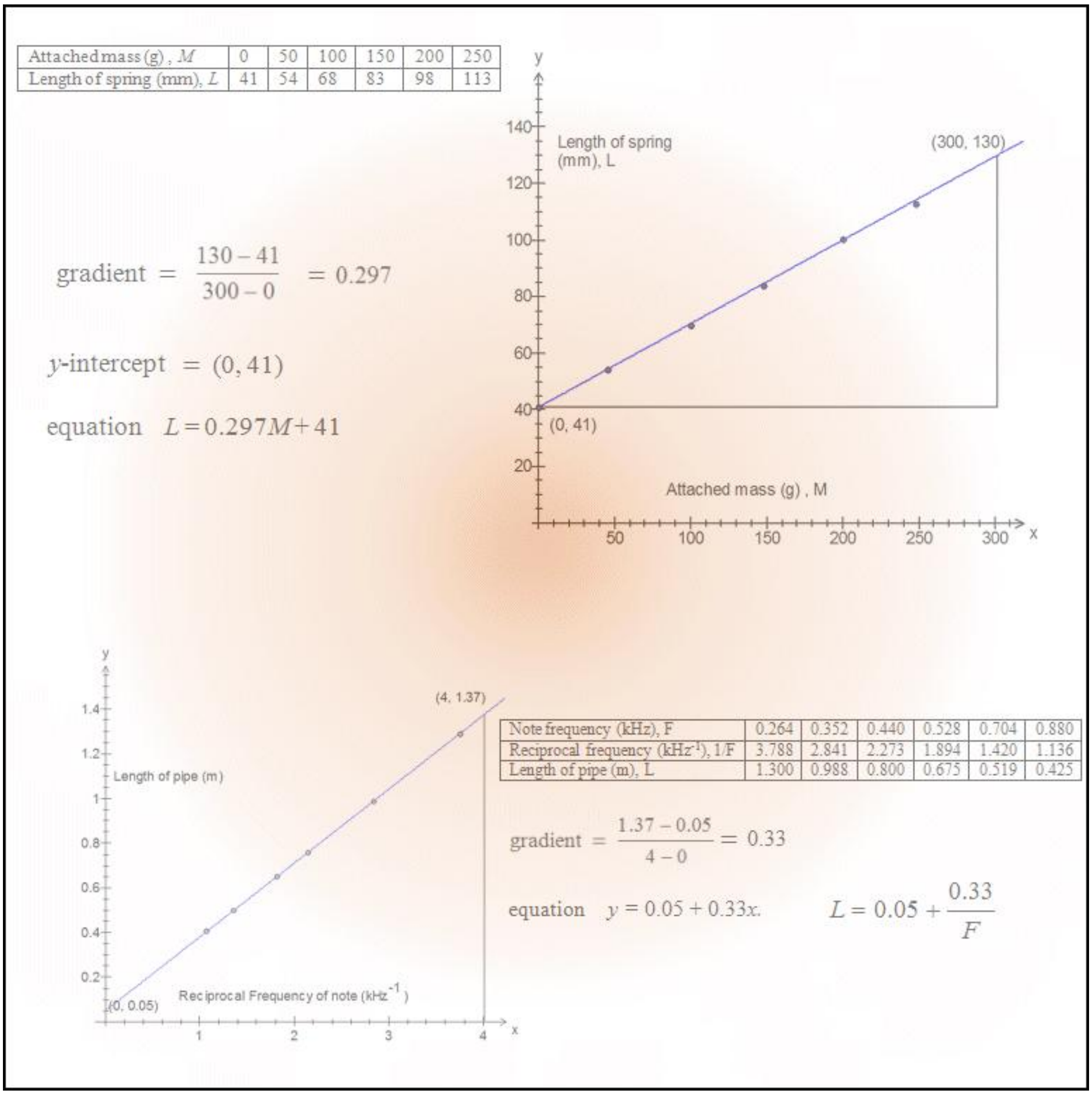
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level - MEI

OCR MEI: **C2**

GRAPH FITTING



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GRAPH FITTING.

Introduction.

A scientific experiment, or a mathematical model of a real-life situation, often consists of investigating related changes between two sets of data, such as car stopping distance against time, the extension of a spiral spring against the mass attached to it, or the period of an oscillating pendulum against its length.

Typically, a set of about five or six readings is made, and the results plotted on a graph. The resulting graph might be linear or it might be a curve.

This section follows on from the study of direct and inverse proportion at GCSE, where there is a substantial amount of crossover.

Linear graphs.

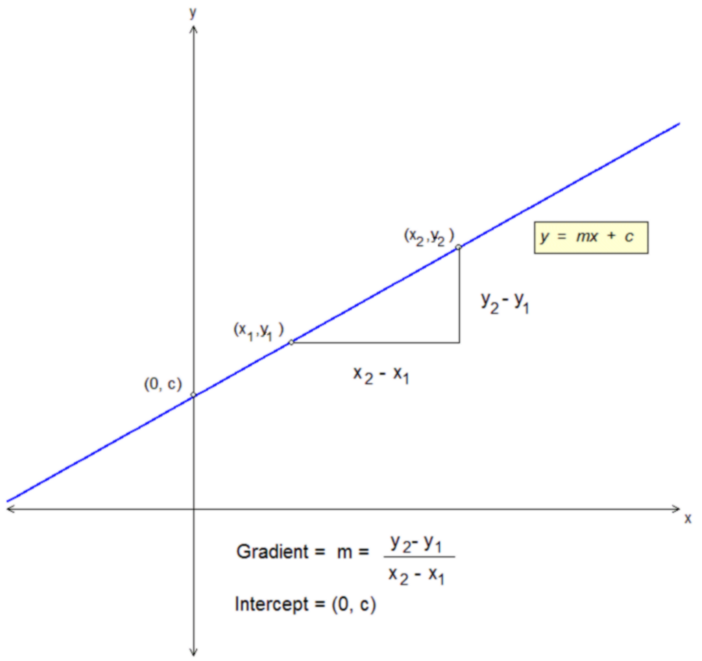
The linear graph is the easiest to analyse.

The gradient of a line connecting two points (x_1, y_1) and (x_2, y_2) is given by the formula

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ - it is the change in the value of *y* divided by the change in the value of *x*.

The *y* – intercept of the line is the point where the graph meets the *y* – axis. Its coordinates are (0, *c*).

Provided that the straight line is not parallel to the *y*-axis, its equation can also be written in the form $y = mx + c$.

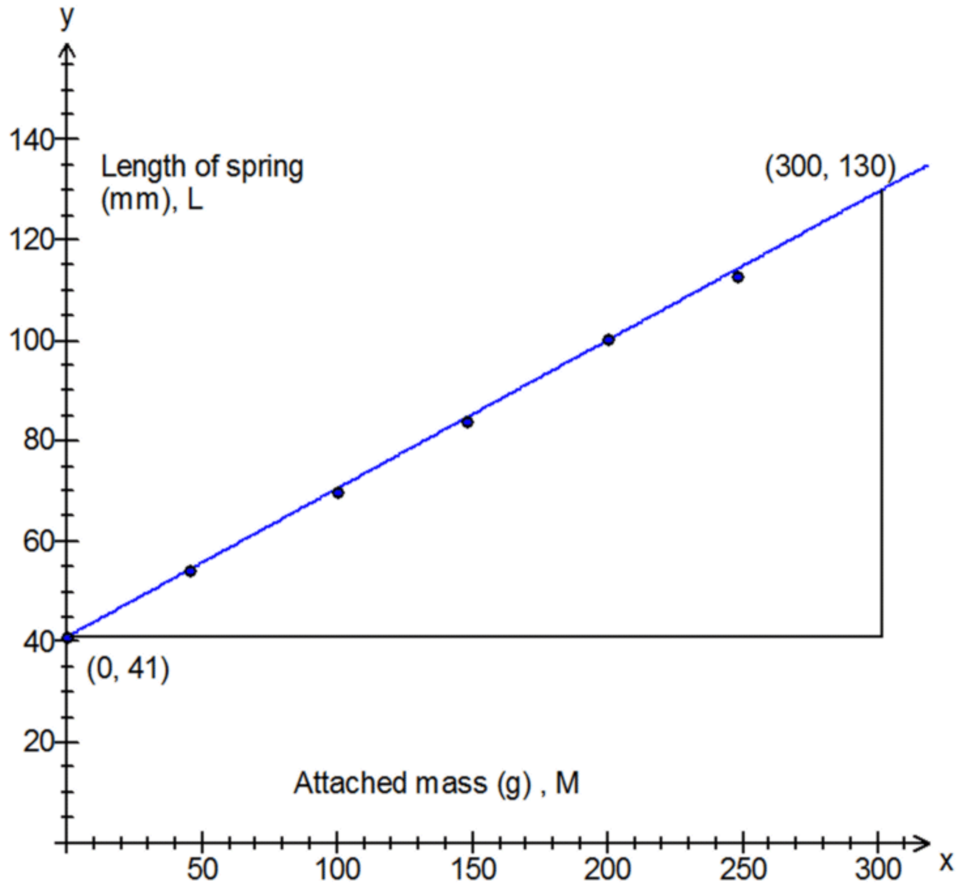


This is known as the **gradient-intercept** equation because the gradient (*m*) and *y*-intercept (*c*) are clearly evident.

Example (1): The following results were obtained in an experiment to determine the rate of extension of a spiral spring.

Attached mass (g) , *M*	0	50	100	150	200	250
Length of spring (mm), *L*	41	54	68	83	98	113

The rate of extension is suspected to be linear. Plot the result on a graph, with the mass on the *x*-axis and the length of the spring on the *y*-axis. What linear equation connects *L* and *M* ?



The gradient of the graph is $\frac{130-41}{300-0} = 0.297$ and the *y*-intercept is at (0, 41).

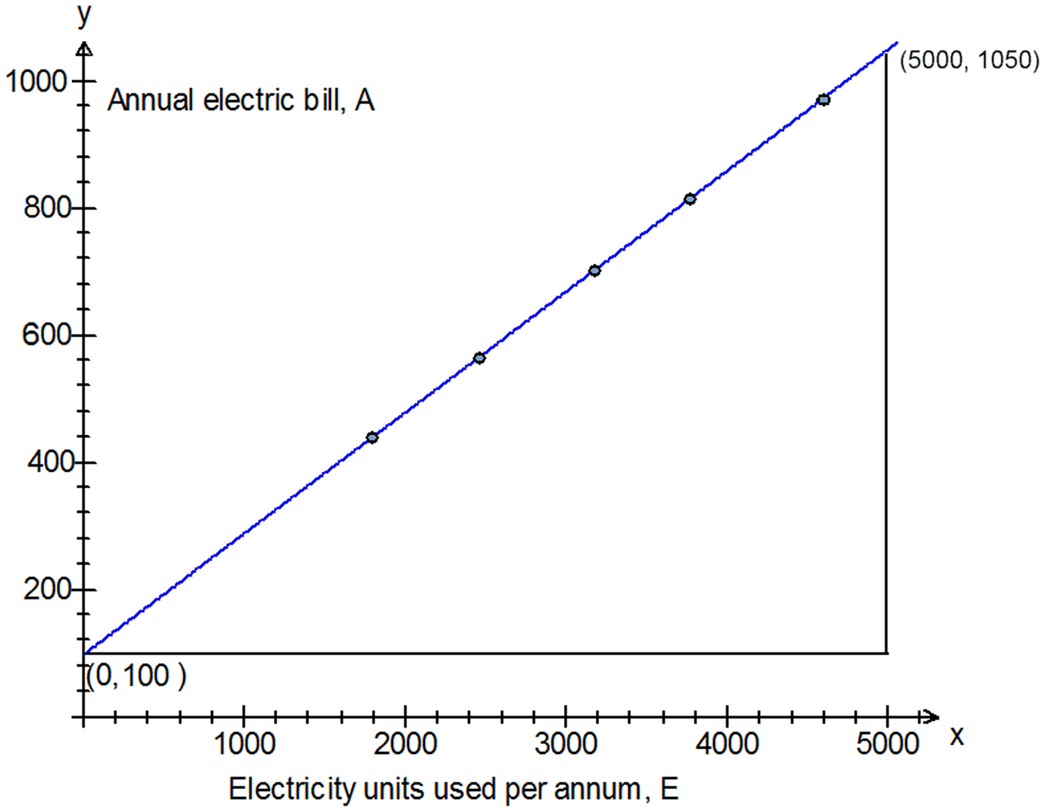
The values of the length *L* of the spring (the *y*-axis variable) and the attached mass *M* (the *x*-axis variable) are therefore connected by the equation $L = 0.297M + 41$.

Example (2):

A business has five electricity accounts with the same firm, and each account is billed according to the same plan – a standing charge plus the actual charge for units used. The units used and the annual bills for each account are as follows:

Electricity units used, *E*	2450	3165	1784	4582	3773
Annual bill (£), *A*	565	700	435	970	810

Plot the results on a straight line graph, and work out its equation. What quantities are represented by the gradient and the *y* – intercept ?



The gradient of the graph is $\frac{1050-100}{5000-0} = 0.19$ and the *y*-intercept is at (0, 100).

The values of the annual bill *A* (the *y*-axis variable) and the electricity units used *E* (the *x*-axis variable) are therefore connected by the equation $A = 0.19E + 100$.

The *y*-value at the intercept (0,100) represents the annual fixed (standing) charge in £, whilst the gradient represents the cost (in £) of a single unit of electricity.

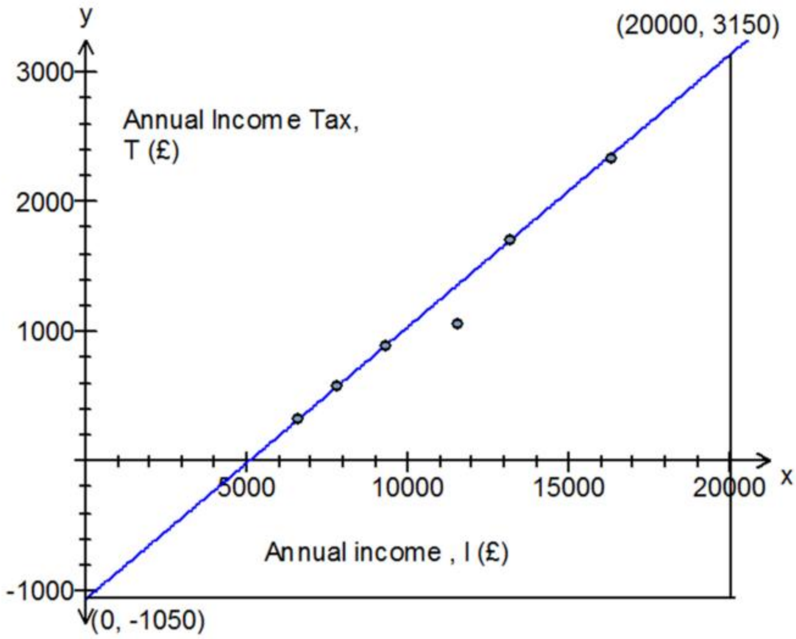
Example (3):

Six self-employed businessmen, all of whom pay the same rate of income tax, have sent self-assessment cheques to the Inland Revenue. (This is a simplified example and does not pretend to reflect the actual UK tax system !)

Annual income (£), I	6550	7900	9400	11550	13450	16150
Income tax paid (£), T	310	590	905	1040	1760	2330

Plot the results on a straight line graph, and work out its equation. One of the businessmen has not been quite as honest as the others – which one ?

What quantities are represented by the gradient and the *x* – intercept ?



One point on the graph, (11550, 1040) is some way below the line connecting the other five points – the particular businessman had not been as honest as the others when declaring his income tax !

The gradient of the graph is $\frac{3150-(-1050)}{20000-0} = 0.21$ and the *y*-intercept is at (0, -1050).
The values of the annual tax bill *T* in £ (the *y*-axis variable) and the annual income *I* in £ (the *x*-axis variable) are therefore connected by the equation $A = 0.21I - 1050$.

The gradient of the graph, here 0.21, represents the standard rate of income tax, or 21%.

The *y*-intercept values (income tax of -£1050 for a zero salary) do not make everyday common sense.

The *x*-intercept values at the point (5000, 0) are more useful. From these, we can see that the businessmen start paying income tax once income exceeds £5000 p.a. – i.e. there is a tax-free personal allowance of £5000 p.a.

Non-linear relationships.

The last three examples all produced linear graphs of the relationships between the quantities on the *x*-axis and the quantities on the *y*-axis.

However, many such relationships will not be linear, as the next cases will show.

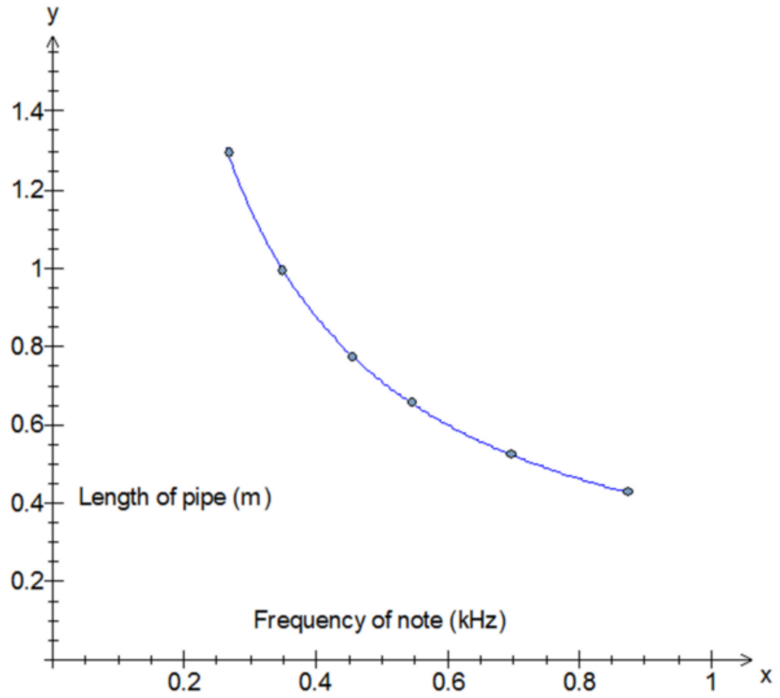
Example (4): A company manufactures organ pipes of different lengths depending on the frequency of the note produced. Some of the factory settings are as follows :

Note frequency (kHz), F	0.264	0.352	0.440	0.528	0.704	0.880
Length of pipe (m), L	1.300	0.988	0.800	0.675	0.519	0.425

Plot the graph of note frequency against the length of the pipe. What do you notice ?

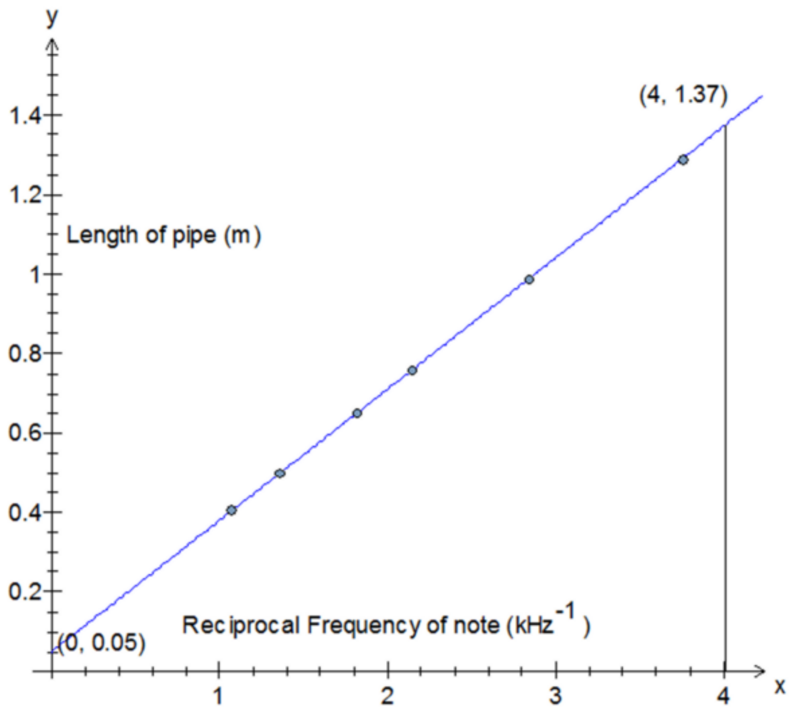
The resulting graph is definitely not linear, and in fact it has a strong resemblance to the reciprocal graph, $y = 1/x$.

The length of the pipe is inversely proportional to frequency of the note.



We therefore redefine the graph by plotting the *reciprocal* of the frequency along the *x*-axis against the length of the pipe on the *y*-axis.

Note frequency (kHz), F	0.264	0.352	0.440	0.528	0.704	0.880
Reciprocal frequency (kHz^{-1}), 1/F	3.788	2.841	2.273	1.894	1.420	1.136
Length of pipe (m), L	1.300	0.988	0.800	0.675	0.519	0.425



This time the graph is linear , and the gradient works out as $\frac{1.37-0.05}{4-0}$ or 0.33, with the *y*-intercept at the point (0, 0.05).

The equation of the graph is therefore $y = mx + c$ with $m = 0.33$, $c = 0.05$, or $y = 0.05 + 0.33x$.

It must be remembered, though, that the *x*-axis is now represented by 1/F and the *y*-axis by L, therefore the formula linking the length of the pipe and the frequency is $L = 0.05 + \frac{0.33}{F}$.

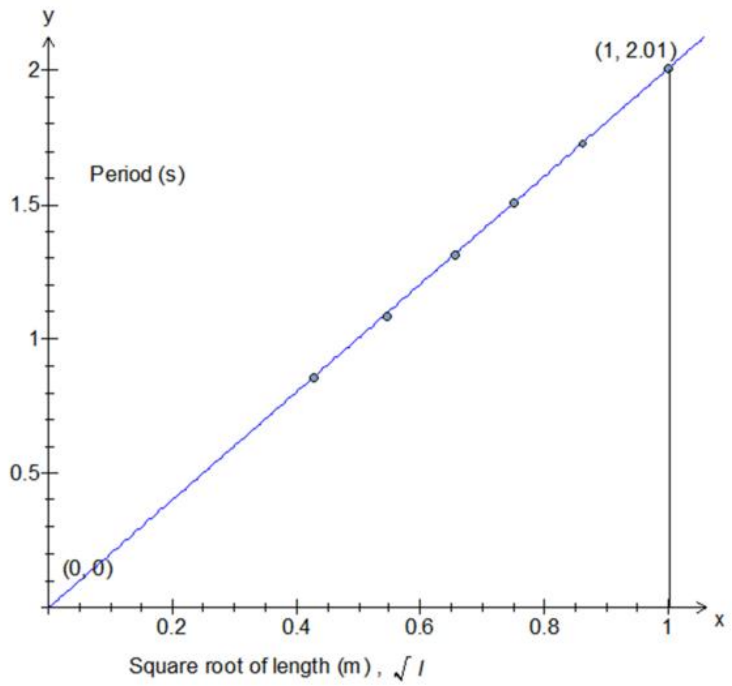
Example (5): An experiment was carried out to note the time period, T, of pendulums of various lengths *l*. The results were as follows :

Length of pendulum (m), *l*	0.20	0.30	0.45	0.60	0.75	1.00
Time period of pendulum (s), T	0.90	1.10	1.35	1.55	1.74	2.01

It is suspected that T is proportional to the square root of *l*. Plot a suitable graph to show that this is the case. What is the constant of proportionality ?

We must therefore take the *square root* of the length of the pendulum and use that for the *x*-axis, whilst plotting the period on the *y*-axis.

Length of pendulum (m), *l*	0.20	0.30	0.45	0.60	0.75	1.00
Square root of length of pendulum (m), $\sqrt{l}$	0.45	0.55	0.67	0.78	0.87	1.00
Time period of pendulum (s), T	0.90	1.10	1.35	1.55	1.74	2.01



The constant of proportionality, i.e. the gradient of the graph,
is $\frac{2.01-0}{1-0} = 2.01$ and the *y*-intercept is at (0,0).

The equation of the graph is therefore $y = mx + c$ with $m = 2.01$, $c = 0$, or $y = 2.01x$.

The time period is therefore related to the length by the formula $t = 2.01\sqrt{l}$. (remember we used the square root of the length to turn the graph into a straight line !)

Example (6): The force of wind resistance experienced by a particular car moving with velocity *V* was recorded for various values of *V* and the results tabulated :

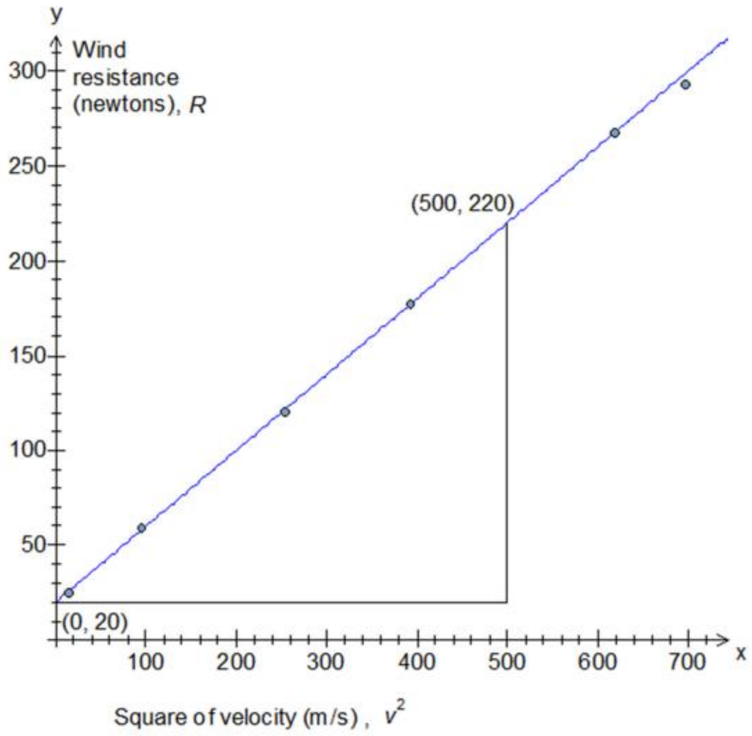
Velocity of car (m/s), *V*	4	10	16	20	25
Wind resistance (newtons), *R*	26	60	122	180	270

By plotting a suitable graph, show that the figures agree approximately with a relationship $R = a + bV^2$. Also find the values of the constants *a* and *b.*

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 3H, Q.5)

We must plot the *square* of the velocity on the *x*-axis against the resistance on the *y*-axis to produce a linear graph.

Velocity of car (m/s), *V*	4	10	16	20	25
Square of velocity, V^2	16	100	256	400	625
Wind resistance (newtons), *R*	26	60	122	180	270



We have a linear graph by plotting the square of the velocity on the *x*-axis and the resistance on the *y*-axis. The gradient is

$\frac{220 - 20}{500 - 0}$ or 0.4, and the

y-intercept is at (0, 20).

The equation of the graph is therefore $y = mx + c$ with $m = 0.4$, $c = 20$, or $y = 20 + 0.4x$.

The *x*-axis shows V^2 and the *y*-axis shows *R*, therefore the two quantities are related by

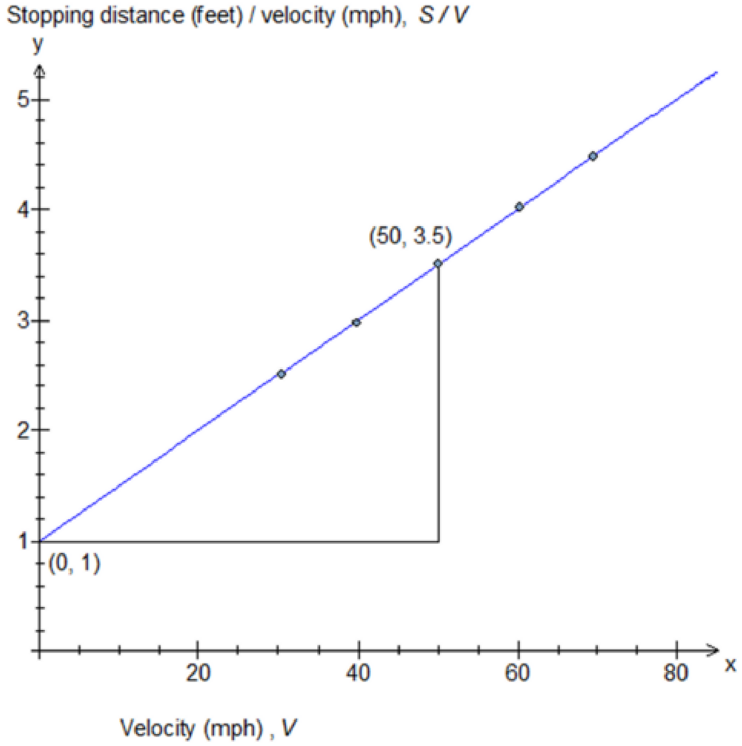
$R = 20 + 0.4\,V^2$.

Example (7): The Highway Code quotes the following values for the total stopping distance (*S*) of a car (in feet) for various velocities (*V*) (in miles per hour) :

Velocity of car (mph), *V*	30	40	50	60	70
Stopping distance (feet), *S*	75	120	175	240	315

The stopping distance and the velocity are related by the formula $S = aV + bV^2$. Plot *V* on the *x*-axis and *S/V* on the *y*-axis. Find the values of *a* and *b* given that the *y*-intercept is (0, *a*) and the gradient is *b*.

Velocity of car (mph), *V*	30	40	50	60	70
Stopping distance (feet), *S*	75	120	175	240	315
Ratio *S* / *V*	2.5	3.0	3.5	4.0	4.5



We have a linear graph by plotting the velocity on the *x*-axis and the stopping distance divided by time on the y-axis.

The gradient is $\frac{3.5-1}{50-0}$ or 0.05, and the *y*-intercept is at (0, 1).

The equation of the graph is therefore $y = mx + c$ with $m = 0.05$, $c = 1$, or $y = 1 + 0.05x$.

The *x*-axis shows *V* and the *y*-axis shows *S/V*, therefore the two quantities are related by $S/V = 1 + 0.05\,V$, or $S = V + 0.05\,V^2$.

This relationship is probably the most difficult of the above.

Logarithmic relationships.

Sometimes the relationship between the *x* and *y* values on a graph is not obvious in cases where the graph is not linear. In such cases we can make use of logarithms to find such a relationship.

The following results are derived from the laws of logarithms:

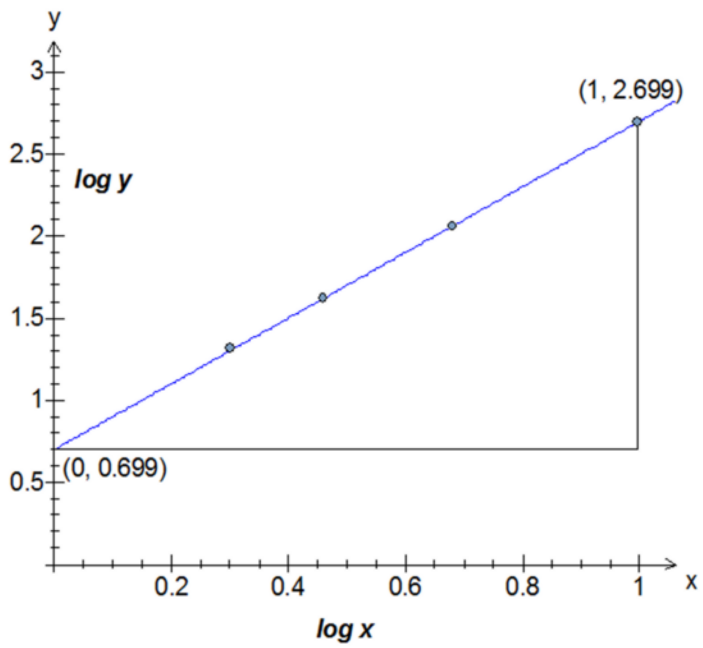
$\log Ax^b = \log A + b \log x$. ($x$ is the **base** in Ax^b).
$\log Ab^x = \log A + x \log b$. ($x$ is the **power** in Ab^x).

These two results enable us to plot straight-line graphs when investigating relationships between x and y of the form $y = Ax^b$ or $y = Ab^x$.

All the examples in this section will use logarithms to base 10.

Example (8): Compute the values of $y = 5x^2$ for x = 2, 3, 5 and 10. Take the logarithms of both x and y, and plot both sets of logs on a graph. What can you say about the gradient and the y-intercept ?

x	*2*	*3*	*5*	*10*
y	*20*	*45*	*125*	*500*
log *x*	0.301	0.477	0.699	1.000
log *y*	1.301	1.653	2.097	2.699



The first thing to notice is that the quadratic graph is no longer a parabola but a straight line when the logarithms of x and y have been plotted against each other.

Note here that the value of x refers to the **base** in the expression Ax^b, not the power.

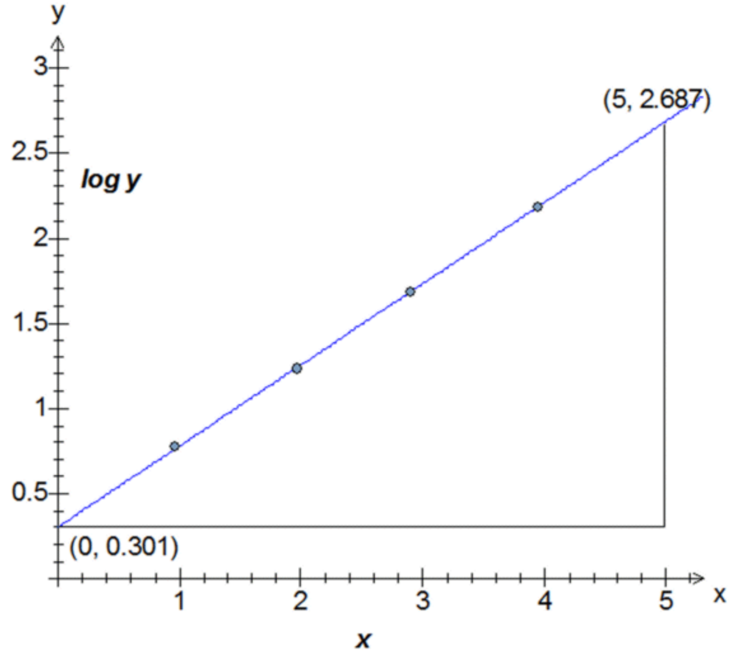
The gradient of the line works out at $\frac{2.699 - 0.699}{1} = 2$, which is the same as the power of x in the expression for y. The y-value at the intercept of (0, 0.699) is also the same as the logarithm of 5 – the multiple of x^2.

This is not a coincidence, since $\log Ax^b = \log A + b \log x$, giving an expression of the form $Y = mX + c$ where $Y = \log y$, $c = \log A$, $m = b$ and $X = \log x$.

The gradient of the transformed graph therefore corresponds to the power b and the y-intercept to the logarithm of the constant multiplier A.

Example (9): Compute the values of $y = 2 \times 3^x$ for x = 1, 2, 3 and 4. Take the logarithms of *y*, and plot the logarithm of *y* against *x* on a graph. What can you say about the gradient and the *y*-intercept ?

x	1	2	3	4
y	*6*	*18*	*54*	*162*
log *y*	0.778	1.255	1.732	2.210



Note here that the value of *x* refers to the **power** in the expression Ab^x, not the base.

The exponential graph is now a straight line when the logarithm of *y* has been plotted against *x*. The gradient of the line works out at $\frac{2.687 - 0.301}{5} = 0.477$, which is the same as the logarithm of 3 – the base of the expression where *x* is the power. The *y*-value at the intercept of (0, 0.301) is also the same as the logarithm of 2 – the multiple of 3^x in the expression.

This is because $\log Ab^x = \log A + x \log b$, giving an expression of the form $Y = mX + c$ where $Y = \log y$, $c = \log A$, $m = \log b$ and $X = x$.

The gradient of the transformed graph therefore corresponds to the log of the base *b* and the *y*-intercept to the logarithm of the constant multiplier *A*.

As can be seen, there are two different cases where logarithms are used to transform a curved graph into a straight line.

We could be looking for an unknown **power** in the relationship, where we plot the logarithm of *y* against the logarithm of *x*. On the other hand we might be looking for an unknown **base** where we plot the logarithm of *y* against *x* itself.

A useful rule of thumb is, if the variable on the *x*-axis can be related to *time*, then we plot the logarithm of *y* against *x*. If there is no obvious 'time' relation but an unknown power, we plot the logarithm of *y* against the logarithm of *x*.

Example (10): The distances from the Sun of several planets and their revolution periods (in years) are given in this table. The distance between the Earth and the Sun is taken as 1 unit, and the Earth's period of revolution is one year.

Planet	Mercury	Venus	Mars	Jupiter	Saturn
Mean distance from Sun (Earth = 1), *D*	0.387	0.723	1.524	5.203	9.539
Revolution period (years), *R*	0.241	0.615	1.881	11.86	29.46

Show graphically that the revolution periods of the planets are connected to their mean distances from the sun by a relationship of the form $R = AD^b$ where *A* and *b* are constants.

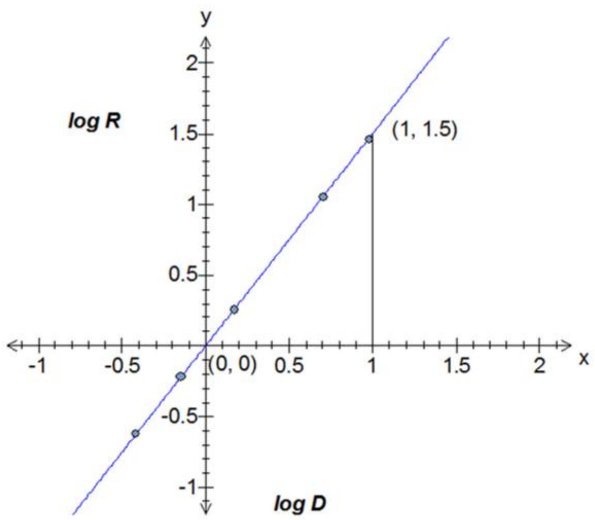
Find the values of *A* and *b*. Which planet's properties correspond to the *y*-intercept ?

The planet Uranus was discovered in 1781 and found to have an orbital period of 84 years. How many times is it further from the Sun than the Earth ?

In this example, the distance *D* is the base of the expression, and the power *b* is a constant. This is a case of "find the missing power" , so the graph will therefore be logarithmic on both axes, so we find logs of both sets of values.

Mean distance from Sun (Earth = 1), D	*0.387*	*0.723*	*1.524*	*5.203*	*9.539*
Revolution period (years), R	*0.241*	*0.615*	*1.881*	*11.86*	*29.46*
log *D*	-0.412	-0.141	0.183	0.716	0.980
log *R*	-0.618	-0.211	0.274	1.074	1.469

The graph has a gradient of $\frac{1.5-0}{1} = 1.5$, which is the required value for *b*. The *y*- intercept corresponds to the origin, and represents the distance and the orbital period of the Earth. The value of *A* in the expression is thus 1.



The distance from the Sun, *D*, is therefore related to the revolution period *R* by the formula $R = D^{3/2}$. The gradient of the transformed graph therefore corresponds to the power *b* and the *y*-intercept to the logarithm of the constant multiplier *A*.

The distance of Uranus from the Sun can thus be calculated by solving $R = D^{3/2}$ or $84 = D^{3/2}$, giving as $D = 84^{2/3}$ or 19.2 times the distance of the Earth from the Sun.

Example (11): This table is used by shipping companies to calculate daily fuel usage based on the speed of the ship. There is a suspected relationship of the form $C = aV^b$ is suspected where V is the speed of the ship and C the fuel consumption. The quantities a and b are constants to be determined.

Speed in knots, V	16	18	20	22	24	27
Tonnes of fuel used per day, C	138	197	270	359	467	664

The ship's speed V is the base of the expression, and the power b is a constant. The graph will therefore be logarithmic on both axes.

Speed of ship (knots), V	*16*	*18*	*20*	*22*	*24*	*27*
Fuel used per day (tonnes), C	*138*	*197*	*270*	*359*	*467*	*664*
log V	1.20	1.26	1.30	1.34	1.38	1.43
log C	2.14	2.29	2.43	2.56	2.67	2.82

The gradient of the graph works out at $\frac{3.04-2.14}{1.50-1.20}$ or 3,

suggesting a relationship of

$C = aV^3$ where the fuel consumption is proportional to the *cube* of the speed.

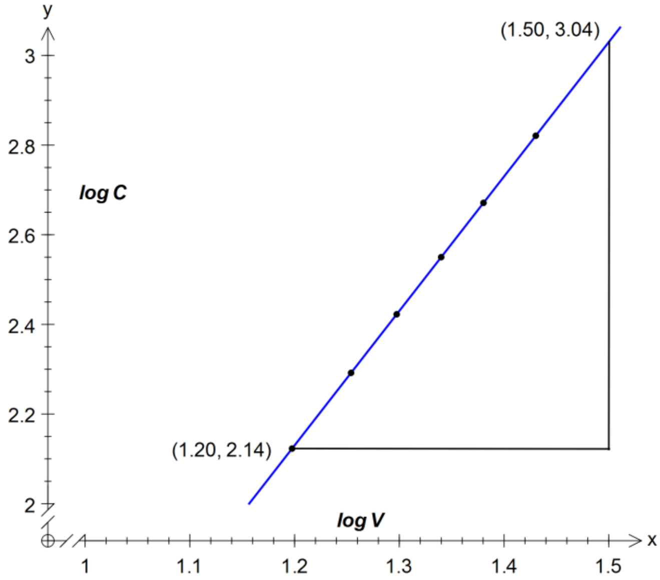
To find the constant a, we rearrange as

$a = \frac{C}{V^3}$ and substitute (say) $C = 270$

and $V = 20$.

Hence $a = \frac{270}{8000} = 0.03375$

and $C = 0.3375V^3$ or $C = \frac{27V^3}{800}$.



Example (12): An investor opens a low-risk five-year investment bond with a one-off sum of £4000, and the value of the total money at the end of each of the first four years is as follows:

Time in years, *t*	1	2	3	4
Value of investment, *V*	4290	4770	5180	5460

Plot a graph to show that the value of the investment is approximately related to the time by a relationship of the form $V = Ab^t$ where *A* and *b* are constants. Find the values of *A* and *b*. What would be the projected value of the account at the end of the fifth year if the same level of growth is maintained ?

Why is it difficult to draw an accurate conclusion ?

The time *t* is the power of the expression, and the base *b* is a constant. The graph will therefore be logarithmic on the *y*-axis only. Notice that the *x*-axis represents *time* here.

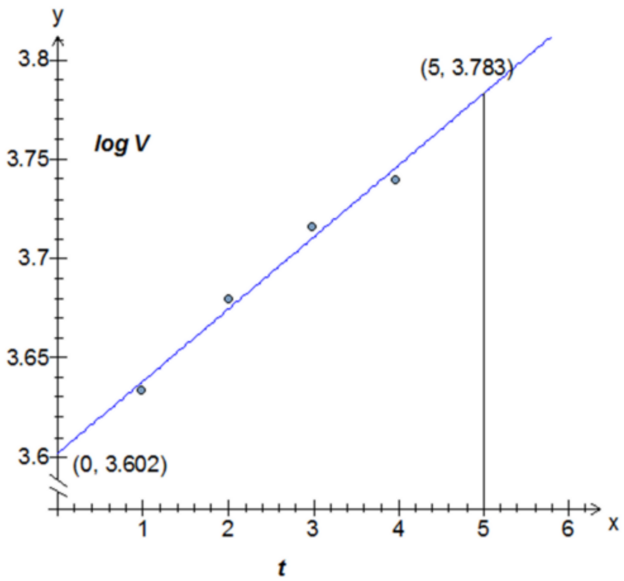
Time in years, *t*	1	2	3	4
Value of investment, V	*4290*	*4770*	*5180*	*5460*
log *V*	3.632	3.679	3.714	3.737

Since the investment was worth £4000 at *t* =0, the *y*-intercept of the graph will be (0, 3.602), the *y*-value being the log 4000.

Thus *A* = 4000.

Notice that the graph does not pass through any of the points – the account growth can only be gauged approximately.

Also, the *y*-scale must be chosen so that the resulting graph does not have an inconveniently shallow gradient.



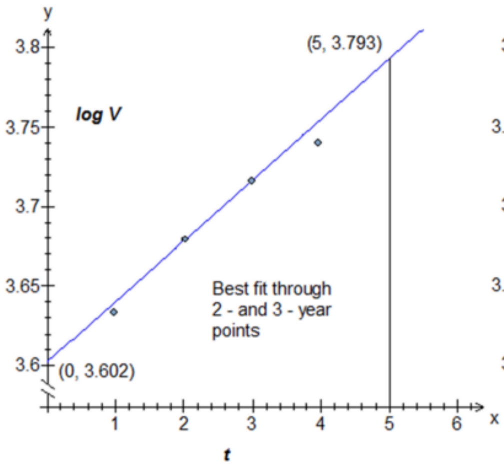
The gradient of the line works out at $\frac{3.783 - 3.602}{5} = 0.0362$, which is the same as the logarithm of 1.087. The value of *b* is 1.087 (corresponding to a growth rate of 8.7% p.a.).

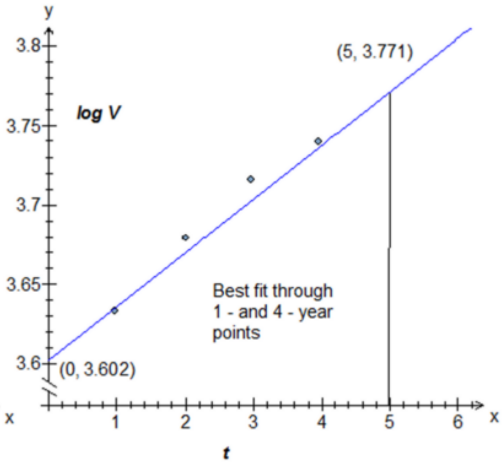
The approximate relationship between *t* and *V* is therefore $V = 4000 \times 1.087^t$.

Substituting $t = 5$ gives a projected value of **£6070** after 5 years (although past performance is no guarantee for the future !)

The following variants show why the interpretation of such projections can be misleading. The graph on the previous page did not pass through any of the points except (0, 3.602) corresponding to the start of the investment.

The following graphs would show what would happen if the attempt at a straight line had passed through a) the 2 – and 3- year points and b) the 1 – and 4- year points.





The gradient of the left-hand graph is $\frac{3.793 - 3.602}{5} = 0.0382$, the logarithm of 1.092, corresponding to an annual growth rate of 9.2% and giving a relationship between t and V of $V = 4000 \times 1.092^t$.

Using that formula, substituting $t = 5$ gives a projected value of **£6211** after 5 years.

The right-hand graph shows the other side of the coin. Its gradient is $\frac{3.771 - 3.602}{5} = 0.0338$.
That value is the logarithm of 1.081, corresponding to an annual growth rate of 8.1% and giving a relationship between t and V of $V = 4000 \times 1.081^t$.

Using that formula, substituting $t = 5$ gives a projected value of **£5904** after 5 years.

The two cases highlighted above show a considerable difference of £300 between the two projected values, depending on the points chosen for plotting. Graphical results of this type can be very sensitive to relatively small changes in gradient.

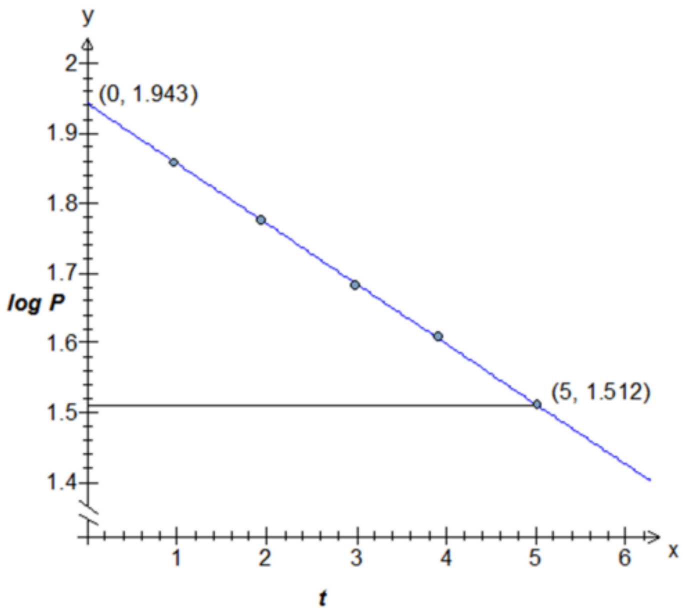
Example (13): A motor dealer uses a table for valuing a certain model of second-hand car based on its age in years and expected mileage for its age. By valuing a brand new car at 100, the following part-exchange values are quoted for a car in top condition:

Age of car in years, *t*	1	2	3	4	5
Part-exchange value as % of brand new price, *P*	72.0	59.0	48.5	39.5	32.5

Plot a graph to show that the part-exchange value of the car is connected to its age by a relationship of the form $P = Ab^t$ where *A* and *b* are constants. Suggest why the *y*-intercept is not at (0, 100).

Again *t* is the power of the expression, and the base *b* is a constant, giving a graph which is logarithmic only on the *y*-axis.

Age of car in years, *t*	1	2	3	4	5
Part-exchange value as % of brand new price, P	*72.0*	*59.0*	*48.5*	*39.5*	*32.5*
log *P*	1.857	1.771	1.686	1.597	1.512



The gradient of the line works out at $\frac{1.512 - 1.943}{5} = -0.0862$, which is the same as the logarithm of 0.82.

$\therefore$ b = 0.82 (corresponding to an annual depreciation of 18%).

The *y*-intercept is at (0, 1.943), and the *y*-value corresponds to log 88 $\therefore$ A = 88.

The approximate relationship between *t* and *P* is therefore $V = 88 \times 0.82^t$.

It looks as if the part-exchange for a brand new car is only 88% of the full value.
This is probably due to a) higher depreciation of a car in its first year, and b) trade-in profit for the dealer.