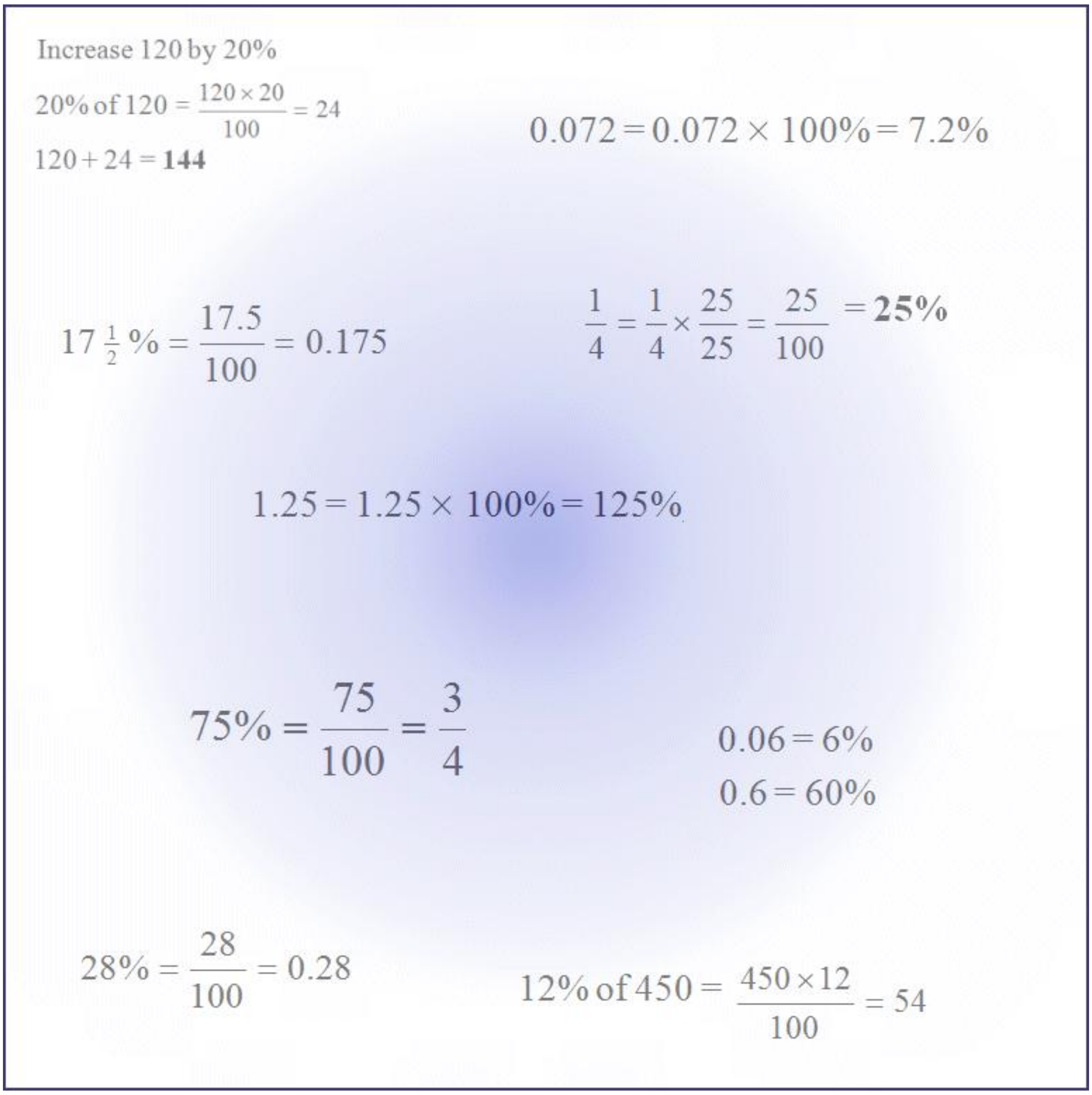
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

PERCENTAGES



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PERCENTAGES.

A percentage is a convenient way of expressing a fraction with 100 on the denominator. Such a quantity can be expressed as a decimal or a fraction, for example 15%, 5.2%, $7\frac{1}{2}\%$, $8.\dot{3}\%$.

The % symbol is short for "÷ 100" or "$\frac{}{100}$".

('Fractional' percentages like $37\frac{1}{2}\%$ are rare at Foundation Level)

Converting between decimals and percentages.
This is very easy and does not need a calculator.

To express a decimal as a percentage, multiply by 100%:

Examples (1).
1.25 = 1.25 × 100% = 125%; 0.072 = 0.072 × 100% = 7.2%

To express a percentage as a decimal, divide by 100. (You may need to convert fractions first !)

Distinguish between 0.6 = 60%; 0.06 = 6%; 0.006 = 0.6% and so forth.

Examples (2).

$28\% = \frac{28}{100} = 0.28$; $17\frac{1}{2}\% = \frac{17.5}{100} = 0.175$; $33\frac{1}{3}\% = \frac{33.\dot{3}}{100} = 0.\dot{3}$

Converting a fraction to a percentage.

To convert a fraction into a percentage, multiply both the numerator and denominator to give 100.

With calculator:
Multiply the top line by 100, and then divide by the bottom line to give the equivalent %

Without calculator:
Multiply both the numerator and denominator by a suitable number to give 100.

Examples (3).

$\frac{1}{4} = \frac{1}{4} \times \frac{25}{25} = \frac{25}{100}$ = **25%** (multiply both sides by 25)

$1\frac{3}{5} = \frac{8}{5} = \frac{160}{100}$ = **160%** (convert to an improper fraction, then multiply both sides by 20)

Sometimes, it might be necessary to make the denominator a multiple of 100 and leave the percentage in fractional form.

$\frac{3}{8} = \frac{75}{200} = \frac{37.5}{100} =$ **37.5%** or $37\frac{1}{2}\%$ (multiply by 25 to get a multiple of 100, here 200, then halve top and bottom.

$\frac{1}{3} = \frac{100}{300} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\%$ (multiply top and bottom by 100, then divide by 3).

(The answer is **not** 33%, because multiplying 33% by 3 would give 99%, not 100% !)

Converting a percentage to a fraction (without a calculator).

To convert a percentage to a fraction, divide by 100 and reduce the result to its lowest terms.

Examples (4).

$15\% = \frac{15}{100} = \frac{3}{20}$ (cancel common factor of 5)

$70\% = \frac{70}{100} = \frac{7}{10}$ (cancel common factor of 10)

$75\% = \frac{75}{100} = \frac{3}{4}$ (cancel common factor of 25)

Sometimes a percentage may be fractional, in which case the top and bottom must be multiplied out to get rid of the fraction.

$12\tfrac{1}{2}\% = \frac{25}{200} = \frac{1}{8}$ (multiply top and bottom by 2 before cancelling out common factor of 25)

$66\tfrac{2}{3}\% = \frac{200}{300} = \frac{2}{3}$ (multiply top and bottom by 3 before cancelling 100)

Expressing one quantity as a percentage of another.

To find out what percentage quantity *A* is of quantity *B*, divide *A* by *B* and multiply the result by 100.

Thus, to find out what percentage 48 is of 64, work out the value of $\frac{48}{64} \times 100\%$. Here it is 75%.

Example (5): A football ground has a capacity of 36,000 spectators. A certain match attracted an attendance of 32,400. How full was the ground, expressed as a percentage ?

We need to divide 32,400 (the attendance) by 36,000 (the capacity) and multiply the result by 100.

The ground is therefore $\frac{32{,}400}{36{,}000} \times 100\%$ full, or 90% full.

Finding a percentage of a quantity.

To find the percentage of a quantity, multiply the quantity by the percentage and divide by 100.

Thus 12% of 450 = $\frac{450 \times 12}{100} = 54$.

Example (6). In a local election, 7800 people were eligible to vote, but only 45% actually turned out. How many people voted ?

We need to multiply 7800 (the number eligible to vote) by 45 (the % turnout) and then divide by 100.

Thus the actual turnout = $\frac{7800 \times 45}{100} = 3510$ voters.

Percentage arithmetic.

Examples (7): i) Increase 120 by 20% ; also ii) decrease 180 by 25%.

(The method shown here is not the quickest, but is recommended for Foundation Tier.)

In part i), we first find 20% of 120, or $\frac{120 \times 20}{100} = 24$. We are dealing with an increase, so we add it on to the original quantity to obtain 120 + 24 = **144.**

In part ii), we first find 25% of 180, or $\frac{180 \times 25}{100} = 45$.
This time, we have a percentage decrease, so we subtract it from to the original quantity to obtain 180 - 45 = **135.**

Example (8): A college had 650 pupils on its books at the start of the 2005 academic year. The authorities wanted to increase the number on the roll by 8% for 2006, and the actual number enrolled was 704. Did the college achieve its target ?

The original number was 650, so we need to add 8% to it to find the target value.

We find 8% of 650, which is 52. Adding it to the original 650 gives us a target of 702.
The college has therefore exceeded the target by 2 pupils.

Example (9): A car loses 22% of its resale value every 12 months. Exactly a year ago it was worth £5250; how much is it worth now ?

The value of the car was £5250 twelve months ago, but we need to subtract 22% find its value now.

We work out £5250 × 22% = £1155, and then subtract it from £5250 to give £4095.

$\therefore$ The car is now worth £4095.

Percentage arithmetic - Alternative method.

If a quantity *A* is increased by a percentage *P*%, then it is possible to find *P*% of *A* and add it to the original, but a more efficient method is to multiply *A* by $1+\frac{P}{100}$ to obtain the same result.

For example, increasing by 8%, 15% and 20% is the same as multiplying by 1.08, 1.15 and 1.2 respectively.

If on the other hand a quantity *A* is decreased by a percentage *P*%, then we subtract *P*% of *A* from the original, or perform the calculation in one go and multiply *A* by $1-\frac{P}{100}$.

For example, decreasing by 5%, 10% and 25% is the same as multiplying by 0.95, 0.9 and 0.75 respectively.

Examples (7) revisited : i) Increase 120 by 20% ; also ii) decrease 180 by 25%.

In part i), we multiply by $1+\frac{20}{100}$ or 1.2, to obtain 120 × 1.08 = **144.**

In part ii), we multiply by $1-\frac{25}{100}$ or 0.75, to obtain 180 × 0.75 = **135.**

Example (8) revisited : A college had 650 pupils on its books at the start of the 2005 academic year. The authorities wanted to increase the number on the roll by 8% for 2006, and the actual number enrolled was 704. Did the college achieve its target ?

The original number was 650, so we need to add 8% to it to find the target value.

Adding 8% is the same as multiplying the original by $1+\frac{8}{100}$, or 1.08, so the new number of college pupils is 650 × 1.08 = 702.

The college has therefore exceeded the target by 2 pupils.

Example (9) revisited : A car loses 22% of its resale value every 12 months. Exactly a year ago it was worth £5250; how much is it worth now ?

Subtracting 22% is the same as multiplying the original by $1-\frac{22}{100}$, or 0.78, giving the new value of the car as 5250 × 0.78 = 4095.

Thus the car is now worth £4095.

Finding the percentage difference between two quantities.

We will assume in this section that that *A* is the original ‘old’ quantity and *B* the changed ‘new’ quantity in question.

To find the percentage change between the two, we subtract the ‘old’ value *A* from the ‘new’ value *B*, divide by the **‘old’ value *A*** and finally multiply by 100.

The formula is $P = \frac{B - A}{A} \times 100\%$.

If $B > A$, we have a positive result, indicating a percentage increase; if $B < A$, we have a negative result, indicating a percentage decrease.

Note that a percentage change of -15% is the same as a decrease of 15%.

Example (10): The population of a village had changed from 720 to 810 over a ten-year period. Express this change as a percentage.

We subtract 720 from 810 to get 90, divide 90 by 720 and multiply by 100:

$$\frac{810 - 720}{720} \times 100\% = \frac{90}{720} \times 100\% = 12.5\% .$$

The ‘new’ value is greater than the ‘old’, so the village’s population had *increased* by 12.5% over the ten years.

Example (11): The number of road accidents in a town had fallen from 560 to 476 between 2009 and 2010. Calculate the percentage change here.

We subtract 560 from 476 to get -84, divide -84 by 560 and multiply by 100:

$$\frac{476 - 560}{560} \times 100\% = \frac{-84}{560} \times 100\% = -15\%$$

Because the ‘new’ value for 2010 is less than the ‘old’ value for 2009, we are dealing with a *decrease* of 15% in the number of road accidents between the two years.

Reversed percentages – finding the original value.

These require a little more care to work out. In those cases, we are given a value *after* a percentage change, but we need to find the original.

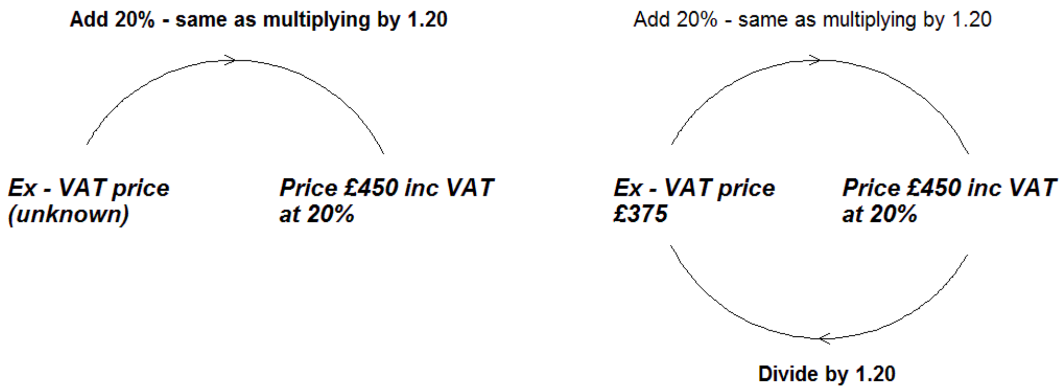
Example (12): Consider this case: a businessman buys a PC for £450 inclusive of VAT at 20%. He can claim the VAT back, so how much will it be, and also what is the ex-VAT price ?

On the face of it, the VAT will be 20% of £450, or £90.

Wrong ! If we take £90 from £450, we have £360.
Adding back the VAT by adding 20% of £360 or £72 would give us £432 !

The correct way of working this problem out is to realise that the full price of the PC, £450, is an unknown original value with 20% added to it, or that original value multiplied by 1.2.

The original cost, excluding VAT, is therefore worked out by reversing the multiplication by 1.2, namely through *dividing* by 1.2.



Note: although multiplying by 1.2 is the same as adding 20%, dividing by 1.2 is *not* the same as subtracting 20% !

The VAT itself is then obtained by finding 20% of the *original* value before the change.

The pre-VAT price is therefore £$\frac{450}{1.2}$, or $£450 \times \frac{100}{120}$, or £ 375.

The VAT itself is 20% of £375 (*not* 20% of £450) , or £75.

Therefore, to find a value *before* a percentage increase *P*%, you must

divide the price *after* the increase by $1 + \frac{P}{100}$ or multiply the price *after* the increase by $\frac{100}{100+P}$.

Example (13): The average price of a British house is now £163,800 according to marketing surveys. They say that prices had increased by 38% in the last three years. What was the price of an average house three years earlier ? Please give the answer to the nearest £ 100.

Do not be tempted to subtract 38% from £163,800 !
Here, the percentage increase, P, is 38%, and thus the current price is (100 + 38)% or 138% of the price we are being asked to find. We must therefore divide the current price by $1+\frac{P}{100}$, or 1.38.

This gives £$\frac{163{,}800}{1.38}$, or £ 118,695.65, or **£118,700** to the nearest £100.

Reversed percentage decreases are handled similarly, but with the plus signs in the last expressions replaced by minus signs.

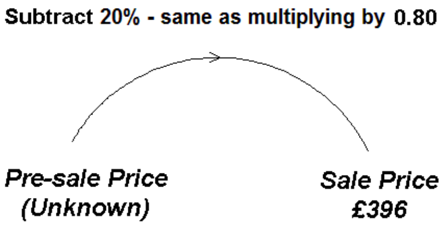
Therefore, to find a value before a percentage decrease P%, you must divide the price after the decrease by $1-\frac{P}{100}$ or multiply the price after the decrease by $\frac{100}{100-P}$.

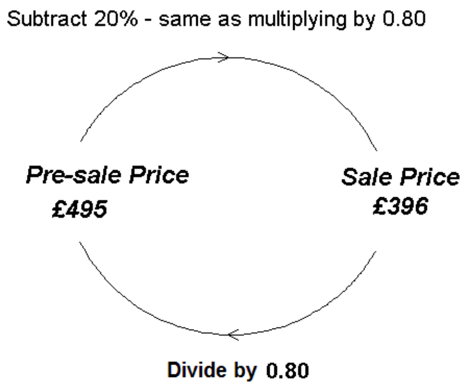
Example (14): The Sofa Shop has announced a sale, where all the stock has been reduced by 20% for one day. A shopper buys a sofa for £396 that day. What price was the sofa before the reduction ?

Here we are trying to find a price before the reduction. Again, do not be tempted to add 20% to get the original price.

This time the percentage decrease, P, is 20%, and thus the current price is (100 - 20)% or 80% of the price we are being asked to find. We must therefore divide the current price by $1-\frac{P}{100}$, or 0.8.

This gives £$\frac{396}{0.8}$, or **£495.**





Non-Calculator Percentage Arithmetic using Fractional Parts.

This is a method of multiplication where a complicated multiplier is divided up into easier parts, which are then added (or sometimes subtracted) . The best fractional parts to work with are halves, quarters, fifths and tenths, as they make for easy division.

Example (15): Find 15% of £246.

We split the 15% into 10% + 5%. .

100% of £246 is obviously £246. Now 10% is one tenth of 100%, so 10% of £246 is one tenth of £246, or £24.60. Finally, 5% is half of 10%, so 5% of £246 is half of £24.60, or £12.30.

Adding those last two sub-totals gives us £24.60 + £12.30 = £36.90.

100% of £246 =	£246.00	
10% of £246 =	£24.60	(Divide £246 by 10, as 10% = one tenth)
5% of £246 =	£12.30	(5 is half of 10, so divide 10% by 2)
15% of £246 =	**£36.90**	(Add last two sub-totals)

Example (16): The rate of VAT in Belgium is 21%.
Calculate the VAT on a television set whose ex-VAT price is €229 in Belgium.

We can split the 21% either as 20% + 1% or 10% + 10% + 1%.

20% of €229 =	€45.80	(Divide €229 by 5, as 20% = one fifth)
1% of €229 =	€2.29	(Divide €229 by 100)
21% of €229 =	**€48.09**	(Add the two sub-totals)

(Alternative)

10% of €229 =	€22.90	(Divide €229 by 10, as 10% = one tenth)
10% of €229 =	€22.90	(Repeat last result)
1% of €229 =	€2.29	(1% is one tenth of 10%, so divide €22.90 by 10)
21% of €229 =	**€48.09**	(Add the three sub-totals)

Example (17): House prices are estimated to rise by 9% over the next year.
Estimate the value of a house next year when the current value is £280,000.

We are being asked to increase £280,000 by 9%, but it is rather tedious to split this 9% into 5% + 1% + 1% + 1% + 1% . It is much easier to treat it as 10% - 1%.

The house value after one year is, of course, 109% of the original. (109 = 100 + 10 – 1)

100% of £280,000 =	£280,000	(The original price)
10% of £280,000 =	£28,000	(Divide original price by 10, as one tenth = 10%)
1% of £280,000 =	£2,800	(Find one tenth of £28,000) SUBTRACT !
109% of £280,000 =	**£305,200**	(Add the first two rows, but subtract the third)

Example (18): The price of a suit is reduced by 35% in a sale. If the original price was £180, find the sale price.

One method is to split 35% up into 20% + 10% + 5%, add those subtotals, and finally subtract from the original

100% of £180 =	£180	(The original price)
20% of £180 =	£36	(Divide original price by 5, as one fifth = 20%)
10% of £180 =	£18	(Find half of 20%, or for that matter, one tenth of original)
5% of £180 =	£9	(Find half of 10%)
35% of £180 =	**£63**	(Add the previous three rows)

This value of £63 is the reduction, so we must finally subtract it from the original £180 to give the sale price of £(180-63) or **£117**.

Another method is to reckon that a 35% reduction in price means a sale price of (100-35)% or 65% of the original. We then split the 65% up into 50% + 10% + 5% and add those subtotals.
:

100% of £180 =	£180	(The original price)
50% of £180 =	£90	(Divide original price by 2, as one half = 50%)
10% of £180 =	£18	(Find one fifth of 50%, or one tenth of original)
5% of £180 =	£9	(Find half of 10%)
65% of £180 =	**£117**	(Add the previous three rows)

Compound percentage arithmetic.

All the examples shown so far involved a single percentage change. Compound changes are handled in the same way.

Example (19): A bank offers an account paying 4% compound interest per annum for a fixed rate for three years. If £3000 is paid in to this account at the start of this term, how much interest will it earn at the end of the third year?

It is wrong to reckon $3000 \times 4\% = 3000 \times 0.04 = £120$ interest per year,
and hence $£120 \times 3 = £360$ interest over the 3 years. That is *simple* interest !

Instead we work as follows:
Year 1: account earns 4% interest on £3000, or £120 interest, for a total account of £3120.
Year 2: account earns 4% interest on **£3120**, or £124.80, for a total account of £3244.80.
Year 3: account earns 4% interest on **£3244.80**, or £129.79, for a total account of £3374.59.

The final amount in the account is £3374.59, so we subtract the original £3000 to find the interest earned, namely **£374.59**.

The difference between compound and simple interest.

Example (19) illustrated the idea of compound interest, where both the original sum *and the interest* earned interest.

In simple interest, only the original sum earns interest. Hence the following:

Example (19a): A bank offers an account paying 4% simple interest per annum for a fixed rate for three years. If £3000 is paid in to this account at the start of this term, how much interest will it earn at the end of the third year?

We are dealing with *simple* interest here, so we *can* reckon $£3000 \times 4\% = 3000 \times 0.04 = £120$ interest per year, and hence $£120 \times 3 =$ **£360** interest over the 3 years.

The simple interest is therefore

(Original sum) × (Annual percentage rate) × (Time).

The units of time and the percentage rate must be consistent; is the interest rate is quoted as an annual rate, then the time units must be years.

Example (20): A car which retails at £16000 when brand new loses 25% of its value every year from new. What is its value at the end of the third year?

Again this is a 'standard' percentage question, as we are given a starting value before any decreases. Do not be tempted to subtract $3 \times 25\% = 75\%$ of the total !

Year 1: loss in value is 25% of £16000, or £4000, leaving £12000.
Year 2: loss in value is 25% of £12000, or £3000, leaving £9000.
Year 3: loss in value is 25% of £9000, or £2250, leaving £6750.

The car is worth **£6750** after 3 years.

Compound percentage arithmetic – Alternative method.

This brings us back to the idea of using multipliers (see Examples 7-9) to perform repeated percentage calculations more rapidly than previously shown.

Example (19) revisited : A bank offers an account paying 4.5% compound interest per annum for a fixed rate for three years. If £3000 is paid in to this account at the start of this term, how much total interest will it earn at the end of the third year?

We are given a value before the increase, so this is a 'standard' percentage question, not a reversed one. Here the percentage change, P = 4.5, so the multiplier, $1+\frac{P}{100}$, is 1.045.

The original sum, £3000, needs to be multiplied by 1.045 three times to obtain the final balance of the account.

The final balance is $£3000 \times (1.045)^3$ or £3423.50, so the interest earned is £3423.50 minus the original sum invested, or **£423.50 .**

The general form of the compound interest equation is

$$N_t = N_0\left(1+\frac{P}{100}\right)^t$$

where N_0 is the amount at the start of the time span, P is the percentage rate, t is the time, and N_t is the amount at the end of the time span.

Again, the units of time and the percentage rate must be consistent; is the interest rate is quoted as an annual rate, then the time units must be years.

In this example, N_0 = 3000, P = 4.5, and t = 3, resulting in N_t = 3423.50.
(To find the interest alone, we subtract N_0 from N_t to give 423.50.)

Example (20) revisited : A car which retails at £16000 when brand new loses 25% of its value every year from new. What is its value at the end of the third year ?

Since we are dealing with percentage decreases here, we modify the compound interest / growth formula to .

$$N_t = N_0\left(1 - \frac{P}{100}\right)^t$$ (also known as compound decay)

where N_0 is the amount at the start of the time span, P is the percentage rate, t is the time, and N_t is the amount at the end of the time span.

The units of time and the percentage rate must be consistent; is the interest rate is quoted as an annual rate, then the time units must be years.

In this example, N_0 = 16000, P = 25, and t = 3. The multiplier is $1 - \frac{25}{100} = 0.75$.

The value of the car after 3 years is $£16000 \times (0.75)^3$ or **£6750**.

Profit and discount (loss) – worked examples.

If an item is said to be sold at a profit, its selling price is more than its cost price; on the other hand, if an item is sold for less than its cost price, it is said to be sold for a loss or a discount.

Example (21): A fruiterer buys a crate of apples for £18 and proceeds to sell them for a 30% profit. If there are 20kg of apples in the case, what should be his selling price per kilogram ?

30 % of £18 is £5.40, so the whole crate of apples should sell for £(18 + 5.40) or £23.40.
As there are 20kg in the crate, this works out at £ $\frac{23.40}{20}$ or £1.17 per kg.

Example (22): A market trader buys 1000 chickens at £2 each and sells 800 of them at a 40% profit, but then is forced to sell the remainder at a 30% loss. What is the percentage profit (or loss) for the whole transaction ?

Firstly, we find 40% of £2, which is 80p. The trader therefore makes 800 × 80p, or £640 profit, on the first 800 chickens sold.

The remaining 200 chickens are sold at a loss of 30% of £2, or 60p each – a total loss of 200 × 60p, or £120, on the remainder.

Combining a profit of £640 with a loss of £120 gives a net profit of £ (640 – 120) or £520.

Since the 1000 chickens originally cost him £2 × 1000, or £2000, the actual percentage profit is

$$\frac{520 \times 100}{2000} = 26\% .$$