# M.K. HOME TUITION

Mathematics Revision Guides
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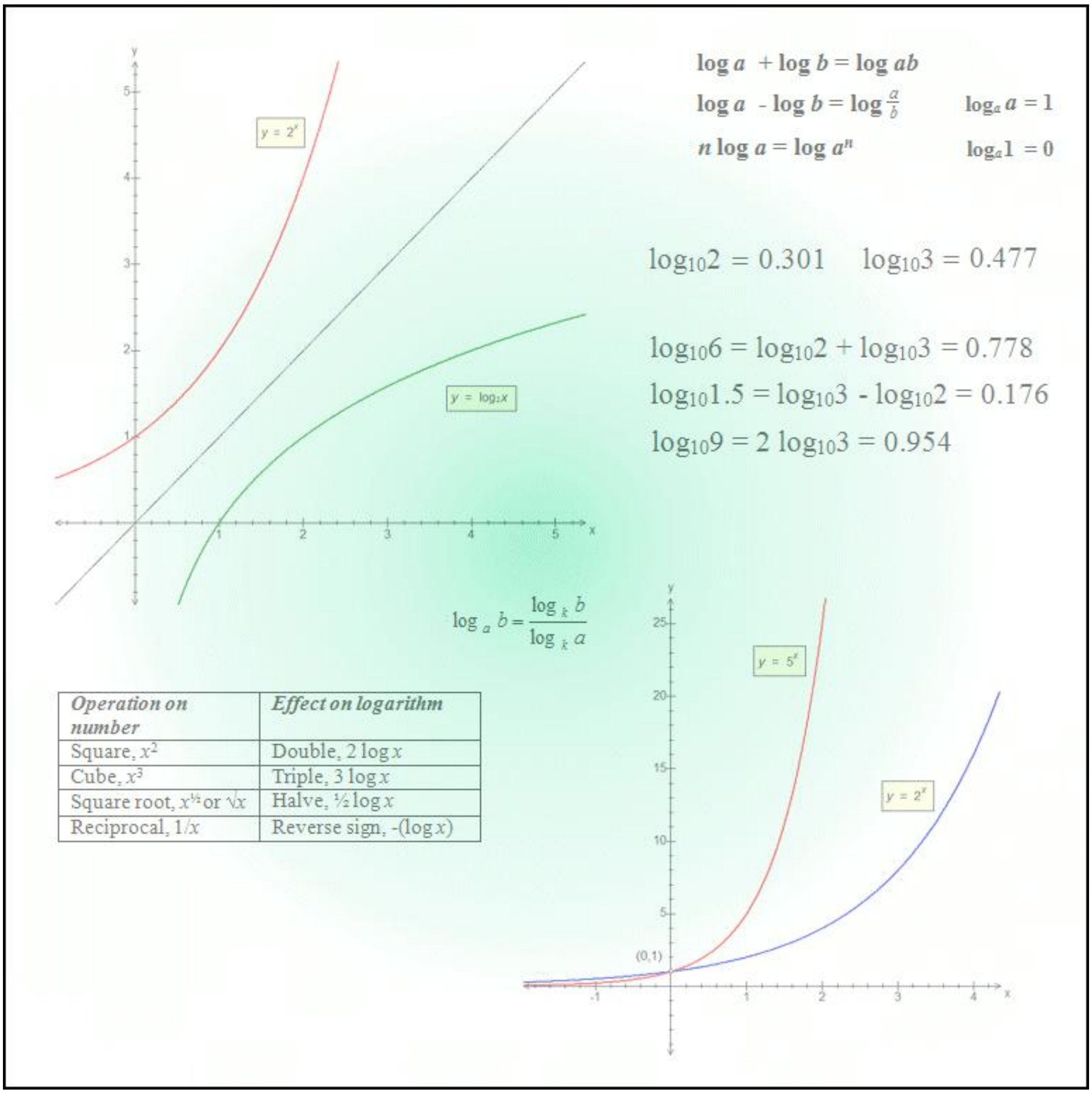
# EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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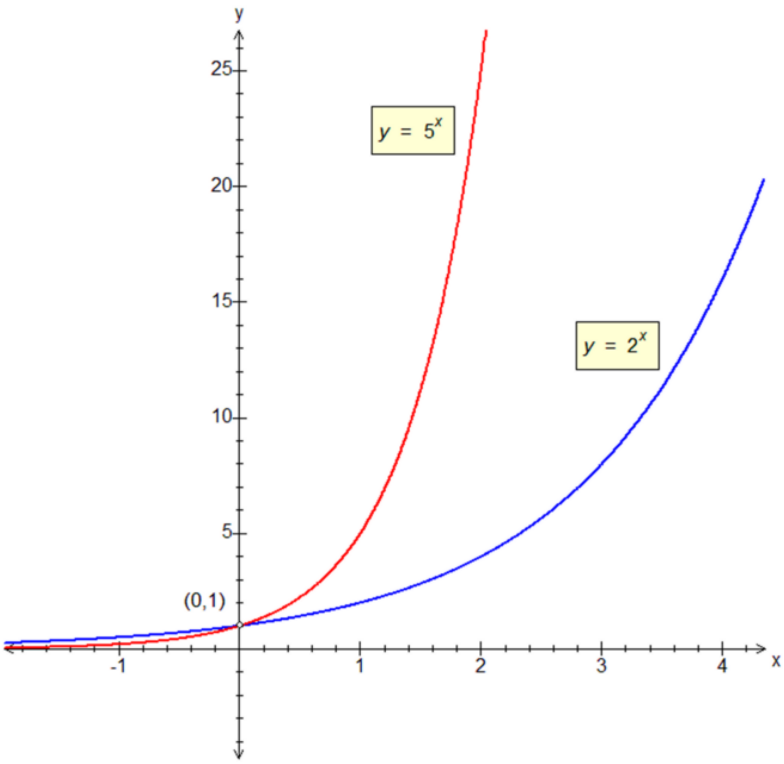
# EXPONENTIAL AND LOGARITHMIC FUNCTIONS

An **exponential function** is one whose variable is a power or exponent. Any function of the form $f(x) = a^x$, where $a$ is a non-zero positive constant, is an exponential function.

The graphs on the right are of two such exponential functions: $y = 2^x$ and $y = 5^x$.



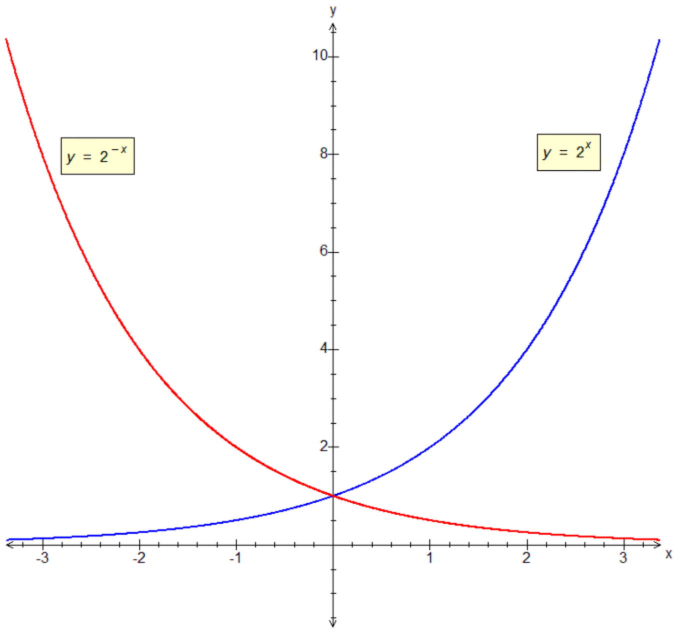

All graphs of this form cross the $y$-axis at the point (0, 1) as would be expected of the zero index law.

Neither graph ends up crossing the $x$-axis, in other words, the function is always positive.

Each exponential graph has a different gradient at the point (0, 1), and the value of this gradient depends on $a$.

There is a particular exponential function whose graph has a gradient of 1 at the point (0, 1), to be studied in detail in later sections.

The next graph shows the relationship between the functions $y = 2^x$ and $y = 2^{-x}$.




$2^{-x}$ is the same as $(½)^x$ – recall the laws of negative indices.

The graph of $y = 2^{-x}$ is a reflection of the graph of $y = 2^x$ in the $y$-axis.

In general, for any positive $a > 1$, the function $a^x$ is always increasing, and the function $a^{-x}$ is always decreasing.

If $a < 1$, the opposite holds true.

Exponential functions are often used for modelling patterns of growth ($a > 1$) or decay ($a < 1$).

**Logarithmic functions** are related to exponential functions.

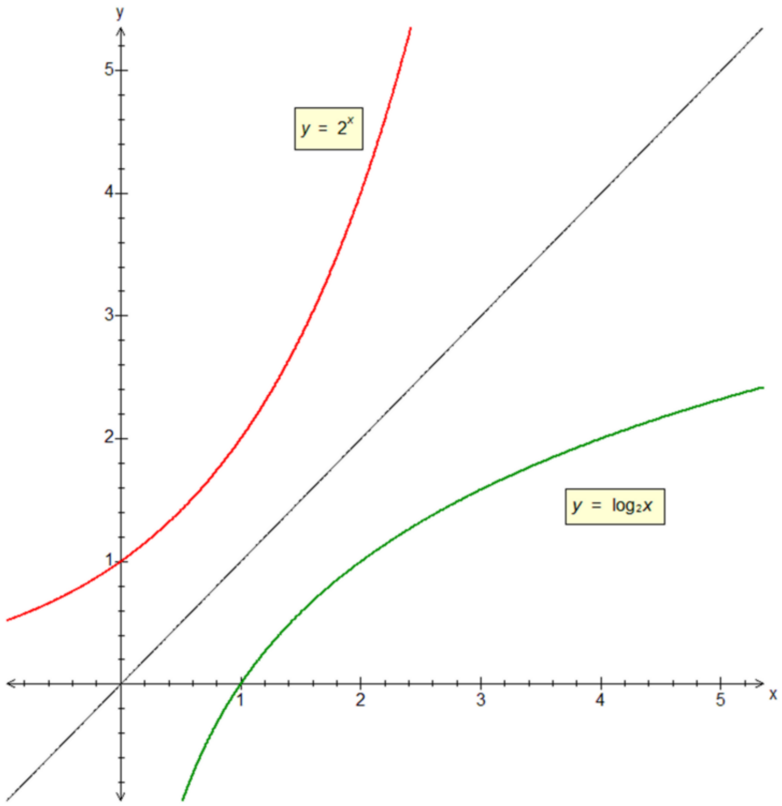
**If a number $x = a^y$, then $y = \log_a x$.**

The number $a$ is the **base** of the logarithm.

The **logarithm** (log) to a given **base** of a number is the **power** to which the base must be raised to equal that number.

Thus, $10^2 = 100$, therefore the logarithm of 100 to base 10, or $\log_{10}100 = 2$.
The following identities hold for any base $a$:

$\log_a a = 1$, since $a^1 = a$; $\log_a 1 = 0$, since $a^0 = 1$.




The example above shows the graphs of $y = 2^x$ and $y = \log_2 x$. Being inverse functions, each graph is a reflection of the other in the line $y = x$.

**Laws of Logarithms.**

There are three important laws of logarithms, all derived from those for indices. They are irrespective of the base used, as long as the same base is used consistently.

The logarithm of a product of two (or more) numbers is the sum of the logarithms of the individual numbers.

$\log a + \log b = \log ab$

The logarithm of a quotient of two numbers is the difference between the logarithms of the individual numbers.

$\log a - \log b = \log \frac{a}{b}$

The logarithm of the $n^{th}$ power of a number is $n$ multiplied by the logarithm of that number.

$n \log a = \log a^n$ .

Examples below:

| ***Operation on number*** | ***Effect on logarithm*** |
|---|---|
| Square, $x^2$ | Double, 2 log $x$ |
| Cube, $x^3$ | Triple, 3 log $x$ |
| Square root, $x^{½}$ or $\sqrt{x}$ | Halve, ½ log $x$ |
| Reciprocal, $1/x$ | Reverse sign, -(log $x$) |

**The 'Change of Base' Rule.**

To find a logarithm of a number to a given base, we use the 'change of base' formula:

$\log_a b = \log_k b \div \log_k a$.

This can be proven as follows:

Using the fact that if $b = a^y$, then $y = \log_a b$, and taking logs of both sides to an arbitrary base $k$, we have

$\log_k b = \log_k (a^y) = y \log_k a$, and since $y = \log_a b$, we have $\log_a b = \dfrac{\log_k b}{\log_k a}$

The intermediate base, $k$, can be any base you care to choose. Calculators generally use base 10 ($\log_{10} x$) or 'natural' logs ($\ln x$). The base of the 'natural' log function is not an integer, but this is not relevant to the use of the formula.

Most current scientific calculators now have the $\log_a b$ function included.

Other logarithm generalisations arising from the laws of indices are:

$\log_a a = 1$ ; (the logarithm of any number to its own base is always 1.)

$\log_a 1 = 0$ ; (the logarithm of 1 to any base is always 0.)

These laws can be applied to simplify expressions involving logarithms and to solve **exponential equations**, where the unknown value is a power.

**Examples (1):** The values of $\log_{10}2$ and $\log_{10}3$ are 0.301 and 0.477 to 3 decimal places respectively.

Use this result to find i) $\log_{10}6$ ; ii) $\log_{10}1.5$ ; iii) $\log_{10}9$ ; iv) $\log_{10}0.5$ ; v) $\log_{10}200$;

vi) $10^{-0.602}$ ; vii) $10^{0.699}$ ; viii) $10^{0.1505}$ ix) $10^{3.477}$ ; x) $10^{-1.778}$

i) Multiplication of numbers corresponds to addition of logarithms, so
$6 = 2 \times 3, \Rightarrow \log_{10}6 = \log_{10}2 + \log_{10}3 = 0.778$ (to 3 dp).

ii) Division of numbers corresponds to subtraction of logarithms, so
$1.5 = 3 \div 2, \Rightarrow \log_{10}1.5 = \log_{10}3 - \log_{10}2 = 0.176$ (to 3 dp).

iii) Squaring a number corresponds to doubling its logarithm, so
$9 = 3^2, \Rightarrow \log_{10}9 = 2 \log_{10}3 = 0.954$ (to 3 dp).

iv) Taking a reciprocal of a number corresponds to reversing the sign of its logarithm, so
$0.5 = ½, \Rightarrow \log_{10}(½) = -( \log_{10}2) = -0.301$ (to 4 dp).

v) Here we use the multiplication law as well as recognising a power of 10 as the multiplier:
$200 = 100 \times 2$, and since $100 = 10^2$, then $\log_{10}100 = 2$ and finally $\log_{10}200 = 2 + \log_{10}2 = 2.301$.

For parts vi) to x), we remember that the statement

$\log_{10}2 = 0.301$ is equivalent to $2 = 10^{0.301}$ , a fact that can be shown by taking logarithms (here to base 10) of the RHS to obtain the expression on the LHS.

vi) We see that $–0.602 = - (2 \times 0.301)$. We have taken $\log_{10}2$ and then doubled and reversed its sign. This sequence of operations corresponds to squaring 2 to give 4, and then taking the reciprocal of the result.
$\therefore\ 10^{-0.602} = \frac{1}{4}$ .

vii) Here we see that $0.699 = 1 – 0.301$, and therefore $10^{0.699} = 10^1 \div 10^{0.301}$ or $10 \div 2 = 5$.
$\therefore\ 10^{0.6990} = 5$.

viii) By inspection, $0.1505 = ½(0.301)$ and $10^{0.301} = 2$, so $10^{0.1505} = \sqrt{(10^{0.301})}$.
$\therefore\ 10^{0.1505} = \sqrt{2}$. (Halving the logarithm corresponds to taking the square root of the number.)

ix) We recognise that $10^{3.477} = 10^3 \times 10^{0.477}$ or $1000 \times 3$
$\therefore\ 10^{3.477} = 3000$.

x) We can spot that $1.778 = 1 + 0.301 + 0.477$,
$\therefore\ 10^{1.778} = 10^1 \times 10^{0.301} \times 10^{0.477} = 10 \times 2 \times 3$ or 60.

The question however asks for $10^{-1.778}$. Reversing the sign on a logarithm corresponds to taking the reciprocal, and so $10^{-1.778} = \frac{1}{60}$ .

**Example (2):** Express as single logarithms: a) $\log x + 3\log y$ ; b) $2\log x - ½\log y$; c) $\log_5 x - 1$

a) $\log x + 3\log y = \log(xy^3)$

b) $2\log x - ½\log y = \log\left(\frac{x^2}{\sqrt{y}}\right)$.

c) $\log_5 x - 1 = \log_5 x - \log_5(5) = \log_5\left(\frac{x}{5}\right)$ .

**Example (3):** Solve the equations i) $5^x = 30$; ii) $2^x = 0.05$ ; iii) $(3^x)(3^{x+1}) = 40$; iv) $7^{3x} = 2000$;
v) $\log_2 x = 1.5$; vi) $\log_5 x = -0.8$.

i) We take logs of both sides to obtain

$$\log(5^x) = \log 30 \Rightarrow x\log 5 = \log 30 \Rightarrow x = \frac{\log 30}{\log 5} = 2.113 \text{ (to 4 s.f.)}$$

This calculation works in any base – either using $\log_{10}$ or ln on the calculator, as long as we're consistent. (Solving $5^x = 30$ is the same as finding the value of $\log_5 30$).

ii) Again taking logs of both sides, we have

$$\log(2^x) = \log 0.05 \Rightarrow x\log 2 = \log 0.05 \Rightarrow x = \frac{\log 0.05}{\log 2} = -4.322 \text{ (to 4 s.f.)}$$

iii) This is slightly harder, but again we take logs of both sides:

$$\log(3^x) + \log(3^{x+1}) = \log 40 \Rightarrow \log(3^{2x+1}) = \log 40 \Rightarrow (2x+1)\log 3 = \log 40$$

$$\Rightarrow 2x+1 = \frac{\log 40}{\log 3} = 3.3578... \Rightarrow x = ½(3.3578 - 1) = 1.179 \text{ to 4 s.f.}$$

iv) Continuing as before, $\log(7^{3x}) = \log 2000 \Rightarrow 3x\log 7 = \log 2000$

$$\Rightarrow 3x = \frac{\log 2000}{\log 7} = 3.9060..... \Rightarrow x = 1.302 \text{ to 4 s.f.}$$

v) Here we take exponents of both sides ( "2 to both sides") to obtain
$\log_2 x = 1.5 \Rightarrow x = 2^{1.5} \Rightarrow x = 2.828$ (to 4 s.f.)

vi) Similarly, we take "5 to both sides":

$\log_5 x = -0.8 \Rightarrow x = 5^{-0.8} \Rightarrow x = 0.2759$ (to 4 s.f.)

Remember, $2^{\log_2 x}$, $5^{\log_5 x}$ .... are equal to $x$ itself - so are $\log_2 2^x$, $\log_3 3^x$ ..... This is because exponential and logarithmic functions (to the same base) are inverses of each other.

**Example (4):** Use the result from Part ii) in the previous example to find:

i) $2^x = 0.4$; ii) $2^x = 20$; iii) $2^x = \sqrt{5}$.

i) We look for a relationship between 0.4 and 0.05 involving a power of 2, and we see that $0.4 = 0.05 \times 8$.

We have found out that $0.05 = 2^{-4.322}$ in example 2, and. using the fact that $8 = 2^3$,
we have $0.4 = 2^{-4.322} \times 2^3 = 2^{-1.322}$

$\therefore x = -1.322$ to 4 s.f.

ii) Here, we notice that 0.05 and 20 are reciprocals of each other.

$20 = \frac{1}{0.05} = 2^{4.322}$
$\therefore x = 4.322$ to 4 s.f

iii) This time we use the surd laws to express $\sqrt{5} = \sqrt{\frac{20}{4}}$.

To find the base 2 logarithm for that, take $\log_2 20$ from part ii) and subtract $\log_2 4$, or 2, from it, to obtain 2.322 to 4 s.f.
Halve that result (remember $\sqrt{x} = x^{½}$) to solve $2^x = \sqrt{5}$, $\therefore x = 1.161$ to 4 s.f

**Example (5):** Evaluate the following: i) $\log_2 8$; ii) $\log_9 3$; iii) $\log_8 \frac{1}{32}$
i) Since $8 = 2^3$, $\log_2 8 = 3$.
ii) Since $3 = \sqrt{9}$, $\log_9 3 = ½$ or 0.5.

iii) The previous two examples were worked out by little more than casual inspection. This time we must use the 'change of base' rule by choosing a suitable intermediate base.
Since $8 = 2^3$ and $\frac{1}{32} = \frac{1}{2^5} = 2^{-5}$, we can use 2 as the working intermediate base.

$$\log_8 \frac{1}{32} = \frac{\log_2 \frac{1}{32}}{\log_2 8} = \frac{-5}{3}.$$

Sometimes a question connected with logarithms or exponentials can be solved using normal algebra.

**Example (6):** Solve $3^{x-4} = 2^{x+3}$**.**

Taking logarithms of both sides, $3^{x-4} = 2^{x+3} \Rightarrow (x-4) \log 3 = (x+3)\log 2$
$\Rightarrow x \log 3 - 4 \log 3 = x \log 2 + 3 \log 2$
$\Rightarrow x \log 3 - x \log 2 = 3 \log 2 + 4 \log 3$ (collecting $x$-terms)
$\Rightarrow x (\log 3 - \log 2) = 3 \log 2 + 4 \log 3$ (factorising)

$$\Rightarrow \quad x = \frac{3\log 2 + 4\log 3}{\log 3 - \log 2} = 15.967.$$

**Example (7):** Solve $\log(a + 15) = 2 \log( a - 15)$.

The interesting thing here is that the base of the logarithm is immaterial !

$\log(a + 15) = 2 \log( a - 15) \Rightarrow \log(a + 15) = \log(( a - 15)^2)$ using log laws.

$\Rightarrow a + 15 = ( a - 15)^2$ (taking antilogs)

This rearranges into a standard quadratic:

$a^2 - 30a + 225 - (a + 15) = 0$

$\Rightarrow a^2 - 31a + 210 = 0$
$\Rightarrow (a - 21) (a - 10) = 0$
$\therefore a = 21$ or 10.

The equation appears to have solutions of $a = 21$ and $a = 10$.

Substituting $a = 21$ into the original would give log 36 = 2 log 6, which is sound enough,
but when $a = 10$, we would have log 25 = 2 log (-5), which is inadmissible since there is no logarithm of a negative number.

$\therefore$ the only solution to the equation is $a = 21$.

**Example (8):** Solve $2^{2x} - 11(2^x) - 80 = 0$.

Since $2^{2x}$ is equivalent to $(2^x)^2$, this expression is a quadratic in $2^x$, i.e. $(2^x)^2 - 11(2^x) - 80 = 0$.

Factorising the quadratic gives $(2^x - 16) (2^x + 5) = 0$.

The equation appears to have solutions of $2^x = 16$ and $2^x = -5$.
Only the solution of $2^x = 16$ and hence $x = 4$, is valid here, since $2^x$ can never be negative.

The graphs of exponential and logarithmic functions can be transformed in the usual way.

**Transformations of graphs of exponential functions of the form $a^x$.**

- The graph of $a^x + k$ is the same as the graph of $a^x$, but translated by the vector $\begin{pmatrix}0\\k\end{pmatrix}$.
- The graph of $a^{x+k}$ is the same as the graph of $a^x$, but translated by the vector $\begin{pmatrix}-k\\0\end{pmatrix}$.
- The graph of $ka^x$ is the same as the graph of $a^x$, but stretched by a factor of $k$ in the $y$-direction.
- The graph $a^{kx}$ is the same as the graph of $a^x$, but stretched by a factor of $1/k$ in the $x$-direction.

The logarithm laws lead to some unexpected results.

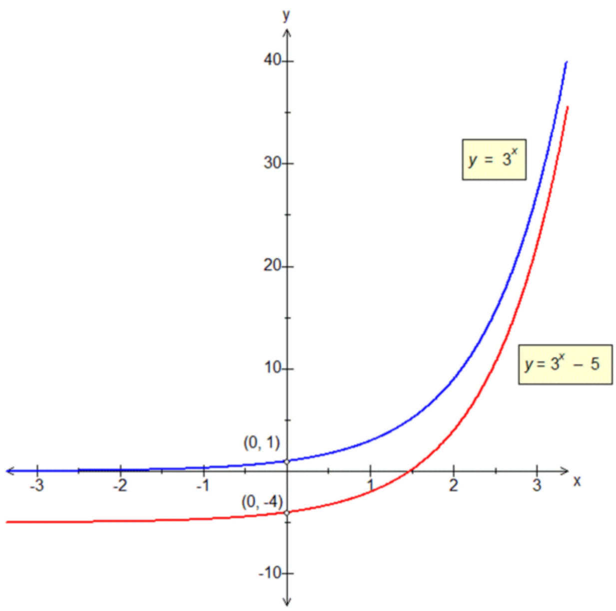
**Examples (9):** Describe the following transformations and sketch them, showing at least one point before and after the transformation:

i) $y = 3^x$ to $y = 3^x - 5$
ii) $y = 2^x$ to $y = 2^{x+2}$
iii) $y = 2^x$ to $y = 4(2^x)$
iv) $y = 2^x$ to $y = 8^x$

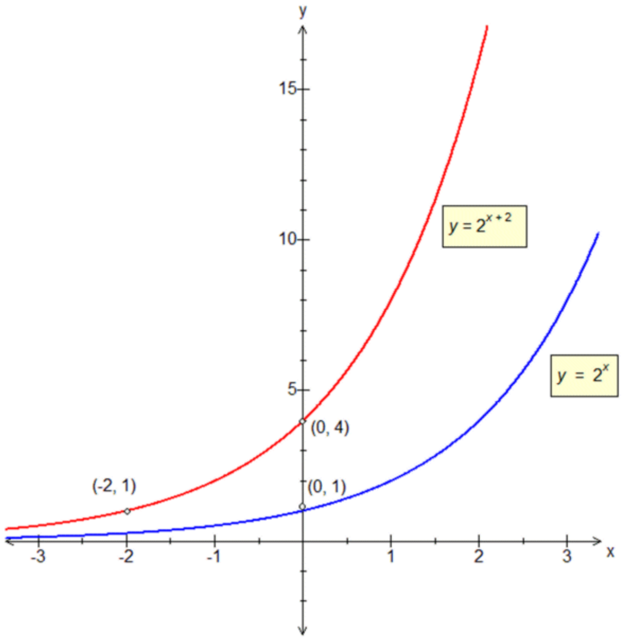
i) $y = 3^x$ to $y = 3^x - 5$

The graph of $y = 3^x - 5$ is a $y$-shift of $y = 3^x$ by 5 units in the negative direction, or a translation by the vector $\begin{pmatrix}0\\-5\end{pmatrix}$.

For example, point (0, 1) on the graph of $y = 3^x$ is transformed to (0, -4) on the graph of $y = 3^x - 5$.



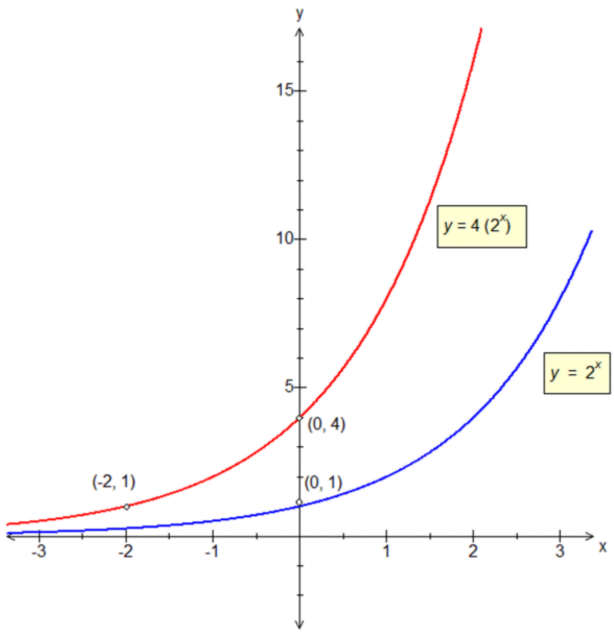
ii) $y = 2^x$ to $y = 2^{x+2}$




The graph of $y = 2^{x+2}$ is an $x$-shift of $y = 2^x$ by 2 units in the negative direction, or a translation with the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

As an example, point (0, 1) on the graph of $y = 2^x$ maps to the point (-2, 1) on the graph of $y = 2^{x+2}$.

iii) $y = 2^x$ to $y = 4(2^x)$




The graph of $y = 4(2^x)$ is a $y$-stretch of $y = 2^x$ by a scale factor of 4 .

For example, point (0, 1) on the graph of $y = 2^x$ maps to the point (0, 4) on the graph of $y = 2^{x+2}$.

The two transformations in ii) and iii) are actually equivalent here.

This is because, by the log laws, $2^{x+2} = 2^x \times 2^2$, or $4(2^x)$.

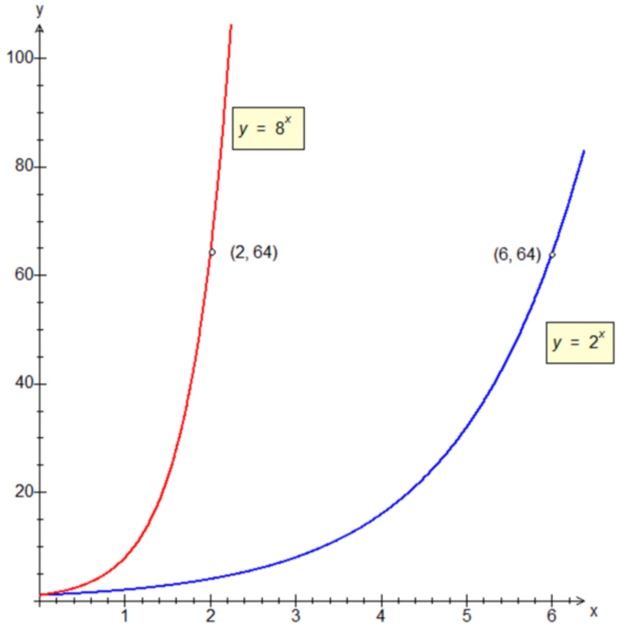
Since $a^{x+k}$ is the same as $a^x \times a^k$, then there are **two** equivalent ways of transforming the graph of $a^x$ to give that of $a^{x+k}$; either a translation through $\begin{pmatrix} -k \\ 0 \end{pmatrix}$, or by a $y$ – stretch with factor $a^k$.

iv) $y = 2^x$ to $y = 8^x$

Here we use the log laws to redefine 8 as $2^3$, and so $8^x = (2^3)^x = 2^{3x}$.

The graph of $8^x$ is an $x$-stretch of $y = 2^x$ by a scale factor of $\frac{1}{3}$.

As an example, point (6, 64) on the graph of $y = 2^x$ maps to the point (2, 64) on the graph of $y = 8^x$.



**Transformations of graphs of logarithmic functions of the form $\log_a x$.**

- The graph of $\log_a x + k$ is the same as the graph of $\log_a x$, but translated by the vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$.
- The graph of $\log_a (x + k)$ is the same as the graph of $\log_a x$, but translated by the vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$.
- The graph of $k \log_a x$ is the same as the graph of $\log_a x$, but stretched by a factor of $k$ in the $y$-direction. Note also that $k \log_a x$ is equivalent to $\log_a (x^k)$.
- The graph $\log_a (kx)$ is the same as the graph of $\log_a x$, but stretched by a factor of $1/k$ in the $x$-direction.

Again, the logarithm laws lead to some unexpected results:

**Examples (10):** Describe the following transformations and sketch them, showing at least one point before and after the transformation:

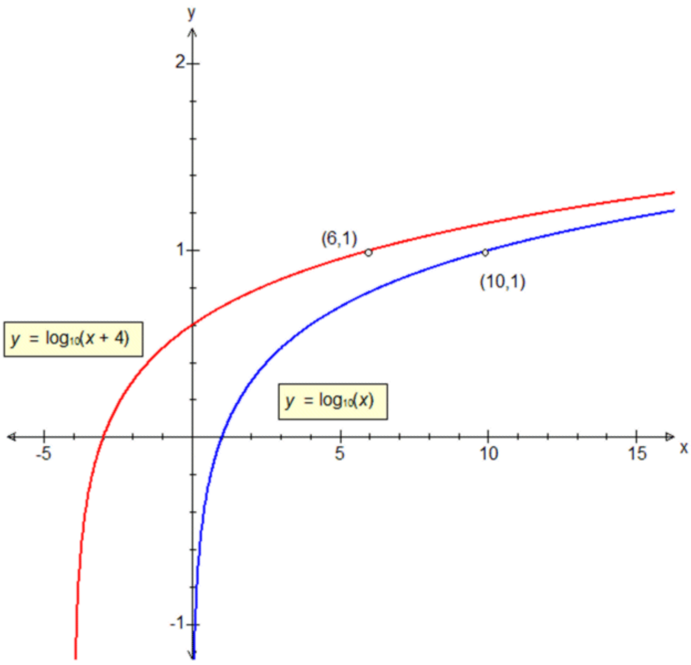
i) $y = \log_{10}x$ to $y = \log_{10}(x + 4)$
ii) $y = \log_5 x$ to $y = \log_5(25x)$
iii) $y = \log_5 x$ to $y = \log_5(x) + 2$
iv) $y = \log_2 x$ to $y = \log_2(x^3)$

i) $y = \log_{10}x$ to $y = \log_{10}(x + 4)$

The graph of $y = \log_{10}(x + 4)$ is an $x$-shift of $y = \log_{10}x$ by 4 units in the negative direction, or a translation using the vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

For example, point (10, 1) on the graph of $y = \log_{10}x$ is transformed to (6, 1) on the graph of $y = \log_{10}(x + 4)$.

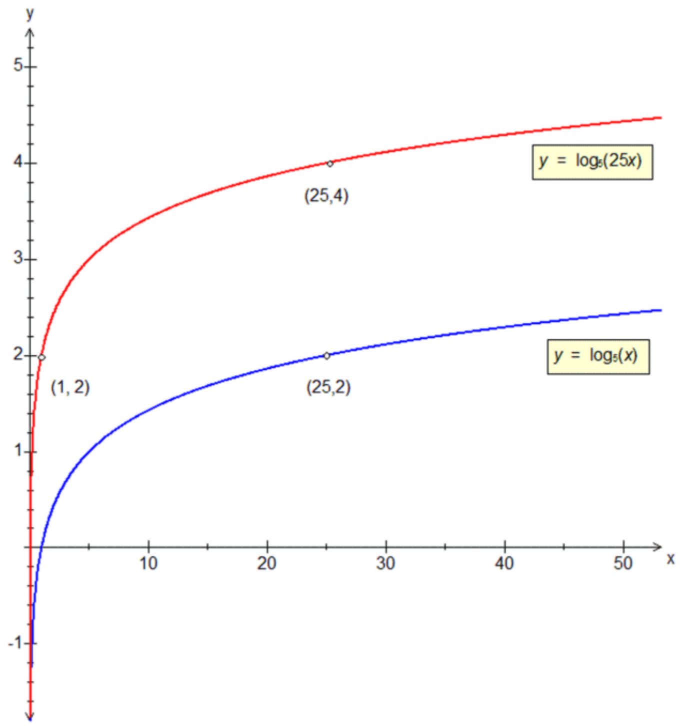
Notice also, how, although $\log_{10}x$ is defined only for $x > 0$, $\log_{10}(x + 4)$ is defined for $x > -4$.



ii) $y = \log_5 x$ to $y = \log_5(25x)$

The graph of $y = \log_5(25x)$ is an $x$-stretch of the graph of $y = \log_5 x$, with a scale factor of $\frac{1}{25}$.

For example, point (25, 2) on the graph of $y = \log_5 x$ is transformed to (1, 2) on the graph of $y = \log_5(25x)$.

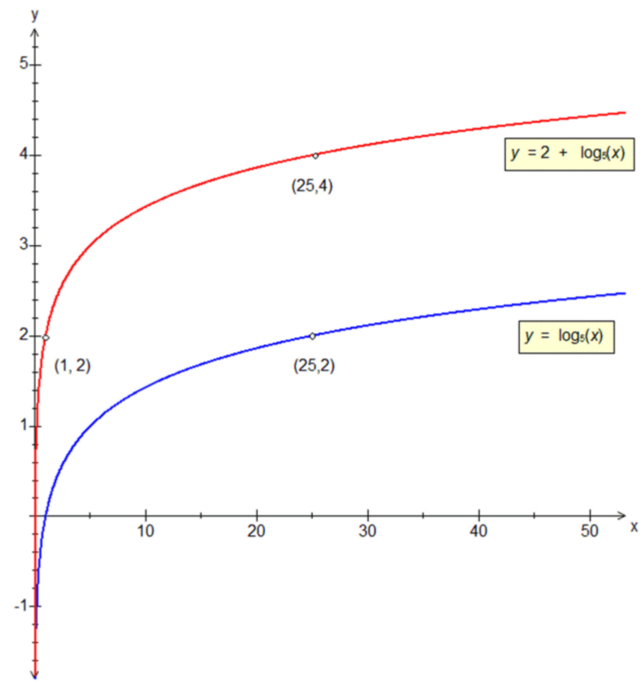



iii) $y = \log_5 x$ to $y = \log_5(x) + 2$

The graph of $y = \log_5(x) + 2$ is a $y$-shift of $y = \log_5 x$ by 2 units in the positive direction, or a translation with the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

As an example, point (25, 2) on the graph of $y = \log_5 x$ is transformed to (25, 4) on the graph of $y = \log_5(x) + 2$.

Again. the transformations in parts ii) and iii) are equivalent.




Since, by the logarithm laws, $\log_a(kx)$ is the same as $\log_a x + \log_a k$, there are **two** equivalent ways of transforming the graph of $\log_a x$ to give that of $\log_a(kx)$; by an $x$-stretch of factor $1/k$, or by a $y$-shift of $\log_a k$ units.

iv) $y = \log_2 x$ to $y = \log_2(x^3)$

Here we use the log laws to redefine $\log_2(x^3)$ as $3 \log_2 x$.

The graph of $y = \log_2(x^3)$ is therefore a $y$-stretch of $y = \log_2 x$ by a scale factor of 3.

As an example, point (4, 2) on the graph of $y = \log_2 x$ is transformed to the point (4, 6) on the graph of $y = \log_2(x^3)$.