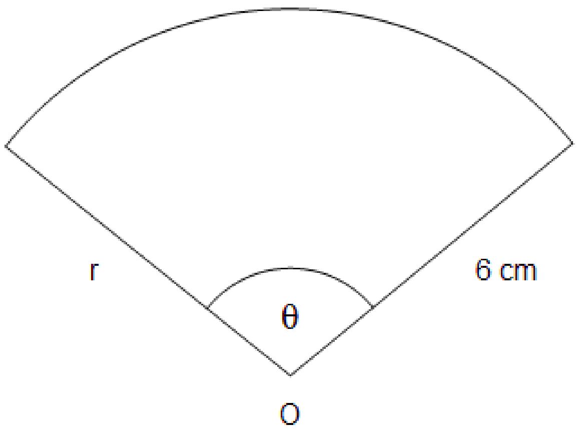
In a) we must first convert 28° to radians, giving $\theta = 0.4887^c$. Given that r = 12 cm, the arc length of the sector is therefore 12 × 0.4887 cm, or 5.86 cm to 2 decimal places.
The area, A is ½($r^2\theta$) or 72 × 0.4887 cm^2 , or 35.19 cm^2 to 2 decimal places.

In (b) we use the fact that $\theta = 45° = \frac{\pi}{4}$. With the radius r equal to 16 cm, the arc length of the sector is 4π cm. The area is ½($r^2\theta$) or 128 × $\frac{\pi}{4}$ cm^2, or 32π cm^2.

Example (3):

The figure on the right shows a sector of a circle of radius r = 6 cm and an angle of θ radians. The sector has an area of 44 cm^2.

Find the value of θ and hence the perimeter of the sector.



We are given the area as A = ½($r^2\theta$), but here it is θ that is missing, so we rearrange the formula as $r^2\theta = 2A$ and finally as $\theta = \frac{2A}{r^2}$.

Hence $\theta = \frac{88}{36} = 2.444^c$.

The perimeter of the sector can therefore be obtained by finding the arc length, here rθ or 14.67 cm, and adding twice the radius, or 12 cm, to the result.

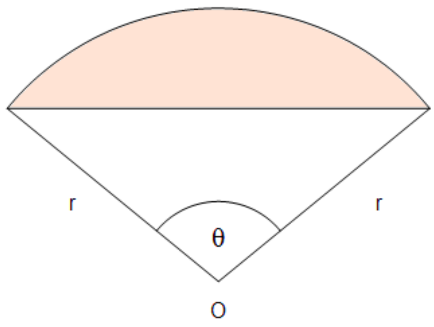
∴ the perimeter of the sector is 26.67 cm.

Other questions of this type might require knowledge of geometry and trigonometric identities.

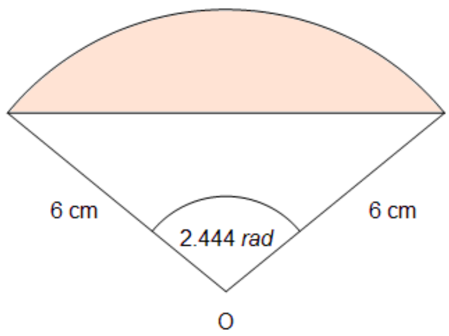
Example (4):

i) Show that the area of a segment subtended by an angle θ can be given as $\frac{1}{2}r^2(\theta - \sin\theta)$.

Use the triangle area formula $= \frac{1}{2}ab \sin C$, where a and b are two sides and C is the included angle.



ii) Using the results from Example (3), find the area and perimeter of the shaded segment in the diagram on the right.



i) The area of the general segment is most easily obtained by subtracting the area of the unshaded triangle from that of the sector.

The formula for the area of a triangle can be adapted here as $A = \frac{1}{2}r^2 \sin\theta$.

Therefore, area of segment = area of sector – area of subtended triangle = $\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$.
Factorising, we have the area of the segment = $\mathbf{\frac{1}{2}r^2(\theta - \sin\theta)}$ **.**

ii) Substituting r = 6 cm and $\theta = 2.444^c$, the area of the segment is $18(2.444 - \sin 2.444^c)$ or 32.44cm^2.

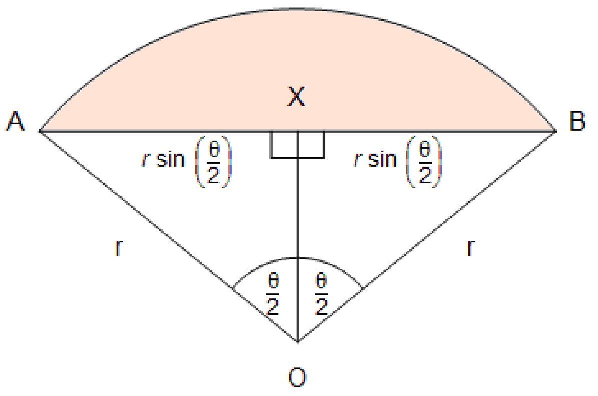
Alternately, we could have obtained the area of the triangle as $\frac{1}{2}(36 \sin 2.444^c) = 11.56$ cm^2, and then subtracted that from the area of the sector in the last example, i.e. 44 cm^2, to obtain the same result .

Finding the perimeter of the segment is a little more tricky – we need to find the chord length AB.

The centre of the circle is at O and the radius is r.

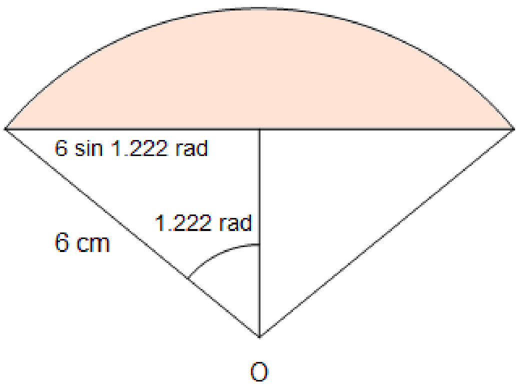
The perpendicular bisector OX of the chord AB is also the bisector of the subtended angle, and so we can apply Pythagoras to the triangle AOX to find the half-chord AX.

AX = r sin ½(θ), and so the length of the chord AB is twice that, or **2*r* sin ½(θ).**



We have already worked out θ = 2.444^c and the arc length , $r\theta$, = 14.67 cm.
The particular chord therefore has a length of 12 sin 1.222^c or 11.27 cm.

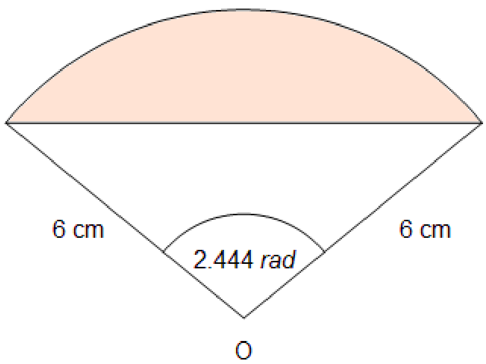
The perimeter of the segment is therefore (14.67 + 11.27) cm or 25.9 cm to 3 s.f.



Another way of finding the length of the chord AB would be to use the cosine formula: $c^2 = a^2 + b^2 - 2ab\cos C$:

Remember a and b are both radii and so can be replaced by r,

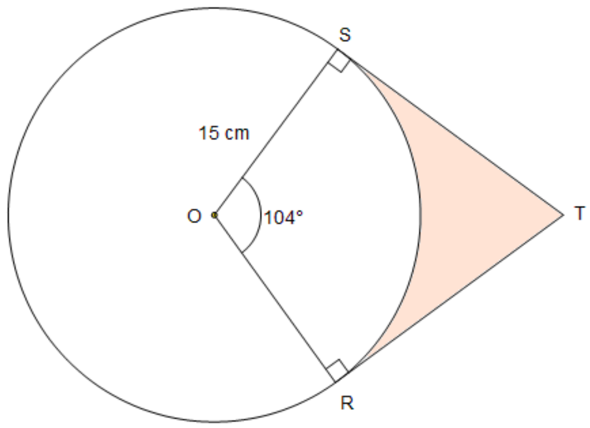
so r = 6 and C = 2.444^c :



$$c^2 = 2r^2 - 2r^2\cos C$$
$$\Rightarrow c^2 = 2r^2(1-\cos C).$$
$$\Rightarrow c^2 = 72(1-\cos 2.444^c) = 127.2 \Rightarrow c = 11.27 \text{ cm.}$$

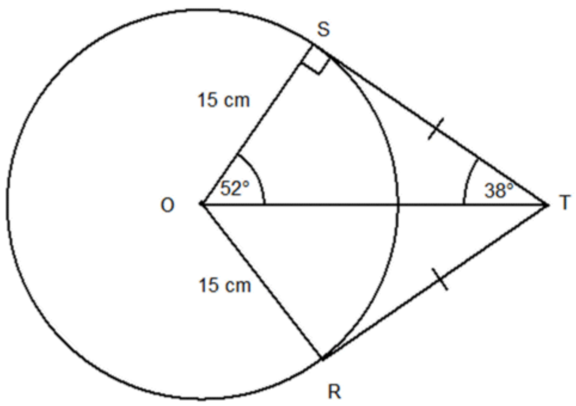
Example (5): The lines ST and RT are tangents to the circle of radius 15 cm centred on O. Angle SOR is 104°.

Find the area of the shaded region .



Recalling the properties of tangents of a circle,
we see that the lengths ST and RT are equal, and that the angles ORT and OST are right angles.

Because OR and OS are both radii, they are also equal, so triangles OST and ORT are congruent.



$\therefore$ $\angle$ SOT = $\angle$ ROT = 52°
and $\angle$ STO = $\angle$ RTO = 38°

The length ST can be found by the sine rule:

$$\frac{ST}{\sin 52^\circ} = \frac{15}{\sin 38^\circ}$$

$$\Rightarrow \; ST = \frac{15 \sin 52^\circ}{\sin 38^\circ} = 19.20$$

$\therefore$ ST = 19.20 cm (4sf)

To find the area of the shaded region in the first diagram, we must first take the combined area of the two triangles OST and ORT, and then subtract the area of the sector SOR from the result.

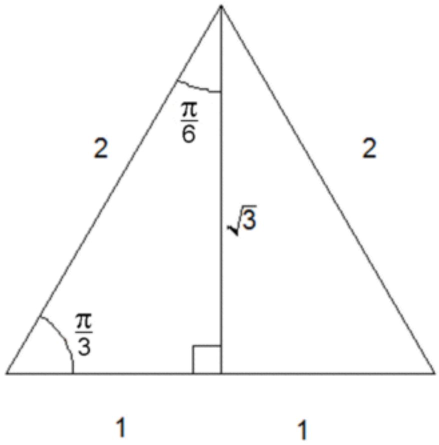
The combined area of the two triangles OST and ORT = 2 × ½(base × height) = **288.0 cm^2**.

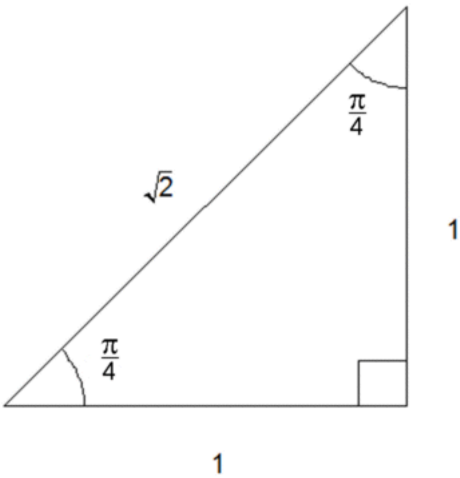
The area of the sector SOR = ½($r^2\theta$) where r = 16 and θ = 104° = $\frac{104\pi}{180}$ radians = 1.815^c.

$\therefore$ the area of the sector SOR = ½ × 225 × 1.815 = **204.2 cm^2**.

From the results above, the area of the shaded region = (288.0 – 204.2) cm^2 or **83.8cm^2**.

Ratios of special angles in radian measure.





sin (π/6) = cos (π/3) = $\frac{1}{2}$

sin (π/3) = cos (π/6) = $\frac{\sqrt{3}}{2}$

tan (π/6) = $\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$

tan (π/3) = $\sqrt{3}$

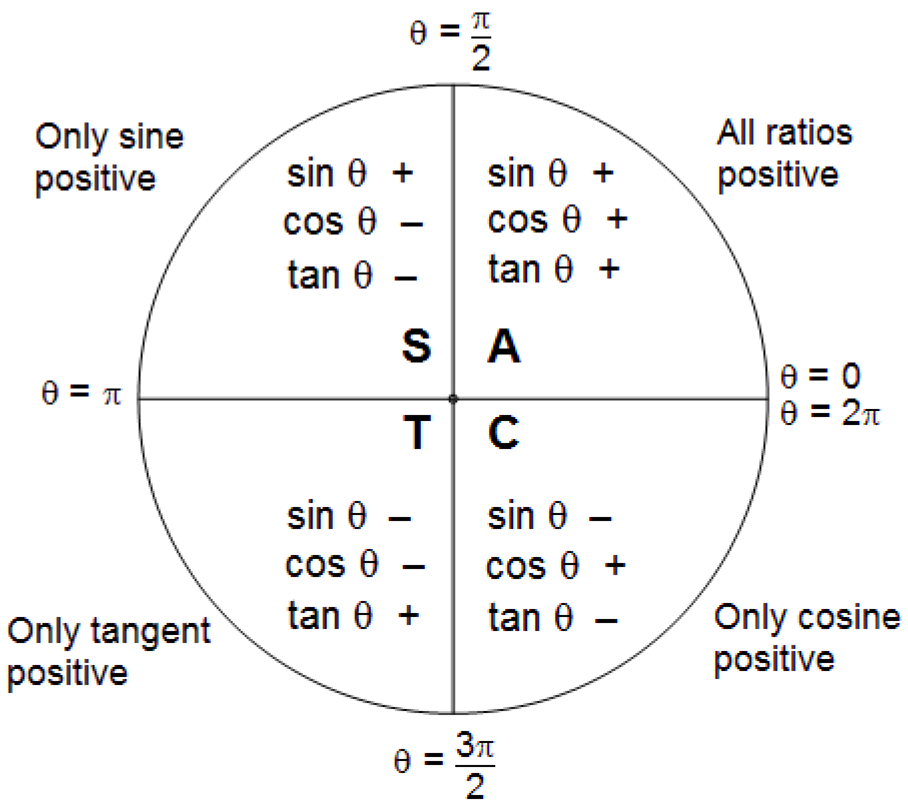
sin (π/4) = cos (π/4) = $\frac{1}{\sqrt{2}}$ = $\frac{\sqrt{2}}{2}$

tan (π/4) = 1

sin (π/2) = cos 0 = 1

sin 0 = cos (π/2) = 0

Quadrant rules in radian measure.



The working range here is $0 \le \theta < 2\pi$.

First quadrant: angle between 0 and π/2.
For any angle θ in the first quadrant, its trig ratios are all positive.

Second quadrant: angle between π/2 and π.
For any angle θ in the second quadrant, we have the trig ratios of the angle $(\pi - \theta)$ related as follows:
$\sin(\pi - \theta) = \sin\theta$; $\cos(\pi - \theta) = -\cos\theta$; $\tan(\pi - \theta) = -\tan\theta$.

Third quadrant: angle between π and 3π/2.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(\pi + \theta)$ related as follows:
$\sin(\pi + \theta) = -\sin\theta$; $\cos(\pi + \theta) = -\cos\theta$; $\tan(\pi + \theta) = \tan\theta$.

Fourth quadrant: angle between 3π/2 and 2π.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(2\pi - \theta)$ related as follows:
$\sin(2\pi - \theta) = -\sin\theta°$; $\cos(2\pi - \theta) = \cos\theta$; $\tan(2\pi - \theta) = -\tan\theta$.

The following ratios hold at the boundaries of the quadrants:

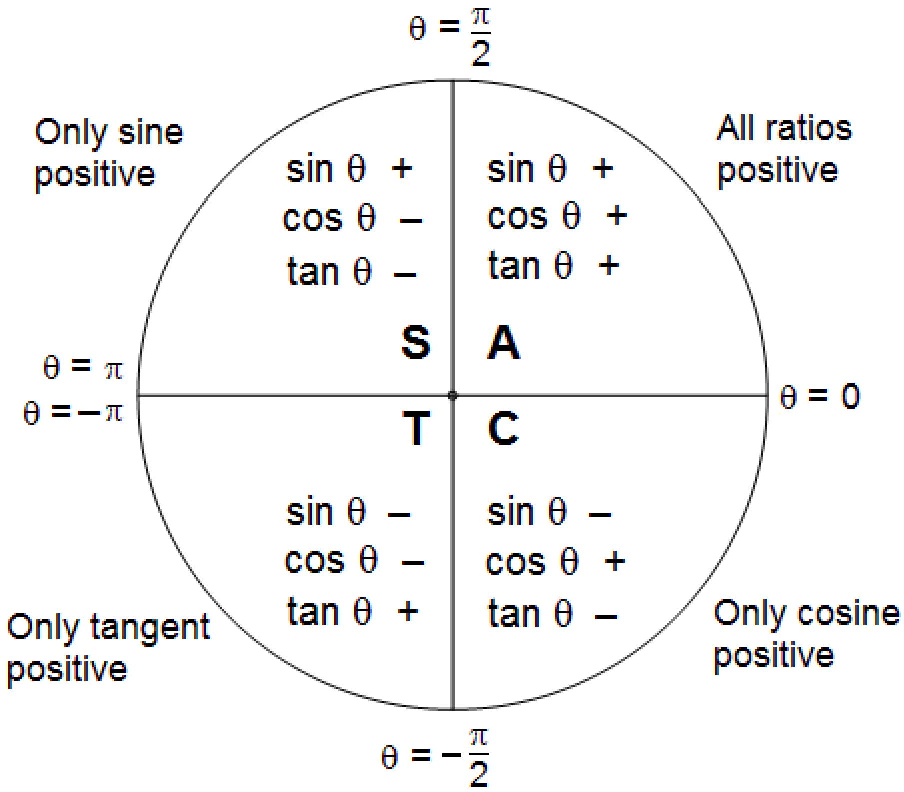
$\sin 0 = 0$; $\cos 0 = 1$; $\tan 0 = 0$

$\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$; $\tan(\pi/2)$ is undefined

$\sin(\pi) = 0$; $\cos(\pi) = -1$; $\tan(\pi) = 0$

$\sin(3\pi/2) = -1$; $\cos(3\pi/2) = 0$; $\tan(3\pi/2)$ is undefined

Here is the alternative CAST diagram for the angle range $-\pi \le \theta < \pi$.



First quadrant: angle between 0 and $\pi/2$.
For any angle θ in the first quadrant, its trig ratios are all positive.

Second quadrant: angle between $\pi/2$ and π.
For any angle θ in the second quadrant, we have the trig ratios of the angle $(\pi - \theta)$ related as follows:
$\sin(\pi - \theta) = \sin\theta$; $\cos(\pi - \theta) = -\cos\theta$; $\tan(\pi - \theta) = -\tan\theta$.

Third quadrant: angle between $-\pi$ and $-\pi/2$.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(-\pi - \theta)$ related as follows:
$\sin(-\pi - \theta) = -\sin\theta$; $\cos(-\pi - \theta) = -\cos\theta$; $\tan(-\pi - \theta) = \tan\theta$.

Fourth quadrant: angle between $-\pi/2$ and 0.
For any angle θ in the fourth quadrant, we have the trig ratios of the angle $(-\theta)$ related as follows:
$\sin(-\theta) = -\sin\theta°$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

The following ratios hold at the boundaries of the quadrants:

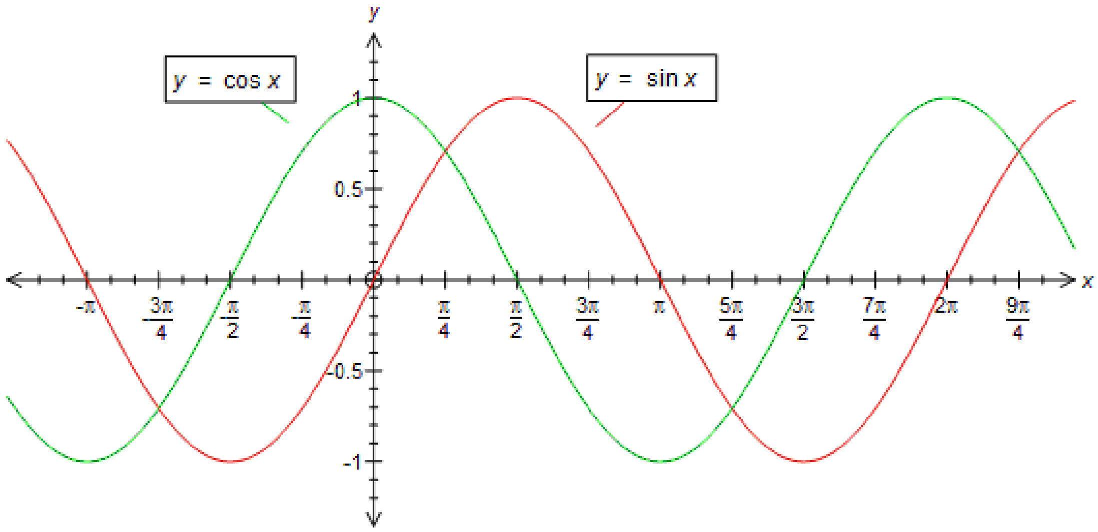
$\sin 0 = 0$; $\cos 0 = 1$; $\tan 0 = 0$

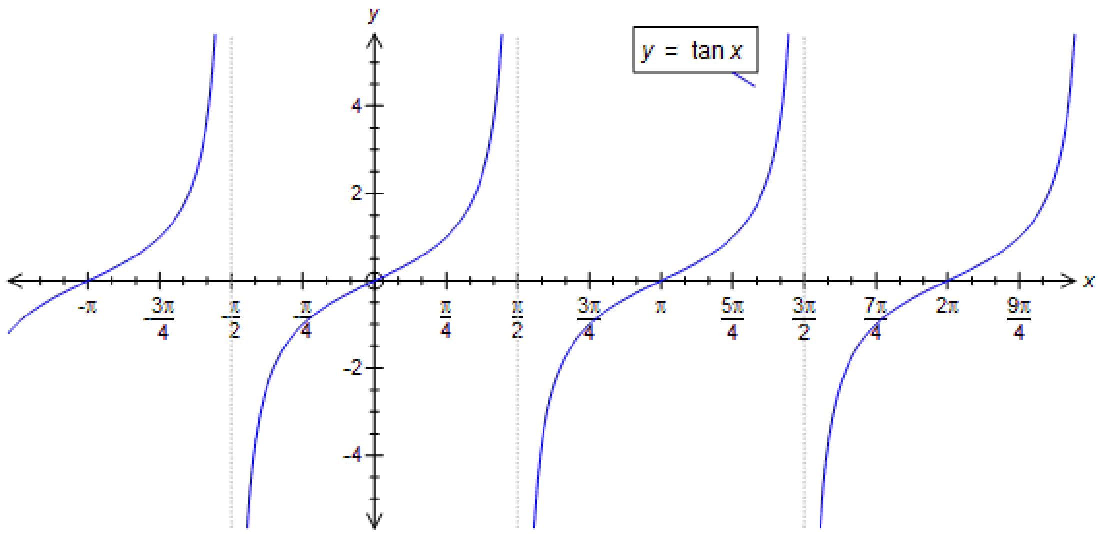
$\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$; $\tan(\pi/2)$ is undefined

$\sin(-\pi) = 0$; $\cos(-\pi) = -1$; $\tan(-\pi) = 0$

$\sin(-\pi/2) = -1$; $\cos(-\pi/2) = 0$; $\tan(-\pi/2)$ is undefined

Graphs of trig functions in radian measure.





(Full details are in the document "Trigonometric Ratios and Graphs", with angles measured in degrees).

The graphs of sin x and cos x each have a repeating period of 2π radians. Indeed, the graph of cos x is the same as that of sin x translated $(\pi/2)$ radians to the left , i.e. by the vector $\begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix}$.

The graph of tan x has a repeating period of π radians, and the function itself is undefined for certain values of x, such as $(\pi/2)$, $(3\pi/2)$, and all angles consisting of an odd number of right angles.

In other words, the tangent graph has asymptotes at $(\pi/2)$, $(3\pi/2)$, and all angles $(n\pi/2)$ where n is an odd integer.