M.K. HOME TUITION

Mathematics Revision Guides
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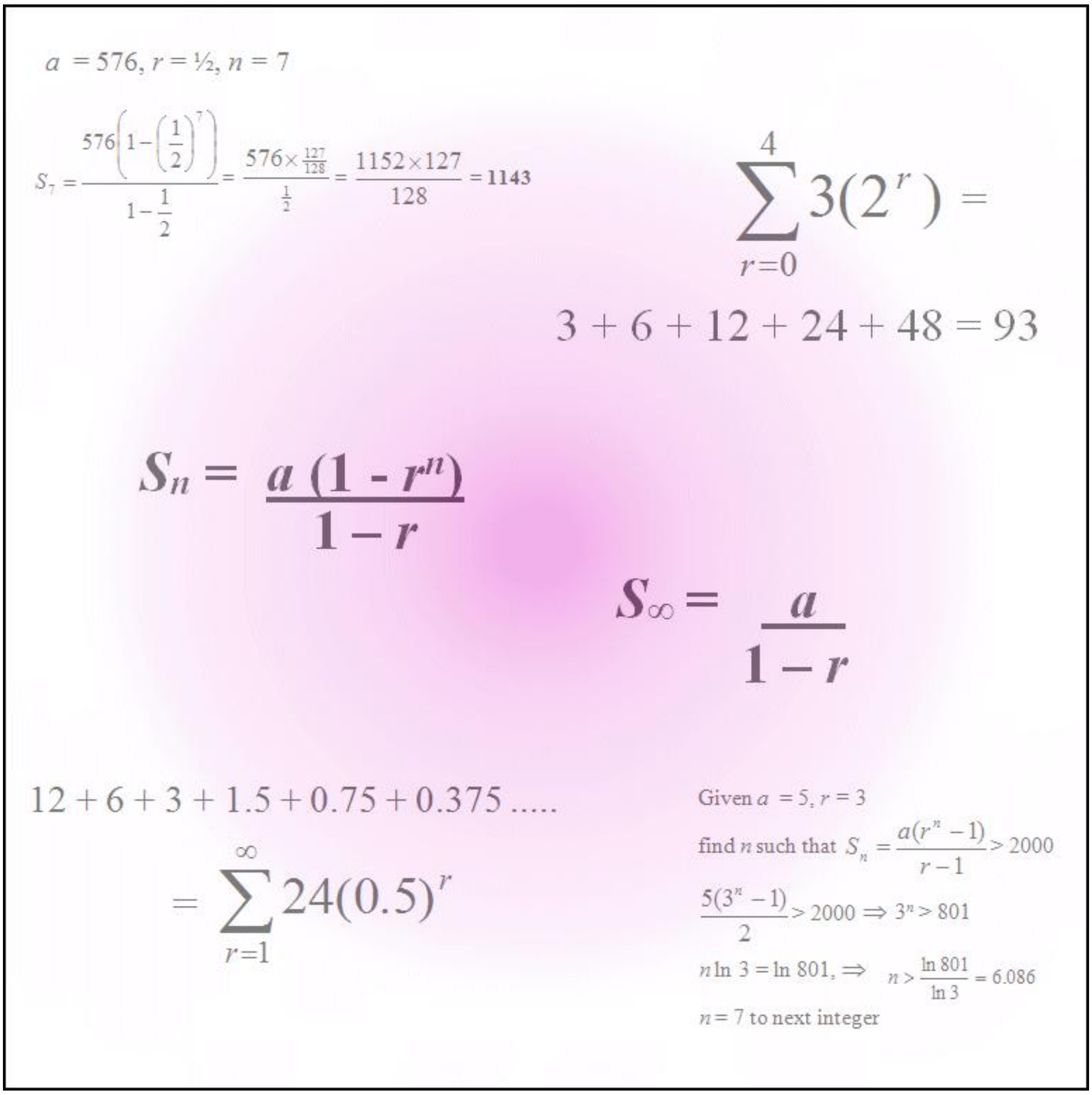
THE GEOMETRIC SERIES



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The Geometric Series. (also known as the Geometric Progression or G.P.)

In a G.P. successive terms are linked by a **common ratio**, such as in 3, 6, 12, 24, 48... The first term is denoted by a and the common ratio is r. A recursive definition of a G.P. is therefore $u_1 = a$; $u_{k+1} = ru_k$; a term is obtained from the previous one by **multiplying by** the common ratio. The example 3, 6, 12, 24, 48... is thus defined as $u_1 = 3$; $u_{k+1} = 2u_k$.

The terms of a G.P. take the form a, ar, ar^2, ar^3, ... and the n^{th} term is ar^{n-1}.

Summing a Geometric Series.

A **geometric series** is formed by adding together the terms of a geometric progression.
Its sum is given by

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Multiplying throughout by r we obtain the result

$$rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^n$$

Subtracting the second result from the first gives
$$S_n - rS_n = a - ar^n$$

Both sides can then be factorised:
$$S_n(1 - r) = a(1 - r^n)$$

This gives the formula for the sum of a geometric series:

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}}$$

When $r > 1$, it is more convenient to write the formula as

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{(r - 1)}}$$

If $|r| < 1$, then r^n tends to zero with increasing n, and therefore $(1 - r^n)$ tends to 1. As n tends to infinity, the sum of the geometric series converges to the value

$$\boldsymbol{S_\infty = \frac{a}{1 - r}}$$

This is known as the **sum to infinity** of the geometric series.

Example(1): Find the sum of a G.P. with 7 terms, whose third term is 144 and common ratio is ½.

We must find the missing first term before substituting into the formula.

$u_3 = ar^2 = 144$. Since $r = ½$, $a = 576$.

Since $r < 1$, we can then enter the values into the general formula $S_n = \frac{a(1-r^n)}{1-r}$ to obtain

$$S_7 = \frac{576\left(1-\left(\frac{1}{2}\right)^7\right)}{1-\frac{1}{2}} = \frac{576 \times \frac{127}{128}}{\frac{1}{2}} = \frac{1152 \times 127}{128} = \mathbf{1143}.$$

Example(2): A G.P. has 6 positive terms, the second being 96 and the fourth being 216. Find its sum.

We have $u_4 = ar^3 = 216$ and $u_2 = 96 = ar$.
Hence $r^2 = \frac{216}{96} = \frac{9}{4}$, and thus $r = \frac{3}{2}$ (positive solution) and $a = 64$.

We will use the second form of the formula, $S_n = \frac{a(r^n - 1)}{r-1}$ to compute the sum.

$$S_6 = \frac{64\left(\left(\frac{3}{2}\right)^6 - 1\right)}{\frac{3}{2}-1} = \frac{(64 \times \frac{729}{64}) - 64}{\frac{1}{2}} = \mathbf{1330.}$$

Example (3): Find the sum of a G.P. whose first term is 12, whose last term is 8748, and whose common ratio is 3.

Here, $u_1 = a = 12$ and $u_n = ar^{n-1} = 8748$.

This therefore gives $r^{n-1} = 3^{n-1} = \frac{8748}{12} = 729$.

From this, $n - 1 = \frac{\ln 729}{\ln 3} = 6$.

The G.P. therefore has 7 terms and its sum can be worked out as

$$S_7 = \frac{12(3^7 - 1)}{3-1} = \frac{12 \times 2186}{2} = \mathbf{13116.}$$

Example(4): Find the sum to infinity of the G.P. in Example (11). The first term is 576 and the common ratio is ½.

Since $r < 1$, there is a sum to infinity, we can then substitute the values into the general formula $S_\infty = \frac{a}{1-r}$ to obtain the final result $S_\infty = \frac{576}{1-\frac{1}{2}} = \mathbf{1152}$.

Example (5): Find the common ratio of a G.P. with first term of 5 and a sum to infinity of 15.

Rearrange $S_\infty = \frac{a}{1-r}$ as $1 - r = \frac{a}{S_\infty}$ and substitute to give $1 - r = \frac{5}{15}$ and finally $r = \frac{2}{3}$.

Example (6): The first three terms of a geometric series are $2k + 5$, k and $k - 6$ where k is a positive integer. Find the value of k and the sum to infinity of the series, giving the result in exact form.

Since we have a G.P. , $\frac{k}{2k+5} = \frac{k-6}{k}$. Cross-multiplying, $(2k + 5)(k - 6) = k^2$.

$\Rightarrow 2k^2 - 7k - 30 = k^2 \Rightarrow k^2 - 7k - 30 = 0 \Rightarrow (k - 10)(k + 3) = 0$.

Hence (positive) $k = 10$ and the first three terms of the G.P. are 25, 10 and 4.

The common ratio, $r = \frac{10}{25} = \frac{2}{5}$.

The sum to infinity is thus $S_\infty = \frac{a}{1-r} = \frac{25}{1-\frac{2}{5}} = 41\frac{2}{3}$.

Again, variant questions on G.P's sometimes come up in exams.

Example (7): A geometric series has first term 5 and common ratio 3. Find the least number of terms the series can have if its sum exceeds 2000.

We need to find a value of n such that $S_n = \frac{a(r^n - 1)}{r - 1} > 2000$.

Substitution gives $S_n = \frac{5(3^n - 1)}{2} > 2000 \Rightarrow 3^n - 1 > 800 \Rightarrow 3^n > 801$

Taking logs of both sides, $n \ln 3 = \ln 801$, $\Rightarrow$ $n > \frac{\ln 801}{\ln 3} = 6.086$.

The next integer is 7, so the G.P. needs to have 7 terms for its sum to exceed 2000.

Example (8): A geometric series has first term 240 and common ratio of $\frac{3}{4}$. Find the least number of terms the series can have for its sum to exceed 900.

We need to find a value of *n* such that $S_n = \frac{a(1-r^n)}{1-r} > 900$.

Substitution gives $S_n = \frac{240\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}} > 900 \Rightarrow 960\left(1-\left(\frac{3}{4}\right)^n\right) > 900$

$\Rightarrow 1-\left(\frac{3}{4}\right)^n > \frac{15}{16} \quad \Rightarrow \quad -\left(\frac{3}{4}\right)^n > -\frac{1}{16} \quad \Rightarrow \quad \left(\frac{3}{4}\right)^n < \frac{1}{16}$ (multiply by –1 and thus reverse sign)

Taking logs of each side, $n\ln\left(\frac{3}{4}\right) < \ln\left(\frac{1}{16}\right)$, and finally $n > \frac{\ln\left(\frac{1}{16}\right)}{\ln\left(\frac{3}{4}\right)} = 9.637...$

The next integer is 10, so the G.P. needs to have 10 terms for its sum to exceed 900.

N.B. Note the reversal of the inequality sign in the last step. A number less than 1 has a negative logarithm, so that step involves dividing by a negative quantity.

Example (9): The sum of the first two terms of a G.P. is 36, and its sum to infinity is 100. Given that the common ratio is positive, find the ratio and the first term.

We have here $S_\infty = \frac{a}{1-r} = 100$, and $S_2 = a + ar = a(1+r) = 36$.

From the expression for S_∞, we have $a = 100(1-r)$ and from the expression for S_2, $a + ar = a(1+r) = 36$.

Solving simultaneously, substituting $a = 100(1-r)$ into the equation $a(1+r) = 36$ gives $100(1-r)(1+r) = 36$, or $100(1-r^2) = 0.36$, and hence a common ratio of $\sqrt{1-0.36} = \sqrt{0.64} = 0.8$.

To find the first term *a*, we rearrange the formula for S_∞ to give

$a = S_\infty(1-r)$, giving $a = 100(1-0.8)$ or $a = 20$.

The first term of the G.P. is therefore 20 and the common ratio is 0.8.

Example (10): The film industry uses an accepted formula to estimate total box-office takings, assuming a 40% week-on-week decline after the first week.

The film *The Golden Ring Chronicles* cost £125 million to produce, and was originally estimated to take £55 million in its first week.

i) Using the "40% decline" formula, after how many weeks was the film expected to break even, with the box-office takings paying back the production costs ?

ii) The actual takings for the film were £39million in the first week and £26 million in the second. Show that the film would be unable to pay for itself if subsequent box-office takings followed the same trend.

i) We have a G.P. whose first term is 55 and whose common ratio is (100-40)%, or 0.6, or $\frac{3}{5}$.

First, we find a value of n such that $S_n = \frac{a(1-r^n)}{1-r} > 125$.

Substituting, $S_n = \frac{55\left(1-\left(\frac{3}{5}\right)^n\right)}{1-\frac{3}{5}} > 125$

$\Rightarrow 55 \times \frac{5}{2} \times (1-\left(\frac{3}{5}\right)^n) > 125 \quad \Rightarrow 1-\left(\frac{3}{5}\right)^n > \frac{10}{11}$

$\Rightarrow -\left(\frac{3}{5}\right)^n > -\frac{1}{11} \quad \Rightarrow \left(\frac{3}{5}\right)^n < \frac{1}{11}$ (multiply by –1 and thus reverse sign)

Taking logs of each side, $n\ln\left(\frac{3}{5}\right) < \ln\left(\frac{1}{11}\right)$, and finally $n > \frac{\ln\left(\frac{1}{11}\right)}{\ln\left(\frac{3}{5}\right)} = 4.694..$

(Inequality sign reversed – the last step involved dividing by a negative number).

The next integer is 5, so the G.P. needs to have 5 terms for its sum to exceed 125.

$\therefore$ The film is expected to break even at the end of the fifth week (i.e. its box-office total is expected to exceed £125 million).

ii) We now have a G.P. whose first term is 39, and whose second term is 26. The common ratio is therefore $\frac{26}{39}$ or $\frac{2}{3}$.

We must try and show that the sum to infinity of this series is less than 125.

Substituting into the general formula $S_\infty = \frac{a}{1-r}$ to obtain the final result $S_\infty = \frac{39}{1-\frac{2}{3}}$ = **117**.

$\therefore$ The film cannot take more than £117million, which is still short of the production cost of £125 million.

Example (11): Tom took out a pension plan at his workplace whereby he would save £100 per month in the first year. In subsequent years, he would increase his monthly contributions by 6.5% of his previous year's contribution.

i) State Tom's annual pension contribution for the first three years, and the common ratio of the geometric sequence formed by his pension contributions.

ii) How much money did Tom pay in the fifteenth year of his pension plan ?

iii) Show that Tom's total pension contributions over fifteen years were almost exactly £29000.

iv) Tom's colleague Steve took out an alternative pension plan where he paid in £125 per month in the first year, and each year later he paid in £x per month more than he did in the previous year. (Steve's monthly payments were only revised once per year, so he paid the same amount over 12 months.)

After 15 years, Steve and Tom had both paid the same amount into their pension funds.

How much money did Steve put into his pension fund in the 15th year of his plan, and what was the increase in the monthly payment, x, to the nearest penny ?

i) Tom paid in £100 per month, or a total of £100 × 12 = £1200 in the first year.
In the second year, he paid in £1200 + 6.5%, or £1278.
In the third year, he similarly paid in £1278 + 6.5% or 1361.07.

Adding 6.5% to a quantity is the same as multiplying it by 1.065, and so the annual contributions form a G.P. with a common ratio, r, of 1.065.

ii) The 15th term of the G.P. with first term a = 1200 and r = 1.065 is
$1200 \times 1.065^{14} = 2897.85$, therefore Tom paid in £2897.85 in total in the 15th year of his pension plan.

iii) We will use $S_n = \frac{a(r^n - 1)}{r - 1}$ to compute Tom's total pension contributions for the 15 years.

$$S_{15} = \frac{1200\left((1.065)^{15} - 1\right)}{1.065 - 1} = \frac{1200 \times (1.571841)}{0.065} = £29{,}018.60.$$

iv) Steve put in £1500 per annum in year 1 and £29018.60 in total (the same amount as Tom).

Steve's annual contributions form an A.P. whose sum to 15 terms, S_{15}, is 29018.60 and whose first term, a, is 1500.

Substitution into the general formula $S_n = ½\, n(2a + (n\text{-}1)\, d)$ gives

$S_{15} = 7.5(3000 + 14d) = 29018.60$

$\Rightarrow 3000 + 14d = 3869.147$
$\Rightarrow 14d = 869.147$
$\Rightarrow d = 62.082$

In the 15th year, Steve paid in £1500 + 14d (from above) or £(1500 + 869.15) = £2369.15.
His annual contributions increased by £62.08 each year, or £5.17 each month, to the nearest penny.

Example (12): The 4th, 7th and 16th terms of an A.P. themselves form a G.P. The first six terms of the A.P. have a sum of 12. What is the common difference of the A.P. and the common ratio of the G.P ?

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 8B, Q.18)

We can enter $n = 6$ and $S_n = S_6 = 12$ into the general formula to give

$\mathbf{3(2a + 5d) = 12}$

For the 4th, 7th and 16th terms to be in G.P, we can represent them as b, br and br^2.

Therefore
$a + 3d = b$ (1)
$a + 6d = br$ (2)
$a + 15d = br^2$ (3)

Subtracting (2) from (3) gives $9d = br(1-r)$ (4)
Subtracting (1) from (2) gives $3d = b(1-r)$. (5)
Dividing (4) by (5) gives $r = 3$.

The common ratio between the 4th, 7th and 16th terms is therefore 3.

From this fact we can establish the value of the common difference, d.

$a + 6d = 3(a + 3d)$ (6)
$a + 15d = 9(a + 3d)$ (7)

Subtracting (6) from (7) gives $9d = 6(a + 3d) \Rightarrow 9d = 6a + 18d) \Rightarrow$ $\mathbf{6a + 9d = 0}$.

The final step is to subtract the last formula from the one established at the beginning.

$\mathbf{6a + 15d = 12}$
$\mathbf{6a + 9d = 0}$

Eliminating a we have $6d = 12$ and $d = 2$.

The common difference of the A.P. is thus 2.

Arithmetic and Geometric Mean (Not all syllabuses).

The **arithmetic mean** of two numbers is equal to half their sum. For example, the arithmetic mean of 10 and 20 is ½(10 + 20) or 15. The numbers 10, 15 and 20 are in arithmetic progression with a common difference of 5, or half the difference between the two original numbers.

The **geometric mean** of two (positive) numbers is equal to the (positive) square root of their product. For example, the geometric mean of 10 and 40 is $\sqrt{(10 \times 40)}$ or 20. The numbers 10, 20 and 40 are in geometric progression with a common ratio of 2, the square root of the ratio of the second number to the first.

Example (13). Find the arithmetic mean of a) 8 and 24; b) –5 and 7; c) 11 and 39. Give the common difference of the resulting three-number progressions.

The arithmetic mean of 8 and 24 is ½(8 + 24) or 16, and the common difference is ½(24 - 8) or 8.

The arithmetic mean of -5 and 7 is ½(-5 + 7) or 1, and the common difference is ½(7 – (-5)) or 6.

The arithmetic mean of 11 and 39 is ½(11 + 39) or 25, and the common difference is ½(39 - 11) or 14.

Example (14). Find the geometric mean of a) 6 and 54; b) 15 and 375; c) 0.4 and 1.6. Give the common ratio of the resulting three-number progressions.

The geometric mean of 6 and 54 is $\sqrt{(6 \times 54)} = \sqrt{324} = 18$, and the common ratio is $\sqrt{(54 \div 6)}$ or 3.

The geometric mean of 15 and 375 is $\sqrt{(15 \times 375)} = \sqrt{5625} = 75$, and the common ratio is $\sqrt{(375 \div 15)}$ or 5.

The geometric mean of 0.4 and 1.6 is $\sqrt{(0.4 \times 1.6)} = \sqrt{0.64} = 0.8$, and the common ratio is $\sqrt{(1.6 \div 0.4)}$ or 2.

More on Sigma Notation.

This method of defining a series has already been encountered in the section on arithmetic series. The Greek letter sigma (Σ) stands for 'sum'.

The series 3 + 6 + 12 + 24 + 48 can be expressed as $\sum_{r=1}^{5} 3(2^{r-1})$

The "r = 1" below the symbol is the starting value for the count variable, r, and the "5" above it is the ending value for r. The "$3(2^{r-1})$" to the right of the symbol is the actual term itself, expressed as a function of r. When r = 1, $3(2^{r-1}) = 3$; when r = 2, $3(2^{r-1}) = 6$, and so on until the last term, 48, corresponding to r = 5.

Another equally valid definition for the same series is $\sum_{r=0}^{4} 3(2^{r})$.

The "4" above the symbol is **not** the same as number of terms in the sum here. There are still 5 terms, but this time the count variable goes from 0 to 4, not 1 to 5, and the formula for the term has been correspondingly modified.

Geometric series with a negative common ratio can be a little more awkward to define:

64 - 32 + 16 – 8 + 4 – 2 can be expressed as $\sum_{r=1}^{6} (-1)^{r+1}(2^{7-r})$ or $\sum_{r=0}^{5} (-1)^{r}(2^{6-r})$

Example (15): Rewrite the following series in sigma notation:

(a) 16 + 8 + 4 + 2 + 1
(b) -1 + 3 - 9 + 27 - 81 + 243 - 729
(c) 12 + 6 + 3 + 1.5 + 0.75 + 0.375 ....

(In each case, we will begin with r = 1).

(a) This is a G.P. with common ratio 0.5 and first term 16, with 5 terms.
The general term is $u_r = 16(0.5)^{r-1}$ or $32(0.5)^{r}$.

Its sigma notation equivalent is $\sum_{r=1}^{5} 32(0.5)^{r}$.

(b) This is a G.P. with common ratio –3 and first term -1, with 7 terms.
The general term can be defined either as $u_r = -(-3)^{r-1}$ or as $u_r = (-1)^{r} 3^{r-1}$. The second method shows the oscillating nature of the series more clearly.

Its sigma notation equivalent is $\sum_{r=1}^{7} (-1)^{r} 3^{r-1}$.

(c) This is a G.P. with common ratio 0.5 and first term 12, with infinitely many terms.
The general term is $u_r = 12(0.5)^{r-1}$ or $24(0.5)^{r}$.

Its sigma notation equivalent is $\sum_{r=1}^{\infty} 24(0.5)^{r}$.

Example (16): Write down the series (do not sum them) corresponding to the following sigma notations:

(a) $\sum_{r=1}^{5} 7(2^r)$

(b) $\sum_{r=1}^{\infty} (-1)^{r-1}(0.2^{r-1})$

(a) is a G.P. with 5 terms with common ratio 2, and whose first term is 14 (not 7 !) ,
i.e. 14 + 28 + 56 +112 + 224.

(b) is a G.P., this time with a sum to infinity. Its first term is $(-1)^0$ $(0.2)^0$ or 1,
the second one is $(-1)^1$ $(0.2)^1$ or –0.2, the third $(-1)^2$ $(0.2)^2$ or 0.04, giving a common ratio of 0.2.

It therefore goes 1 – 0.2 + 0.04 – 0.008 + 0.0016 – 0.00032 ....