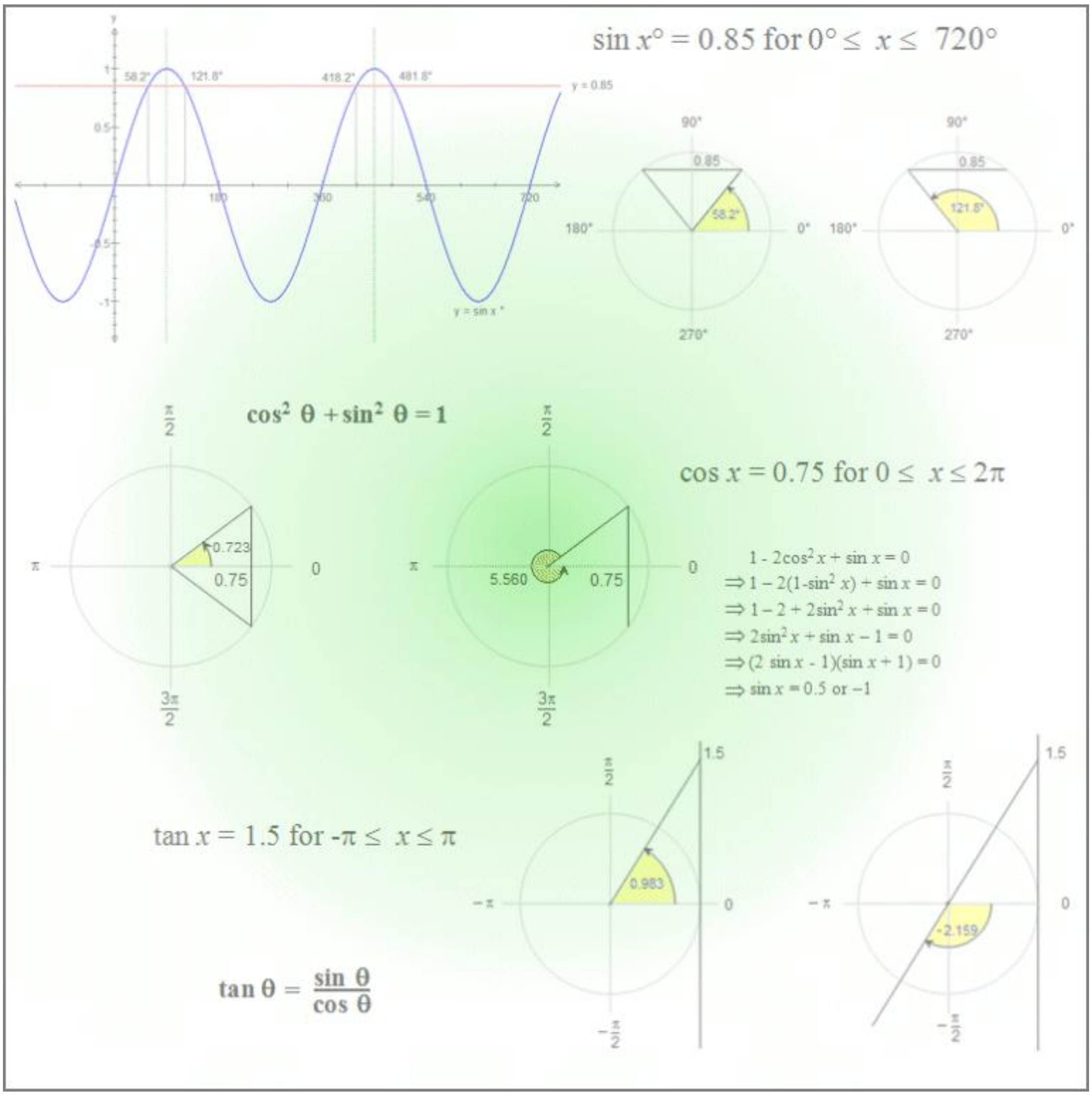
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C2** Edexcel: **C2** OCR: **C2** OCR MEI: **C2**

SOLVING TRIGONOMETRIC EQUATIONS



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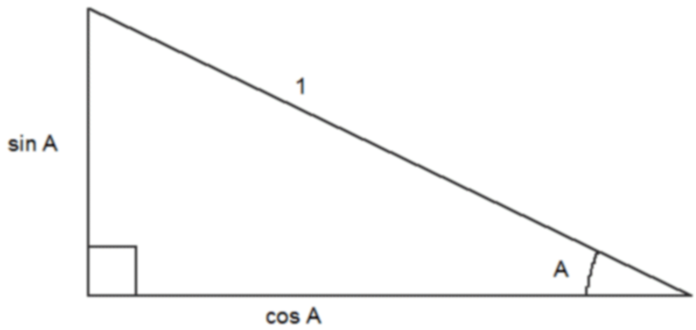
Trigonometric identities.

There are two very important trigonometric identities.

For all angles θ :

$\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$ (This is the Pythagorean identity)



The square on the hypotenuse in the triangle above is 1.
The sum of the squares on the other two sides = $\cos^2 \theta + \sin^2 \theta$.

The length of the opposite side to $A = \sin \theta$ and the length of the adjacent to $A = \cos \theta$.

Since the tangent is the opposite divided by the adjacent, it also follows that

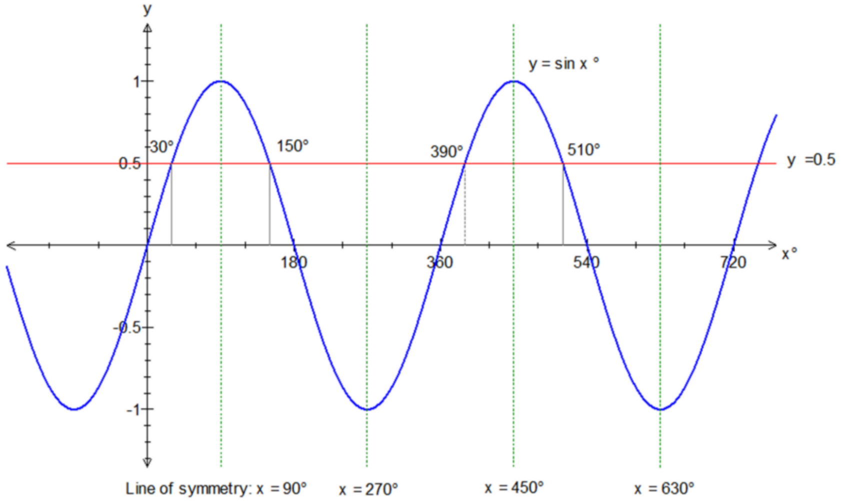
$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

These two identities are important when simplifying expressions or solving various types of equations.

SOLVING TRIGONOMETRIC EQUATIONS

Any trigonometric equation of the form sin x° = a (where $-1 \le a \le 1$), cos x° = a (where $-1 \le a \le 1$) or tan x° = a can have an infinite variety of solutions.

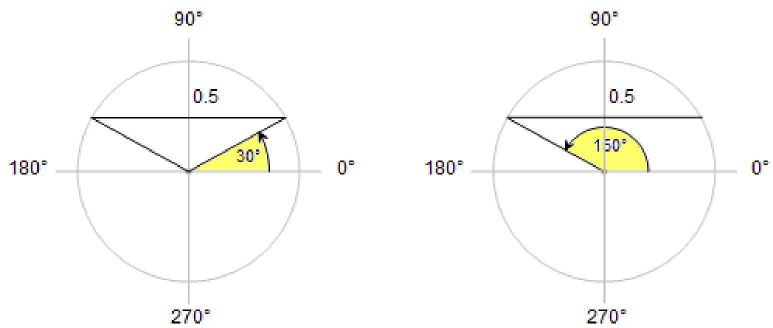
To solve such equations using a calculator, it must be remembered that only *one* of an infinite number of possible values will be given – we must find *all* solutions within the range of the question.



The graph above shows several of the possible solutions of sin x° = 0.5. Apart from the one of 30° given on the calculator (the principal value), the possible solutions also include (180-30)° or 150° because of the reflective symmetry of the graph about the line x = 90°. Moreover, as the graph has a repeating period of 360°, other solutions are 390°, 510° and so on in the positive x-direction, and –210°, -330° and so forth in the negative x-direction.

We can also use CAST diagrams to show the two solutions in the range 0° - 360°, and we can add or subtract multiples of 360° as desired for any further values, should the question ask for them.

Positive angles are measured anticlockwise from the origin; negative ones clockwise.

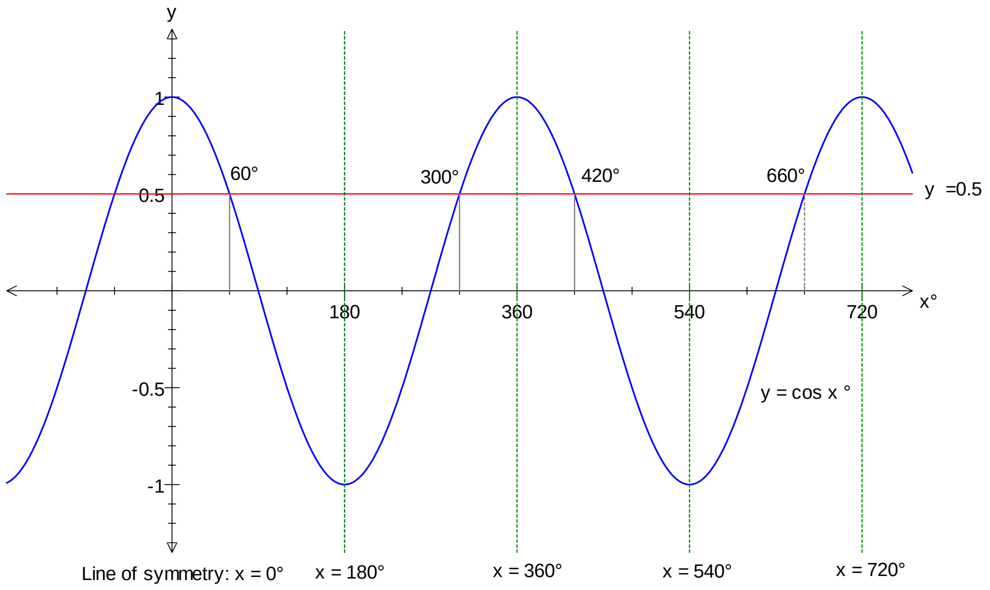


In general, if sin x° = a, then :
sin (180-*x*)° = *a*
sin (180n + x)° = a for even integer n ; sin (180n - x)° = a for odd integer n.

Therefore, for instance, sin (360+x)°, sin (540-x)° and sin (-180-x)° are also equal to sin x.

In radians, the solutions are given as $\pi/6$ and $(\pi - \pi/6)$ or $5\pi/6$. The graph repeats every 2π radians, and other solutions are $13\pi/6$, $17\pi/6$ (positive x-direction) and $-7\pi/6$, $-11\pi/6$ (negative x-direction).

In general, if sin x = a, then :
sin (π -*x*) = *a*
sin ($n\pi$ + x) = a for even integer n ; sin ($n\pi$ - x) = a for odd integer n.
Therefore, for instance, sin ($2\pi+x$) sin ($3\pi-x$) and sin ($-\pi-x$) are also equal to sin x.

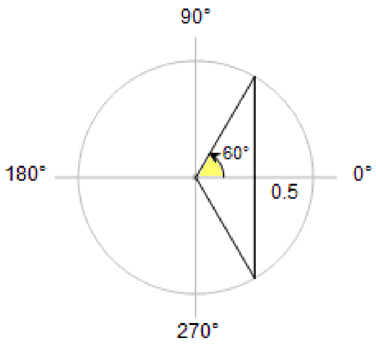


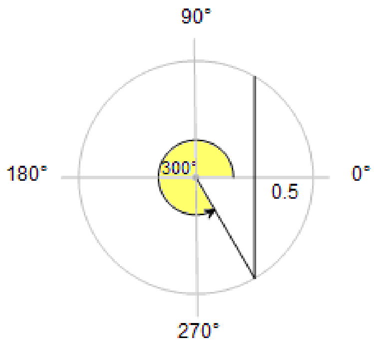
The cosine function also shows similar periodicity.

Several of the possible solutions of cos x° = 0.5 are shown in the graph above. Apart from the one of 60° given on the calculator(the principal value), the possible solutions also include 300° because the graph has reflective symmetry about the line x = 180°. Again, there is a repeating period of 360°, and so other solutions are 420°, 660° and so on in the positive x-direction, and -60°, –300°, -420° and so forth in the negative x-direction.

The CAST diagrams to show the two solutions in the range 0° - 360°. Again, we can add or subtract multiples of 360° if required.

The relationship between cos 60° and cos -60° is evident from the left-hand diagram.





In general, if cos x° = a, then :
cos (-*x*)° = *a* or **cos (360-*x*)° = *a***
cos (360n + x)° = a for any integer n
cos (360n - x)° = a for any integer n.

Therefore, for instance, cos (360+x)°, cos (720-x)° and cos (-360-x)° are also equal to cos x.

In radians, the solutions are given as π/3 and –π/3 (or 5π/3). The graph repeats every 2π radians, and other solutions are 5π/3, 7π/3 (positive x-direction) and -5π/3, -7π/3 (negative x-direction).

In general, if cos x = a, then :
cos (-*x*) = *a* or **cos (2π-*x*)° = *a***
cos (2nπ + x) = a for any integer n
cos (2nπ - x) = a for any integer n.

Therefore, for instance, cos (2π+x), cos (4π–x) and cos (-2π-x) are also equal to cos x.

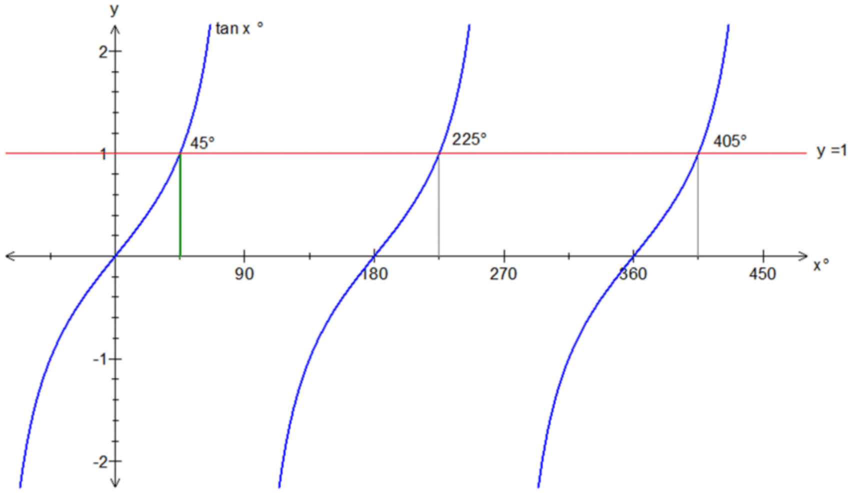
The symmetrical pairing of solutions in the range $0° \le x < 360°$ (or $0 \le x < 2\pi$ in radians) is common to both the sine and cosine graphs, for all angles on either side of the graphs' lines of symmetry.

(An exception occurs when sin x and cos $x = \pm1$; then there is only one solution.in that range.)

Thus sin 115° = sin 65° (both equidistant from line of symmetry at 90°), and cos 155° = cos 205° (both equidistant from line of symmetry at 180°).

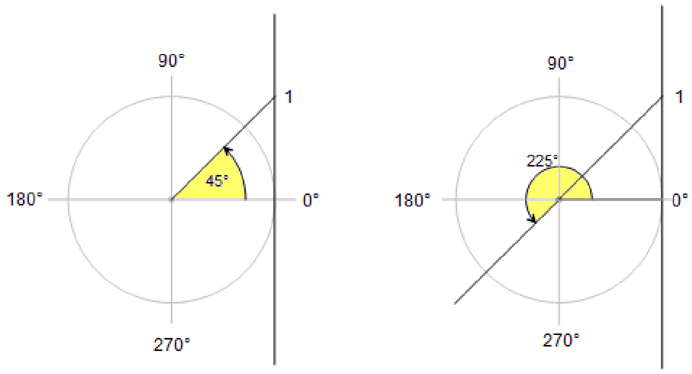
The tangent function, on the other hand, has a period of 180° rather than 360° as in the sine and cosine functions. Also, there is no linear symmetry, so there is no 'doubling-up' of solutions (as in sine and cosine examples).

The graph shows several solutions of the equation tan x° = 1.



Apart from the one of 45° given on the calculator, there are other possible solutions because the graph has a repeating period of 180°. Those solutions are 225°, 405° and so on in the positive x-direction, and –135°, -315° and so forth in the negative x-direction.

The CAST diagrams for the tangent function are simpler in form than the others – you merely add multiples of 180° to obtain all the solutions needed.



In general, if tan $x° = a$, then tan $(180n + x)° = a$ for any integer n.

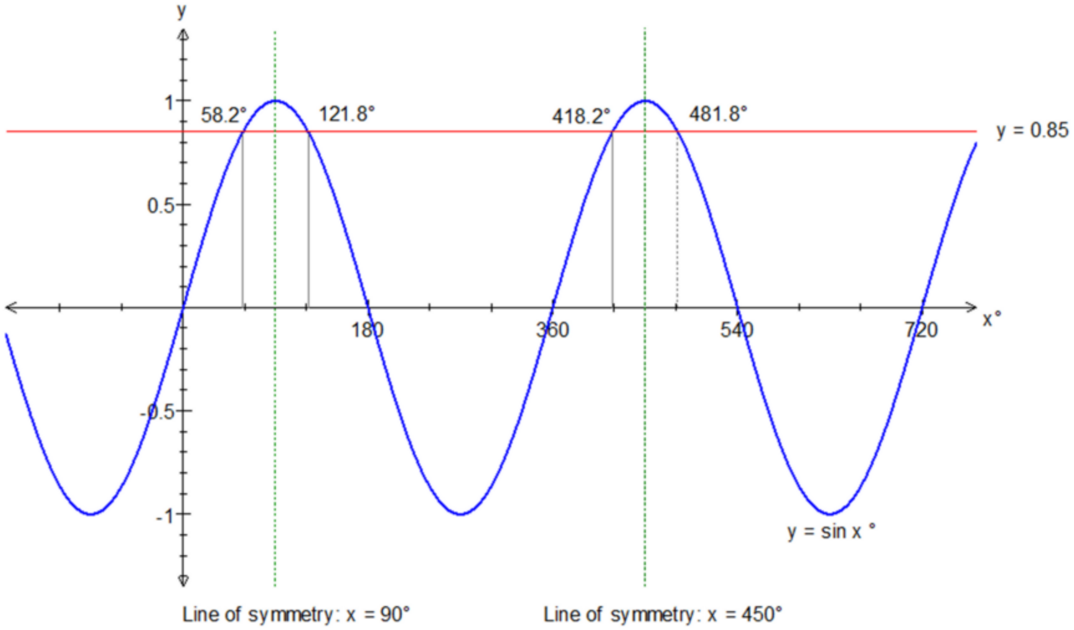
In radians, the calculator solution is $\pi/4$. The graph repeats every π radians, and other solutions are $5\pi/4$, $9\pi/4$ (positive x-direction) and $-3\pi/4$, $-7\pi/4$ (negative x-direction).

In general, if tan $x = a$, then tan $(n\pi + x) = a$ for any integer n.

Example (1): Solve sin x° = 0.85 for 0° ≤ x ≤ 720°, giving answers in degrees to one decimal place.

The principal value is **58.2°**, but care must be taken to ensure that all solutions within the range are given.

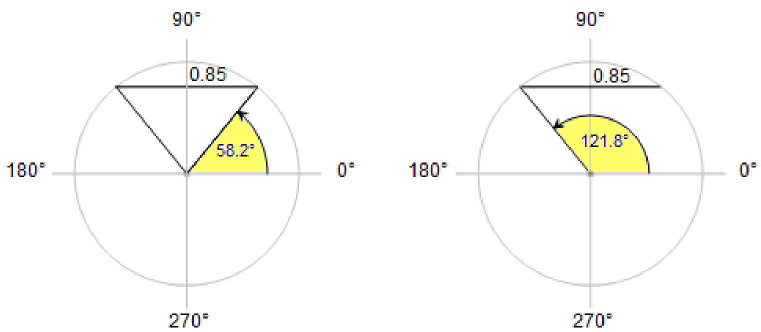
The substitution formulae can be used, but a sketch graph or CAST diagram will probably be easier in finding extra solutions.



The two important lines of symmetry here are x = 90° and x = 450°.
Since 58.2° is 31.8° *less* than 90°, the solution on the other side of that line of symmetry must be 31.8° *greater* than 90°, or **121.8°.**

Alternatively we can use sin (180-x)° = sin x°, and 180° - 58.2° = 121.8°.

CAST diagram illustration:



Having obtained the two solutions above, it is a simple matter of adding and subtracting multiples of 360° as required. Subtracting 360° is no help as there will be no new values found within the range, but adding 360° will give two other solutions:

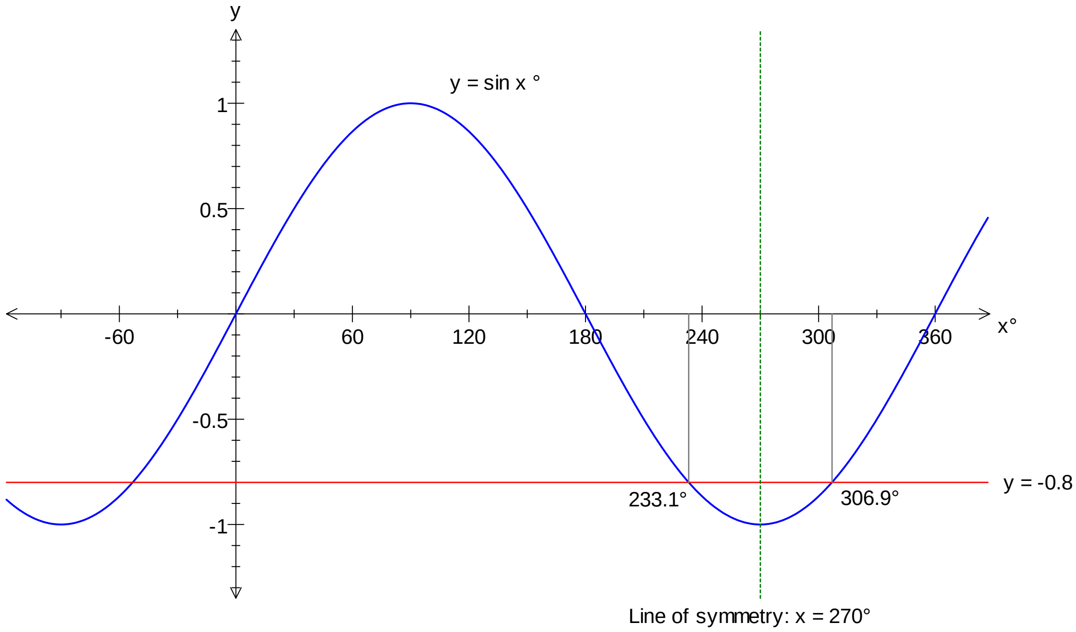
(58.2 + 360)° or **418.2°.**
(121.8 + 360)° or **481.8°.**

∴ the solutions of sin x° = 0.85 for 0° ≤ x ≤ 720° are **58.2°, 121.8°, 418.2°** and **481.8°**.

Example (2): Solve sin x° = -0.8 for 0° ≤ x ≤ 360°, giving answer in degrees to one decimal place.

The principal value, and the one given on a calculator, is **-53.1°**, from which we can derive the other solutions. Note that this solution is not in the quoted range, and so we must add an appropriate multiple of 360° (the period of sin x) to it.

Here, adding 360° gives one solution, i.e. **306.9°**.



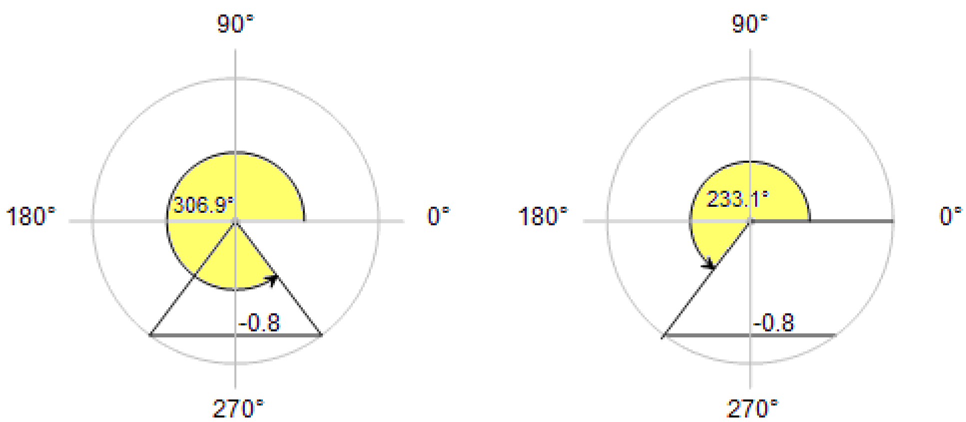
The important line of symmetry here is x = 270°.

Since 306.9° is 36.9° *greater* than 270°, the solution on the other side of that line of symmetry must be 36.9° *less* than 270°, or **233.1°.**

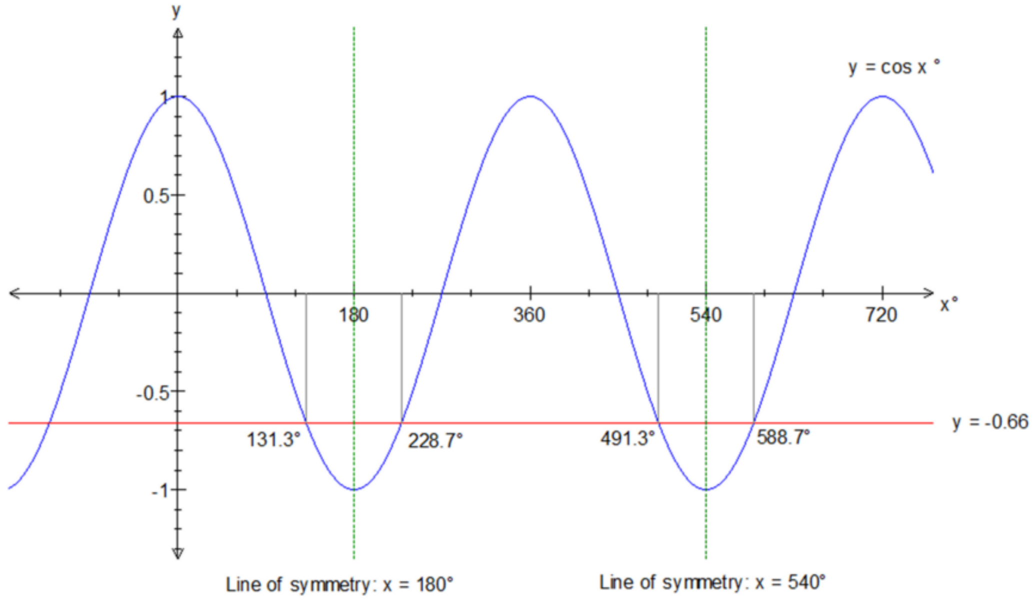
∴ the solutions of sin x° = -0.8 for 0° ≤ x ≤ 360° are **233.1°** and **306.9°.**

Again we could have used sin (180-x)° = sin x°, and 180° - (-53.1°) = 233.1°, although it is less obvious from the diagram.

CAST diagram illustration:



Example (3): Solve cos $x°$ = -0.66 for $0° \leq x \leq 720°$, giving answer in degrees to one decimal place.



The two important lines of symmetry here are $x = 180°$ and $x = 540°$.

Here, the principal value, and the one given on a calculator, is 131.3°.

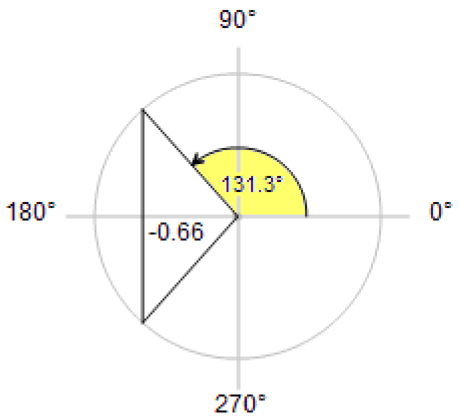
Since 131.3° is 48.7° *less* than 180°, the solution on the other side of that line of symmetry must be 48.7° *greater* than 180°, or **228.7°.**

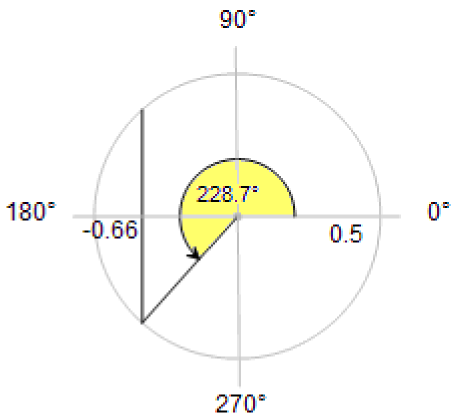
We could have used cos $(360-x)°$ = cos $x°$, and 360° - 131.3° = 228.7°.

Having obtained the two solutions above, it is a simple matter of adding and subtracting multiples of 360° as required. Subtracting 360° is no help as there will be no new values found within the range, but adding 360° will give two other solutions, i.e. **491.3°** and **588.7°.**

$\therefore$ the solutions of cos $x°$ = -0.66 for $0° \leq x \leq 720°$ are 131.3°, 228.7°, 491.3° and 588.7°.

CAST diagram illustration:



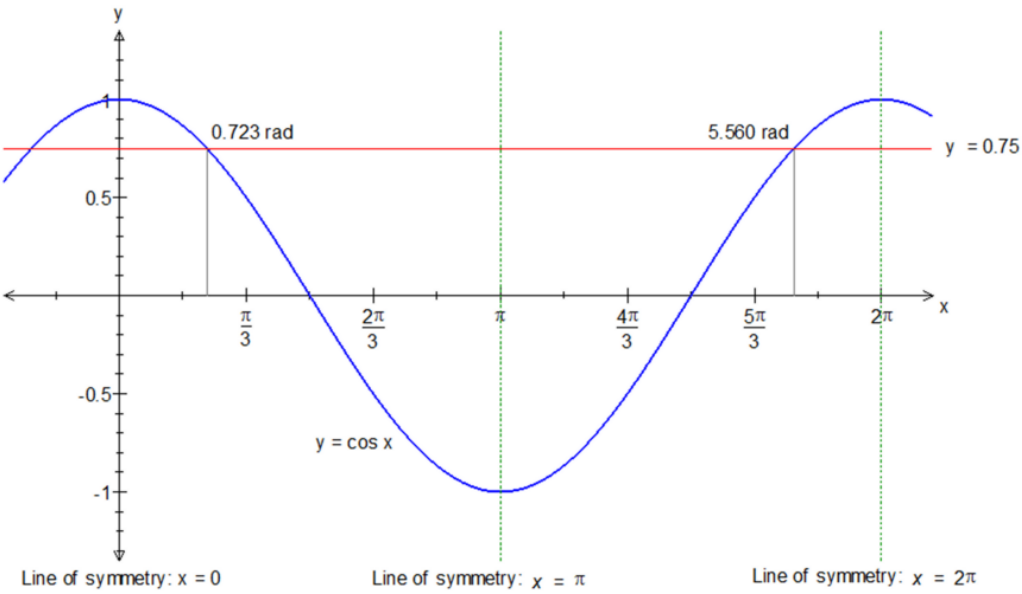


Example (4): Solve cos x = 0.75 for $0 \leq x \leq 2\pi$, giving answers in radians to 3 decimal places.

The principal value is **0.723 ^{c}**, but to find the other solution, we can make use of the lines of symmetry at $x = 0$ and $x = 2\pi$. Since the principal value is 0.723 *greater* than zero, the other required value must be 0.723 *less* than 2π , or **5.560^c**. (This is the same as using the identity cos (2π-x) = cos x.)

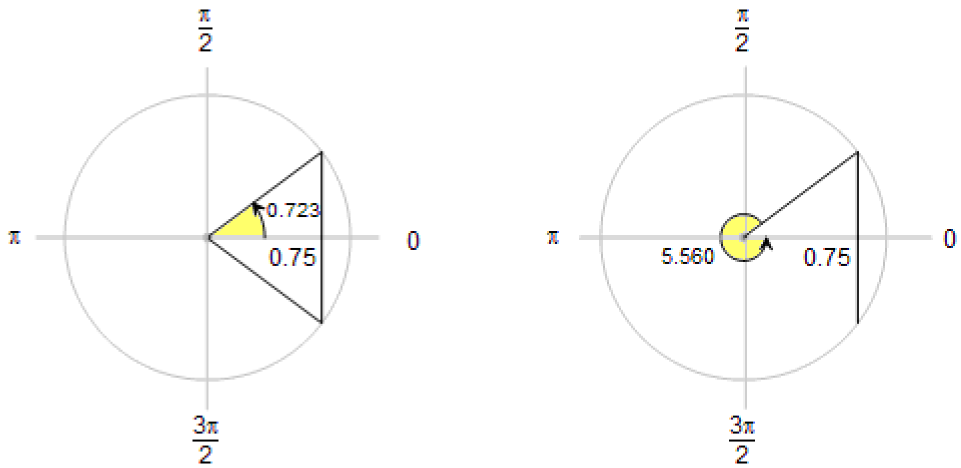
We could have also used the line of symmetry at $x = \pi$, and worked the value as π + (π -0.723) , leading to the same conclusion.

(This, and the substitution identity sin (π-x) = sin x, are perhaps easier to use when solving sin $x = k$ or cos $x = k$ in radians.).



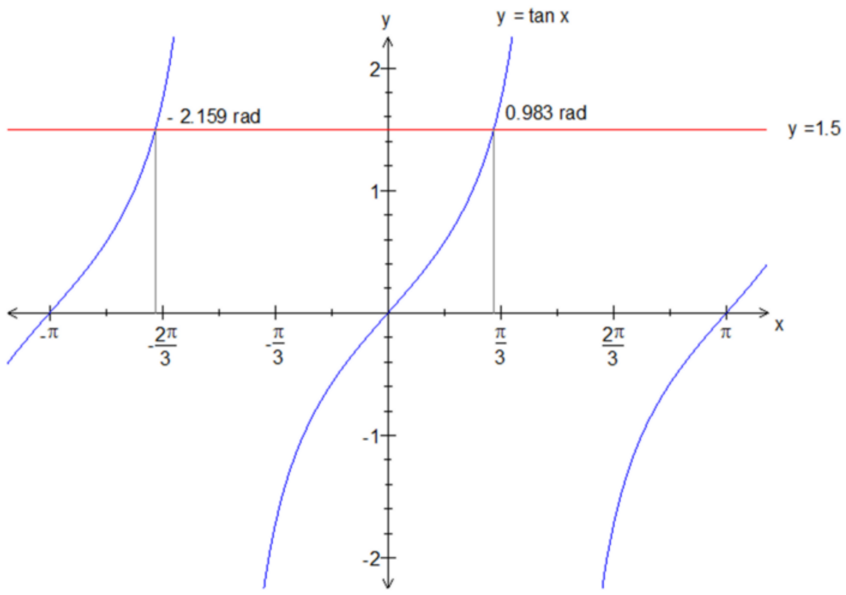
$\therefore$ the solutions of cos x = 0.75 for $0 \leq x \leq 2\pi$ are 0.723 ^{c} and 5.560^c.

CAST diagram illustration:



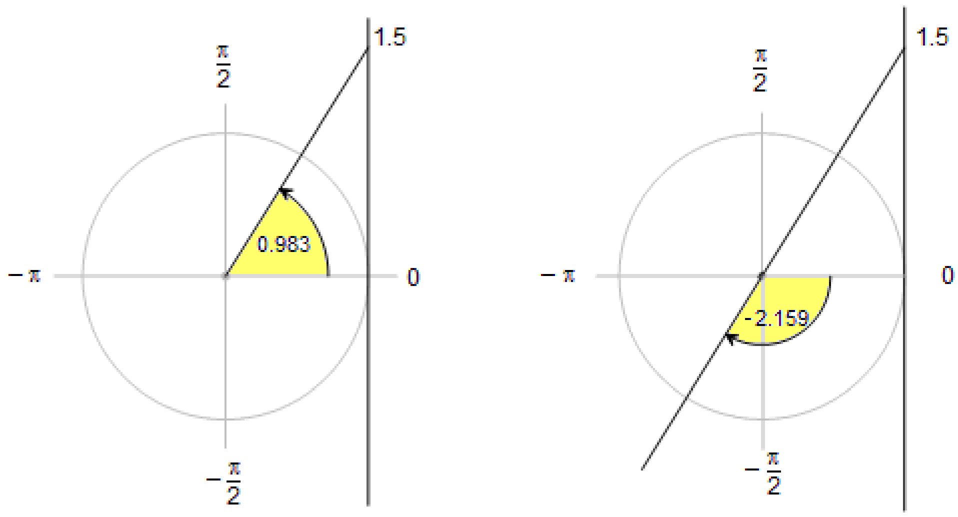
Solution of equations of the form tan $x = k$ is easier: we just add or subtract multiples of 180° (or multiples of π if using radians) to the principal value.

Example (5): Solve tan $x = 1.5$ for $-\pi \le x \le \pi$, giving the answer in radians to 3 decimal places .



The principal value is 0.983^c, but because the tangent function repeats itself every π radians, there is another solution at $(0.983 - \pi)^c$ or -2.159^c.

CAST diagram illustration:



Note that the negative angle is denoted by a clockwise arrow.

Summary. (The principal solution is the one displayed on the calculator.)

The rules are designed to find solutions in the range $-180° \leq x < 180°$ or $-\pi \leq x < \pi$.
This range has the virtue of having the principal solution coincide with the calculator display.

If a different range is stipulated, it is only a matter of adding multiples of 360° or 2π to the solutions so obtained (when solving equations of the form $\sin x = k$ or $\cos x = k$), or multiples of 180° or π (when solving equations of the form $\tan x = k$). In each case, n is an integer.

Solving $\sin x = k$.

Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = 0°$ or 0 rad	(none)	Add/subtract $180n$ ° or $n\pi$
1	$x = 90°$ or $\pi/2$	(none)	Add/subtract $360n$ ° or $2n\pi$
-1	$x = -90°$ or $-\pi/2$	(none)	Add/subtract $360n$ ° or $2n\pi$
positive	$0° < x < 90°$ or $0 < x < \pi/2$	$180° - x$ or $\pi - x$	Add/subtract $360n$ ° or $2n\pi$
negative	$-90° < x < 0°$ or $-\pi/2 < x < 0$	$-180° - x$ or $-\pi - x$	Add/subtract $360n$ ° or $2n\pi$

Solving $\cos x = k$.

Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = 90°$ or $\pi/2$	(none)	Add/subtract $180n$ ° or $n\pi$
1	$x = 0°$ or 0 rad	(none)	Add/subtract $360n$ ° or $2n\pi$
-1	$x = -180°$ or $-\pi$	(none)	Add/subtract $360n$ ° or $2n\pi$
positive	$0° < x < 90°$ or $0 < x < \pi/2$	$-x$	Add/subtract $360n$ ° or $2n\pi$
negative	$90° < x < 180°$ or $\pi/2 < x < \pi$	$-x$	Add/subtract $360n$ ° or $2n\pi$

Solving $\tan x = k$.

Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = 0°$ or 0 rad	(none)	Add/subtract $180n$ ° or $n\pi$
positive	$0° < x < 90°$ or $0 < x < \pi/2$	(none)	Add/subtract $180n$ ° or $n\pi$
negative	$-90° < x < 0°$ or $-\pi/2 < x < 0$	(none)	Add/subtract $180n$ ° or $n\pi$

Sometimes we need to use the idea of transformations to solve slightly more complex trig equations.

Example (6): Use the results from Example (4) to solve $\cos 2x = 0.75$ for $0 \leq x \leq \pi$, giving your answers in radians to three decimal places.

Here the limits for x are $0 \leq x \leq \pi$, but we are asked to solve for $2x$. To ensure that no values are omitted, we must substitute A for $2x$ and multiply the limiting values by 2 to get the transformed limit of $0 \leq A \leq 2\pi$.

From Example (4), we see that the solutions of $\cos A = 0.75$ for $0 \leq A \leq 2\pi$ are 0.723^c and 5.560^c. To convert the A-values back to x-values, we must divide by 2.

$\therefore$ the solutions of $\cos 2x = 0.75$ for $0 \leq x \leq \pi$ are 0.361^c and 2.780^c.

Example (7): Solve $\tan(2x + 60°) = 1$ for $0 \leq x \leq 360°$.

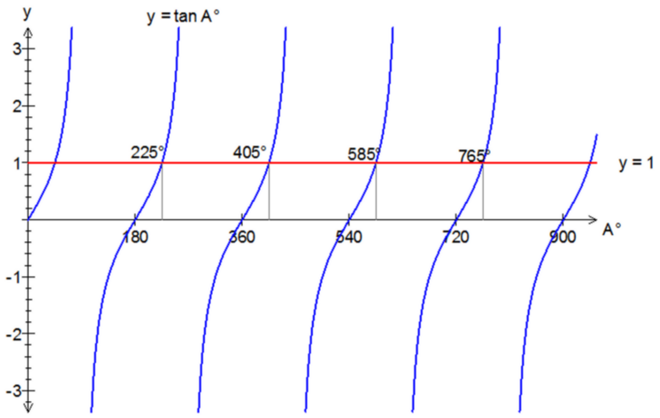
By letting A stand for $2x + 60°$, we first transform the x-limits to A-limits:

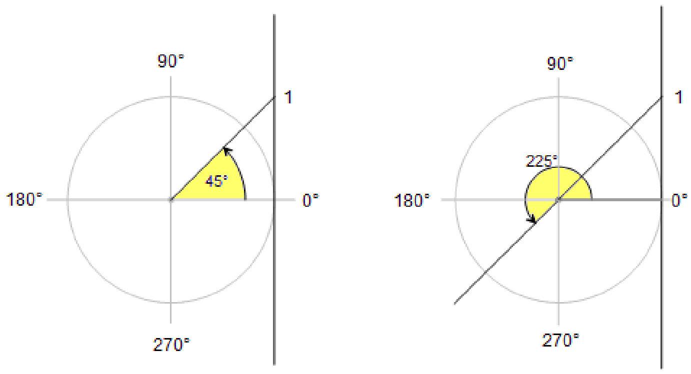
$0 \leq x \leq 360° \Rightarrow 60° \leq A \leq 780°$ (x-values doubled, 60° added).

When drawing sketch graphs or CAST diagrams, draw them with respect to the transformed variable A, and not the original variable x.

The substitution back to x-values must be done afterwards.

The principal value of A where $\tan A = 1$ is 45°, but that value is not within the A-limits.
We therefore keep adding multiples of 180° to obtain A = 225°, 405°, 585° and 765°.





Since we doubled the x-values and then added 60° to get the A-values, we must perform the inverse operations to change the A-values back to x-values – i.e. subtract 60° and then halve the result.

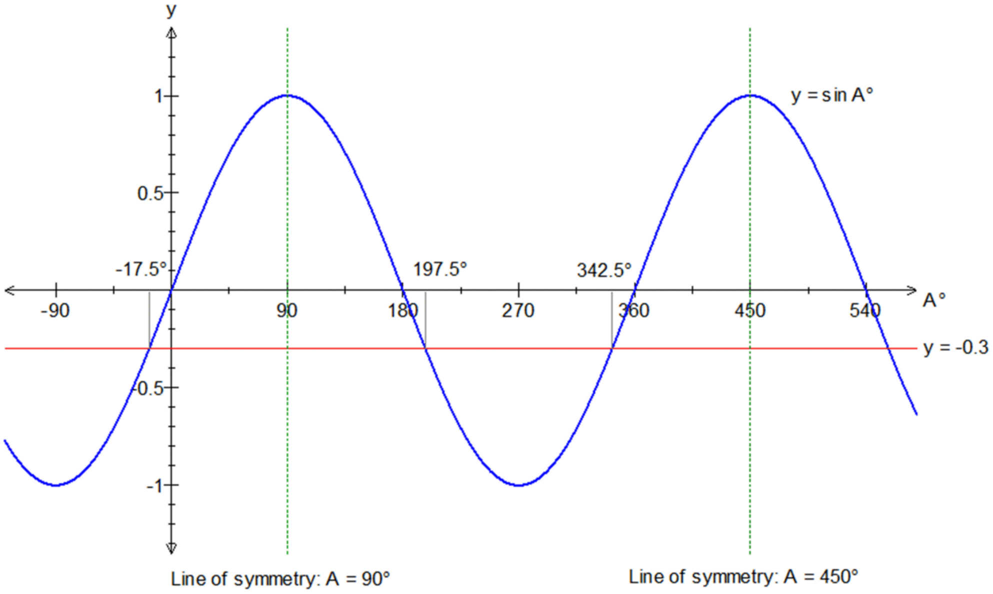
Therefore $A = 225° \Rightarrow x = ½(225-60)° = 82.5°$.
Similarly $A = 405° \Rightarrow x = 172.5°$, $A = 585° \Rightarrow x = 262.5°$ and finally $A = 765° \Rightarrow x = 352.5°$.

The solutions in the range are therefore x = 82.5°, 172.5°, 262.5° and 352.5°.

Example (8): Solve sin (3*x* - 40)° = -0.3 for 0 ≤ *x* ≤ 180° giving answers in degrees to one decimal place.

Again, we must substitute the *x*-limits of 0 ≤ *x* ≤ 180° with *A* – limits. By using *A* = 3*x*- 40, the *A*-limits transform to –40° ≤ *A* ≤ 500°.

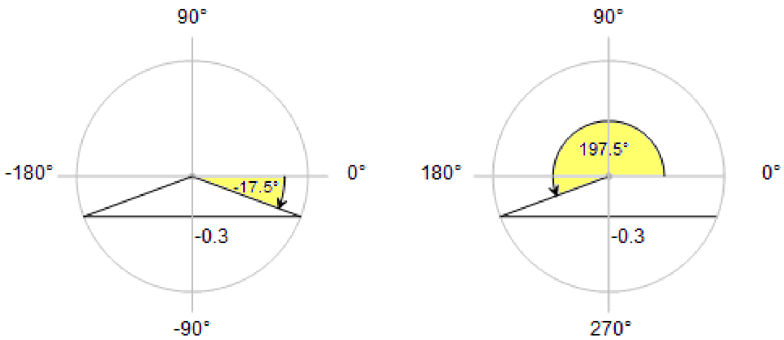
The graph below uses the *A*-limits.



The principal value of *A* is here –17.5°,. which is inside the *A*-limits.
This value is 107.5° to the left of the line of symmetry at *A*= 90°, so another solution will be 107.5° to the right, i.e. at *A* = 197.5°
(We could also have used sin (180-*x*)° = sin *x*°, and 180° - (-17.5°) = 197.5°.)

CAST diagram working:

Note that the negative angle is labelled with a clockwise arrow.



To complete the full set of solutions for *A* in the required range, we add multiples of 360° to the two above values. In fact the only additional one is *A* = (-17.5 + 360)° = 342.5°.

These three solutions will then need converting back from *A*-values to *x*-values by the inverse operation of

$x = \frac{A+40}{3}$, so when *A* = -17.5°, *x* = 7.5°.

Similarly, when *A* = 197.5°, *x* = 79.2° and when *A* = 342.5°, *x* = 127.5°.

Hence the solutions of sin (3*x* - 40)° = -0.3 for 0 ≤ *x* ≤ 180° are **7.5°, 79.2°** and **127.5°.**

Quadratic equations in the trigonometric functions are solved in the same way, but care must be taken when factorising them.

Note also that the graphs accompanying the solutions are for illustration only – you will not be asked to sketch them !

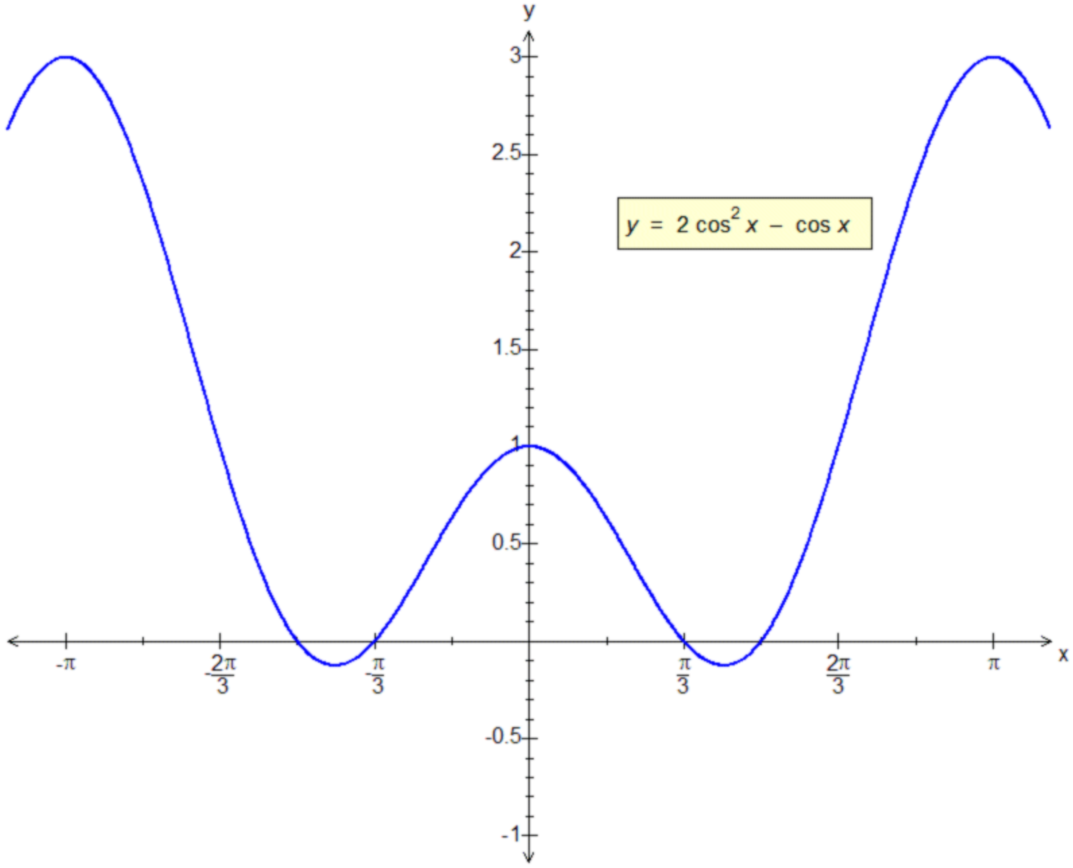
Example (9) : Solve the equation $2\cos^2 x - \cos x = 0$ for $-\pi \leq x \leq \pi$.

This factorises at once into $(\cos x)(2\cos x - 1) = 0$.

$\cos x = 0$ when $x = \pi/2$ or $x = -\pi/2$.

$2\cos x = 1$, or $\cos x = 0.5$, when $x = \pi/3$ or $x = -\pi/3$.

The solutions to the equation are therefore $x = \pm\pi/2$ and $\pm\pi/3$ (illustrated below).



Important: we cannot simply cancel out $\cos x$ from the equation as follows:

$2\cos^2 x - \cos x = 0 \Rightarrow 2\cos^2 x = \cos x \Rightarrow 2\cos x = 1 \Rightarrow x = \pm\pi/3$.

The final division of both sides by $\cos x$ has led to a loss of the solutions satisfying $\cos x = 0$, i.e. $x = \pm\pi/2$.

Example (10) : Solve the equation $\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$. Give the answer in radians to three decimal places.

This quadratic does not factorise and so the general formula must be used.

Substitute $x = \cos x$, a = 1, b = -1 and c = -1 into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos x = \frac{1 \pm \sqrt{5}}{2} \ .$$

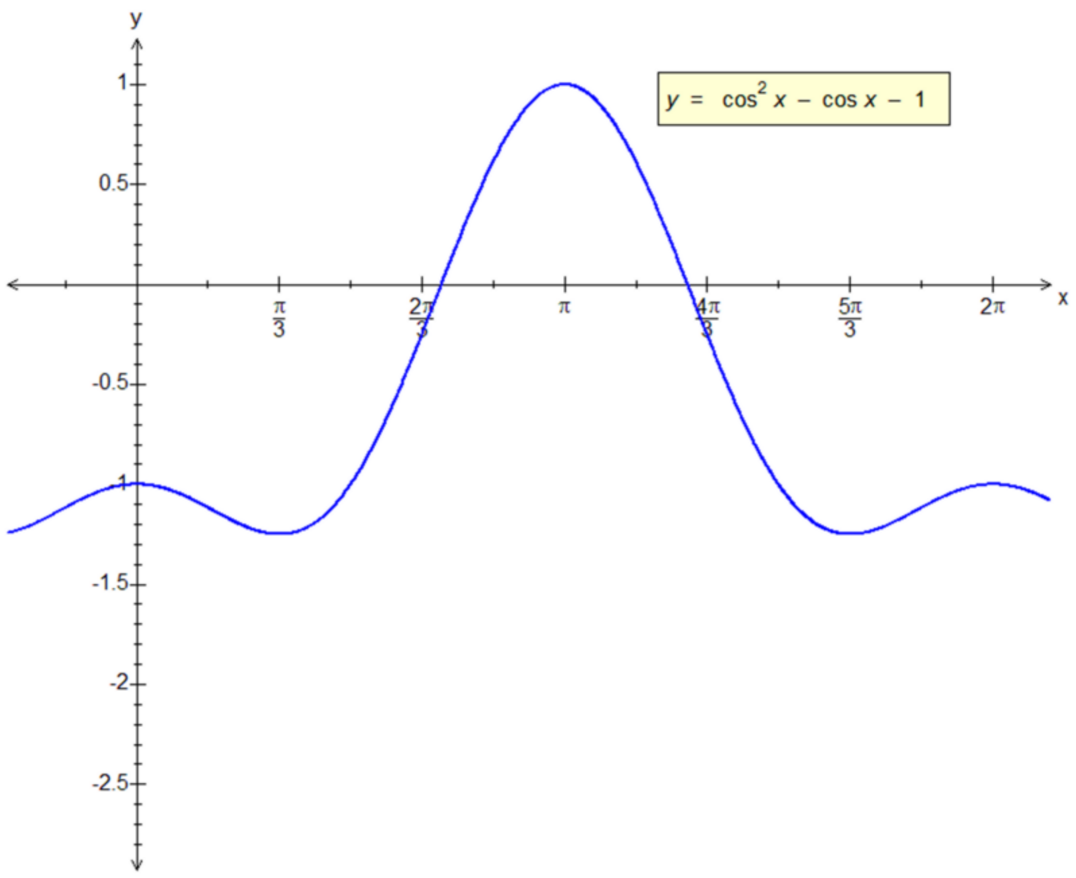
These are the possible solutions, and are 1.618 and -0.618 to 3 decimal places.

The first value can be rejected, since the cosine function cannot take values outside the range $-1 \le \cos x \le 1$.

The only solutions are those where $\cos x = \frac{1-\sqrt{5}}{2}$. The principal value of x is 2.237^c .

Since $\cos(2\pi - x) = \cos x$, another solution would be $(6.283–2.237)^c$ or 4.046^c.

$\therefore$ the solutions of $\cos^2 x - \cos x - 1 = 0$ where $0 \le x \le 2\pi$, are $\mathbf{2.237^c}$ and $\mathbf{4.046^c}$ to 3 decimal places.



Example (11): Find the angle(s) between 0° and 360° satisfying the equation

$1 - 2\cos^2 x + \sin x = 0$

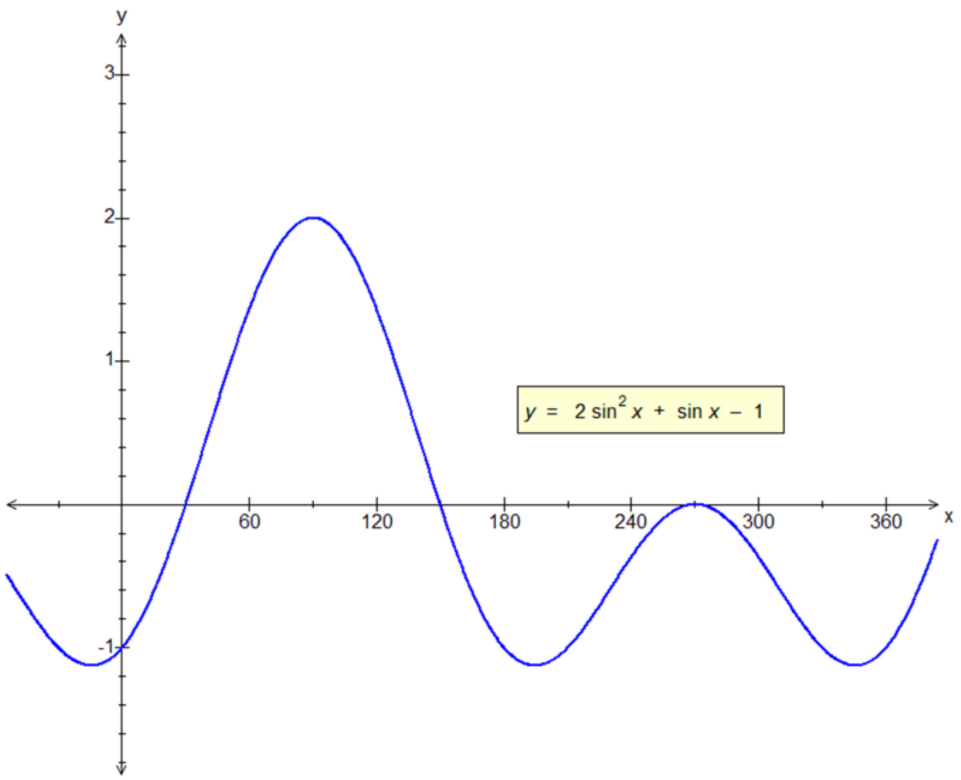
This equation can be manipulated into a quadratic in sin *x* by replacing $2\cos^2 x$ with $2(1 - \sin^2 x)$ using the Pythagorean identity.

$\Rightarrow 1 – 2(1-\sin^2 x) + \sin x = 0$
$\Rightarrow 1 – 2 + 2\sin^2 x + \sin x = 0$
$\Rightarrow 2\sin^2 x + \sin x – 1 = 0$

This is now a quadratic in sin *x* which factorises into $(2 \sin x - 1)(\sin x + 1) = 0$

Hence $\sin x = 0.5$ or $–1$.

This gives $x = 30°$ or $150°$ (where $\sin x = 0.5$); also $x = 270°$ (where $\sin x = -1$).



Example (12): Prove that $\dfrac{(1+\cos x)(1-\cos x)}{\sin x \cos x} = \tan x$.

The top line of the left-hand expression can be recognised as a ‘difference of squares’ result, namely $1-\cos^2 x$. This in turn can be replaced by $\sin^2 x$ using the Pythagorean identity.

The expression on the left thus becomes $\dfrac{\sin^2 x}{\sin x \cos x}$, and dividing both sides by sin *x* we finally obtain $\dfrac{\sin x}{\cos x}$ or tan *x*.

Example (13): Show that the equation $\sin x - \cos^2 x - 5 = 0$ has no solutions.

Substituting $\cos^2 x$ by $1 - \sin^2 x$ gives $\sin x - (1 - \sin^2 x) - 5 = 0$

$\Rightarrow \sin x - 1 + \sin^2 x - 5 = 0$
$\Rightarrow \sin^2 x + \sin x - 6 = 0$
$\Rightarrow (\sin x + 3)(\sin x - 2) = 0$
$\Rightarrow \sin x = 2$ or -3.

The quadratic in $\sin x$ factorises all right, but the solutions are impossible because the sine function cannot take values outside the range $-1 \le \sin x \le 1$.

Example (14): Solve the equation $\sin x = \sqrt{3}\cos x$ for angles between 0° and 360°.

We can divide the equation by $\cos x$ to give $\frac{\sin x}{\cos x} = \sqrt{3}$, or $\tan x = \sqrt{3}$.

The principal value of x is 60°, but since the tangent graph repeats every 180°, another solution is x = 240°.