M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

RATIO AND PROPORTION

Divide £400,000 in the ratio 3 : 5 .

5 + 3 = 8 shares in total.

One share = $\frac{1}{8}$ of £400,000 = £50,000

3 shares = $\frac{3}{8}$ of £400,000 = £150,000

5 shares = $\frac{5}{8}$ of £400,000 = £250,000

80 : 120 = 8 : 12 = **2 : 3**

Map scale : 10 cm : 6 km
= 10 cm : 6000 m
= 1 cm : 600 m
= 1 cm : 60,000 cm
= **1: 60,000**

Model: Wingspan 468 mm, length 380 mm
True: Wingspan **11.23m**, length **9.12 m**
(Multiply by 24)

If 400g of beans cost 36p, then 100g cost $\frac{1}{4}$ of 36p, or 9p.

If 600g of beans cost 59p, then 100g cost $\frac{1}{6}$ of 59p, or 9.8p (to nearest 0.1p).

If 800g of beans cost 75p, then 100g cost $\frac{1}{8}$ of 75p, or 9.4p (to nearest 0.1p).

BURY HOT-POT Serves 4

600 g boneless lamb, diced
1 kg potatoes, peeled and cut into chunks
2 large carrots, sliced
1 large onion, chopped
4 slices of Bury black pudding

BURY HOT-POT Serves 10

1.5 kg boneless lamb, diced
2.5 kg potatoes, peeled and cut into chunks
5 large carrots, sliced
2-3 large onions, chopped
10 slices of Bury black pudding

Ratio of 4 : 10 = 2 : 5 Quantities multiplied by 2½

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RATIO AND PROPORTION.

A ratio is a way of comparing two different quantities by size.

Thus, if a college had an intake of 80 Arts students and 120 Science students in a certain year, then we can say that the ratio of Arts to Science students is 80 : 120.

Similarly, if the label on a drinks bottle said "Dilute to taste – 1 part concentrate to 5 parts water", we say that the ratio of concentrate to water is 1 : 5.

Ratios can be simplified in various ways. One way is to cancel out common factors on each side in the same manner as ordinary fractions are cancelled.

Example (1): A college had an intake of 80 Arts students and 120 Science students in a certain year. Express the ratio of Arts to Science students in its simplest form.

Since 80 and 120 both have a common factor of 40, we can cancel 40 out on each side.
80 : 120 :: 8 : 12, and **8 : 12 :: 2 : 3.**

Notice how the signs of proportion are used.
The statement **8 : 12 :: 2 : 3** is equivalent to saying "8 is to 12, as 2 is to 3."

Also, the product of the outer terms, 8 and 3, is equal to the product of the inner terms. 12 and 2.

Alternatively, we could have used fractions to express the same: $\frac{80}{120} = \frac{8}{12} = \frac{2}{3}$.
(This document favours the fraction form.)

A ratio can be taken as being a scale factor, obtained by dividing the second number by the first.
Thus a ratio stated as 5 : 2 is equivalent to a scale factor of $\frac{5}{2}$ or 2.5.

Example (2):

Increase 300 in the ratio 5 : 2.
We multiply 300 by a scale factor of $\frac{5}{2}$ or 2.5, to obtain a result of 750.
We can also say, as a result, that 5 : 2 and 750 : 300 are the same ratio.

The next example has a tricky point.

Example (3): The catering requirements at a children's charity event include 500ml of diluted fruit squash for 480 children. The recommended dilution for fruit squash concentrate to water is 1 : 5.
How many litres of concentrate will be required to cater for the party ?

The required amount of squash after dilution is 0.5 × 480 = 240 litres, and given the squash : water ratio of 1 : 5, we might think incorrectly that $\frac{240}{5}$ litres or 48 litres of squash would be needed.

The dilution is 1 part of squash to 5 parts of *water*, which gives us 6 parts of *diluted* squash in total.

The ratio of squash concentrate to the *diluted squash* is therefore 1: 6, or a scale factor of 6.
We are given the amount of squash *after* dilution, so the correct amount of squash concentrate needed is $\frac{240}{6}$ litres or **40 litres**.

Example (4): This model of a Spitfire fighter plane has a wingspan of 468 mm and a length of 380 mm.

i) Given that this is a 1:24 scale model, what are the actual wingspan and length of a Spitfire in metres, to the nearest centimetre ?

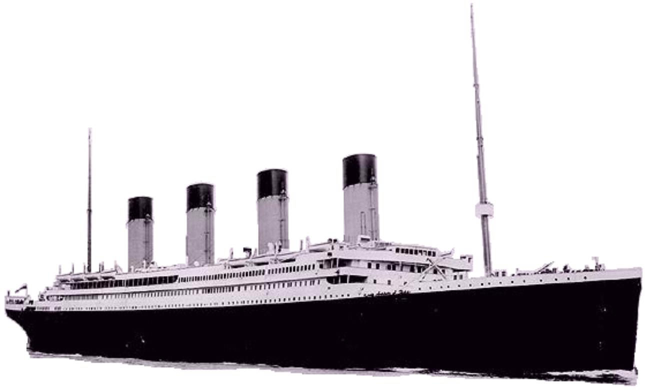
ii) If the height of an actual Spitfire is 3.48 m, calculate the height of the model in millimetres.

i) The scale factor of actual : model is 24 : 1, so all **linear** dimensions of the actual Spitfire will be 24 times those of the model.

The wingspan of an actual Spitfire is therefore (24×468) mm, or **11.23 m**, and the length is (24×380) mm or **9.12m.**

ii) This time we have the height of the actual Spitfire, so we divide 3.48 m, or 3480 mm, by 24 to obtain the height of the model, namely $\frac{1}{24} \times 3480$ mm, or **145 mm**.

Example (5): The film studio mock-up of the *Titanic* liner was built to seven-eighths the scale of the actual ship.



i) If the actual length of the *Titanic* was 270 metres, what was the length of the studio mock-up ?

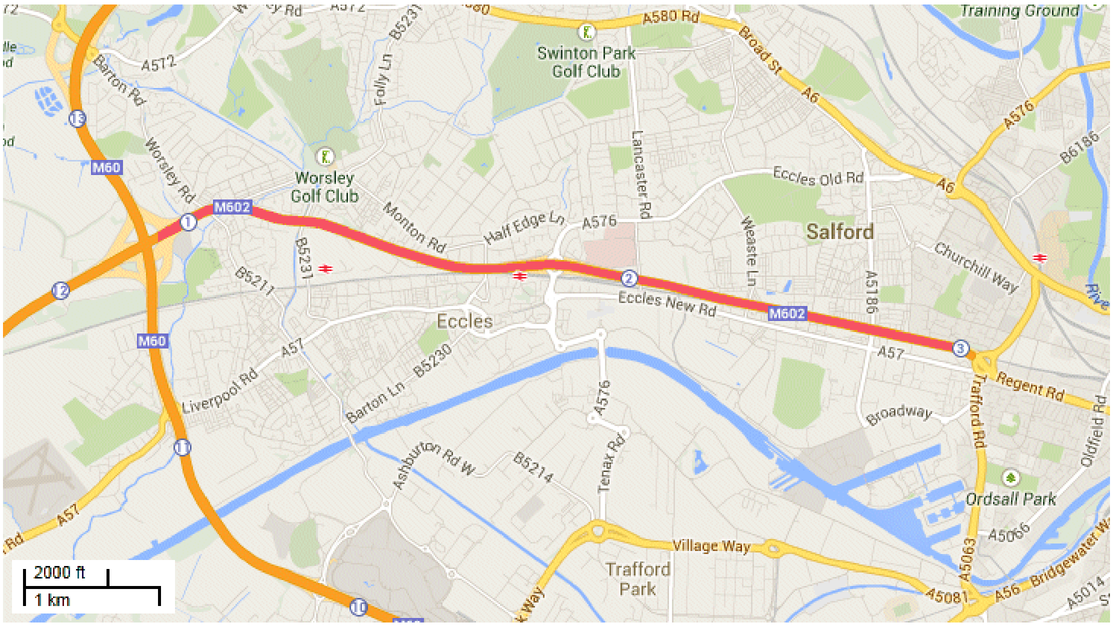
ii) If the funnels of the studio mock-up were 16.8 metres tall, how tall were the funnels of the actual *Titanic* ?

i) The length of the studio mock-up is $\frac{7}{8} \times 270$, or **236 metres** long to the nearest metre.

ii) This time we must divide the funnel height of the mock-up by $\frac{7}{8}$, or multiply it by $\frac{8}{7}$, to obtain the funnel height of the real ship, i.e. $\frac{8}{7} \times 16.8$ metres, or **19.2 metres**.

Example (6) : The M602 motorway appears 10 cm long on a map, but is 6 kilometres long in reality.

i) What is the scale of the map ? Give the result in the form 1 : *n* where *n* is a whole number.



The ratio of the map to reality is 10 cm : 6 km, or 0.1m : 6000 m.
(We have converted both distances to metres for consistency.)

This can in turn be redefined as 1 m : 60,000 m, and finally as 1 : 60,000.

The map scale is therefore **1 : 60,000**.

Example (6b) : Another map is issued to a scale of 1 : 25,000.

i) The true distance between Manchester Piccadilly and Oxford Road stations is 825 metres. How far apart are they on the map ?

ii) Oxford Road and Deansgate stations are 23 mm apart on the map. How far apart are they in reality ?

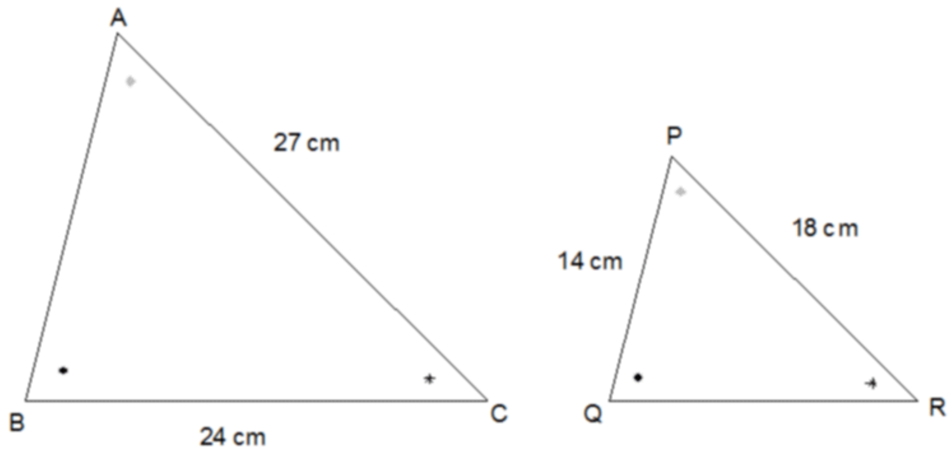
i) A scale of 1 : 25,000 can be redefined as 1 mm: 25,000 mm, or **1 mm : 25 m.**

The distance of 825 metres in real life is therefore represented by $\frac{825}{25}$, or 33 millimetres.
$\therefore$ Piccadilly and Oxford Road stations are **33 mm** apart on the map.

ii) As 1 mm on the map represents 25 metres in real life, 23 mm represents 23 $\times$ 25, or 575 metres.
$\therefore$ Oxford Road and Deansgate stations are **575 m** apart in reality.

Ratio can also be used to find measurements of similar figures, i.e. figures of the same **shape** but not of the same size.

Example (7): The triangles shown in the diagram are similar.
Find the lengths of the missing sides *AB* and *QR.*



The triangles have all 3 angles the same, which is a requirement for similarity. We need to find a pair of corresponding sides, where the length has been stated for both triangles.

Such a pair of sides is *AC* with a length of 27cm and *PR* with a length of 18 cm. They correspond because they are both opposite the angle marked with a solid dot.

The ratio *AC* : *PR* = 27 : 18 or 3 : 2 in its simplest form.

The sides of *ABC* are therefore $\frac{3}{2}$ or $1\frac{1}{2}$ times the length of the corresponding sides of *PQR*, so we multiply by $\frac{3}{2}$ to find the length of side *AB.* Sides *AB* and *PQ* correspond, so ***AB*** **=** $\frac{3}{2}$ ***PQ*** **= 21 cm.**

Conversely, the sides of *PQR* are $\frac{2}{3}$ as long as those of *ABC*, so we multiply by $\frac{2}{3}$ (or divide by $\frac{3}{2}$). to find the length of side *QR.* Hence ***QR*** **=** $\frac{2}{3}$ ***BC*** **= 16 cm.**

Division in a given ratio.

Here we add the proportional parts of a ratio to obtain the number of shares. Each part of the ratio then becomes the top line of a fraction whose bottom line is the number of shares.

Example (8): Alf and Bert buy 5 and 3 National Lottery tickets respectively each week and agree to share out their winnings in proportion to their weekly outlay.

One week, they win a jackpot of £400,000. How much does each player receive ?

Adding the contributions gives 5 + 3 = 8 shares in total.
Each share of the winnings is therefore $\frac{1}{8}$ of the total £400,000, or £50,000.
Alf therefore receives 5 shares , namely $\frac{5}{8}$ of the total, or £(5 × 50,000). or £250,000.
Bert receives 3 shares or $\frac{3}{8}$ of the total, or £(3 × 50,000). or £150,000.

Example (9): The top three cash prizes in a raffle are distributed in the ratio 5 : 2 : 1.
The first prize is £120 more than the second prize. What is the value of each prize ?

There are 5 + 2 + 1 or 8 shares in total, and we also know that the first prize is valued at 5 shares and the second at 2 shares. Their difference is 5 – 2, or 3 shares.
Also, as 3 shares = £120, one share is one-third of £120, or £40.

∴ The first prize is £(5 × 40), or £200; the second, £(2 × 40), or £80, and the third prize is £40.

Example (10):

A bricklayer has prepared 160 kg of mortar by mixing 1 part of cement to 3 of sand.
His foreman then asks him to change the ratio of the mix to 2 parts of cement to 5 of sand.

Assuming that the bricklayer does not use any additional sand, how much cement does he need to add to the existing mortar mixture to produce the required cement to sand ratio of 2 : 5 ?

Since 1 part + 3 parts = 4 parts, the original mortar mixture is one-quarter, or 40 kg cement, and three-quarters, or 120 kg, sand.

After the foreman's request to change the cement : sand ratio to 2 :5, this 120 kg of sand now represents 5 parts, and so one part of the new mixture is $\frac{120}{5}$ kg or 24 kg.

The bricklayer therefore needs 2 parts, or 2 × 24 kg, or **48 kg total cement** in the mixture.

As the original mixture had 40 kg of cement in it, the bricklayer needs to add another 8 kg of cement to change the ratio to the requested 2 : 5.

Example (11) : The outside of a football is made by stitching together a number of panels, whose faces are regular pentagons and regular hexagons. The ratio of pentagons to hexagons for a complete football is 3 : 5.

i) Work out how many pentagons and hexagons make up the outside of a complete football, given that a complete football has 32 panels in total.

ii) A workshop currently has 560 hexagonal panels and 360 pentagonal panels available for assembling the footballs. Calculate how many complete footballs can be assembled out of the panels.

i) Dividing 32 in the ratio 3 : 5 gives 8 parts, with one part is $\frac{1}{8}$ of 32, or 4 panels.
Hence 3×4, or 12, of the panels on a football are pentagons, and 5×4, or 20, are hexagons.

$\therefore$ The outside of a football consists of **12 pentagons and 20 hexagons.**

ii) There are 20 hexagonal panels on a football, so 560 of these panels would be enough to assemble $\frac{560}{20}$, or 28 footballs.

Similarly, there are 12 pentagonal panels on a football, so 360 of these panels would be enough to assemble $\frac{360}{12}$, or 30 footballs.

Since we do not have enough hexagonal panels to assemble 30 footballs, the total number of completed footballs that could be assembled with the given numbers of panels is the smaller number of the two, namely **28 footballs**.

Everyday Ratio Problems.

Ratio methods are often used to solve practical ‘everyday’ problems, such as ‘value for money’ price comparisons and altering recipes for differing numbers of people.

Example (11): The following recipe has appeared in the Lancashire Cooking magazine:

BURY HOT-POT **Serves 4**

600 g boneless lamb, diced
1 kg potatoes, peeled and cut into chunks
2 large carrots, sliced
1 large onion, chopped
4 slices of Bury black pudding

Bob is organising a party where he needs to make enough Bury Hot-Pot to cater for 10 people. How will he need to adjust the amounts of ingredients he uses ?

The recipe quoted is for 4 people but Bob needs to cater for 10.
We therefore need to divide by 4 to find the amounts for one person, and then multiply by 10 to find the amounts for all.

600g of lamb for 4 people therefore equals $\frac{1}{4}$ of 600g or 150g for one, hence 1.5kg for 10 people.
1kg of potatoes similarly works out as 250g for one, and 2.5kg for 10.

The rest is worked out similarly:
2 large carrots for 4 works out as half a carrot each, or 10 halves = 5 carrots for all.
1 large onion for 4 works out as 2½ large onions for 10 (Bob can use 2 or 3 whole) .
Finally, 4 slices of Bury black pudding for 4 works out as 10 slices for 10.

The adjusted recipe is thus:

BURY HOT-POT **Serves 10**

1.5 kg boneless lamb, diced
2.5 kg potatoes, peeled and cut into chunks
5 large carrots, sliced
2-3 large onions, chopped
10 slices of Bury black pudding

Alternatively, we could have deduced that 10 is 2½ times as large as 4, and multiplied all the amounts by 2½ to get the same results.

Example 11(b): Bob has had more people interested in his hot-pot meal than he originally thought, and he has checked his kitchen for ingredients.

He was able to find 3 kg of boneless diced lamb, three 2 kg bags of potatoes, a bag of 16 large carrots, 6 large onions and a bulk pack of 30 slices of Bury black pudding.

What is the maximum number of people he was able to feed by following the original recipe and not buying any extra items ?

Here is the original recipe :

BURY HOT-POT **Serves 4**

600 g boneless lamb, diced
1 kg potatoes, peeled and cut into chunks
2 large carrots, sliced
1 large onion, chopped
4 slices of Bury black pudding

We need to find the "critical" ingredient in Bob's kitchen whose availability is the smallest multiple of the recipe for 4 people.

He has 3 kg of lamb, and as 3 kg is 3000g , he has $\frac{3000}{600}$ or 5 times the quantity specified in the recipe. The recipe asks for 1 kg of potatoes, but Bob has three 2 kg bags, or 6 kg in total, which is 6 times that. He needs two carrots for the recipe, but has 16, or 8 times the recipe amount. The recipe calls for one onion, but Bob has 6. Finally, he has 30 slices of black pudding, which is $\frac{30}{4}$, or 7½, times the recipe amount.

The lamb is the "critical" ingredient here, as the 3 kg he has is 5 times the recipe amount, which is the smallest multiple among the ingredients.

The amounts in the original recipe were suggested to serve 4 people, so the contents of Bob's kitchen are enough to serve 5 × 4, or 20 people.

The "Best Buy" problem.

Example (12): A supermarket sells the same brand of baked beans in three different-sized cans; 400g for 36p, 600g for 59p and 800g for 75p. Which offers the best value for money ?

The method is to find a suitable 'weight unit' (100g would be a good choice here) and then find the prices for that 'unit weight'.

If 400g of beans cost 36p, then 100g cost $\frac{1}{4}$ of 36p, or 9p.

If 600g of beans cost 59p, then 100g cost $\frac{1}{6}$ of 59p, or 9.8p (to nearest 0.1p).

If 800g of beans cost 75p, then 100g cost $\frac{1}{8}$ of 75p, or 9.4p (to nearest 0.1p).

By comparing the prices per 100g, we can see that the 400g can offers the best value for money.

If the quantities are more awkward (as in a calculator question), there are two methods available.

Example (13): A supermarket sells the same brand of baked beans in three different-sized cans; 405g for 36p, 595g for 59p and 822g for 75p. Which offers the best value for money ?

This time, the weights are more awkward to handle, so we have two choices.

Method (1): Divide the price by the weight to find the **lowest price per gram (**or kg) :

Thus, the 405g can costs $\frac{36}{0.405}$ = 88.9p per kg (we have converted weight to kilograms for convenience)

Likewise the 595g can costs $\frac{59}{0.595}$ = 99.2p per kg and the 822g can $\frac{75}{0.822}$ or 91.2p per kg.
Again, the 400g size offers the best value for money.

Here, best value for money = lowest price per g (or kg).

Method (2): Divide the weight by the price to find the **highest weight per penny** (or £):

Thus, a person buying the 405g can buys $\frac{405}{36}$ = 11.3 g per penny.
Likewise, someone buying the 595g can buys $\frac{595}{59}$ = 10.1 g per penny, and someone buying the 822g can buys $\frac{822}{75}$ or 11.0 g per penny.

Here, best value for money = highest weight per penny (or £).

There is little to choose between the two methods, but the first one is more commonly used by supermarkets when comparing prices.

The next example of "shopping economics" is not a ratio problem as such, but features quite commonly in exams, so we have included it here.

Example (14): Barbara needs to buy 6 bottles of lemonade for a family party, and she sees the following offers for the same-size bottle of the same brand at three supermarkets:

Aldi: 49p per bottle
Tesco: 69p per bottle, but on a 'buy 3 for the price of 2' offer
Lidl: 66p per bottle, but on a 'buy one, get second half price' offer

Work out how much money Barbara would have to spend at each of the supermarkets, and hence find out which supermarket gives the best value for money.

If Barbara were to shop at Aldi, she would be paying 6 × 49p or **£2.94**.

At Tesco, she would be able to buy 3 bottles at 2 × 69p or £1.38 using the '3 for 2' offer
Since 6 is twice 3, she would be paying 2 × £1.38 or **£2.76**.

At Lidl, Barbara would be buying one bottle at 66p and the second one at half that or 33p, so she would be paying 99p for 2 bottles. Since 6 is three times 2, she would be paying 3 × 99p or **£2.97**.

$\therefore$ Tesco gives the best value for money of the three supermarkets.

Direct and Inverse Proportion.

Example (15): Laura is training on an exercise bike set to a constant speed. She cycles the equivalent of 3 km in 5 minutes. How long would it take her to cycle i) 12 kilometres, ii) 8 kilometres ?
Also, iii) how far will she cycle in 25 minutes ?

i) We see that 12 km is 4 times longer than 3 km, and if Laura's speed remains constant, then the time taken will also be 4 times greater.

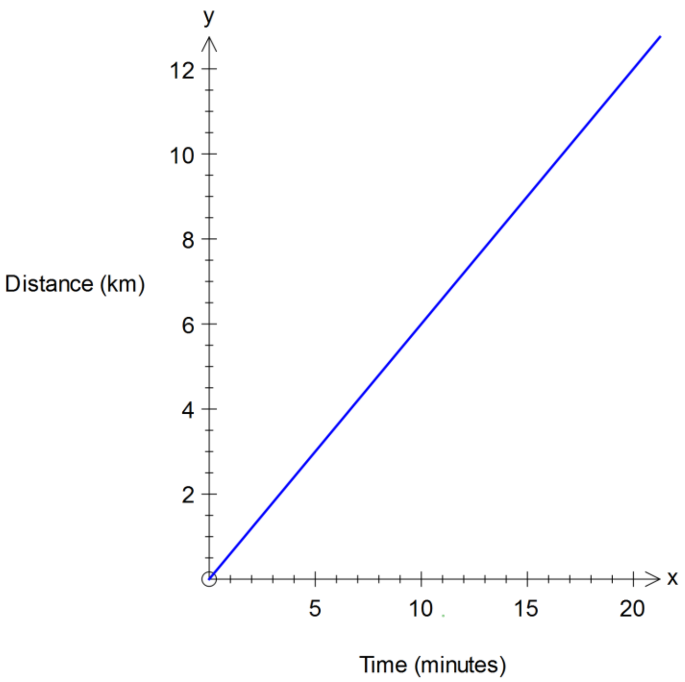
Thus, if she cycles 3 km in 5 minutes, the time taken for her to cycle 4 × 3 km, or 12 km, will be 4 × 5 minutes, or **20 minutes.**

ii) This time the ratio between 3 km and 8 km is fractional, where we would have to multiply by $\frac{8}{3}$.
Laura will cycle the 8 km in $\frac{8}{3}$ × 5 minutes = $13\frac{1}{3}$ minutes = **13 minutes and 20 seconds**.

iii) Because 25 minutes is 5 times as long as 5 minutes, Laura will cycle 5 × 3 km, or 15 km, in 25 minutes.

This example was one of **direct** proportion; when one quantity (in this case, distance) was **increased** by a certain ratio, the other quantity (the time) was **increased by the same ratio**.

All relationships leading to direct proportion lead to straight-line graphs passing through the origin, with a general equation of $y = kx$ where *k* is any positive number.



Example (16): A non-stop express passenger train travelling at 120 miles per hour takes 25 minutes to cover the distance from Rugby to Stafford. How long would it take i) a goods train travelling non-stop at 60 mph, and ii) a slower passenger train limited to 75 mph, to cover the same distance ?

iii) A slower goods train takes 1 hour and 15 minutes to cover the same distance. What is its speed ?

i) A speed of 60 mph is half as fast as 120 mph, and therefore the time taken to cover the same distance will be twice as long.

Therefore if the express train takes 25 minutes to cover the Rugby-Stafford distance, the goods train will take twice 25 minutes, or **50 minutes**, to do the same. So, as speed is halved, time is doubled.

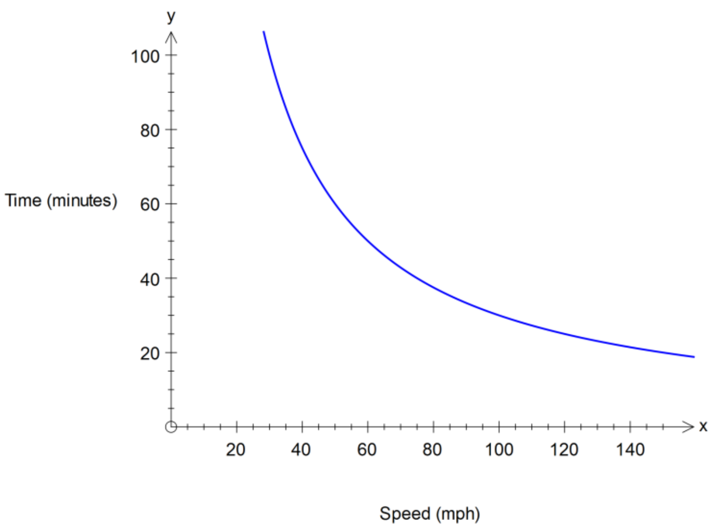
ii) The ratio between the speeds of the slow passenger train and the express train is 75 : 120, simplifying to 5 : 8. Because the slower train is travelling at $\frac{5}{8}$ of the speed of the express train, the time taken to cover the same distance will be $\frac{8}{5}$ times longer.

The express train takes 25 minutes, so the slow passenger train will take $\frac{8}{5} \times 25$ minutes = **40 minutes**.

iii) Since 1 hour and 15 minutes are equal to 75 minutes, the slow goods train takes three times as long to cover the same distance the express train travelling at 120 mph does in 25 minutes. As the time is tripled, the speed is divided by 3, and so the slow goods train is travelling at **40 mph.**

Unlike the previous example, we are dealing with **inverse** proportion; when one quantity (in this case, speed) was **decreased** by a certain ratio, the other quantity (the time) was **increased by the same ratio**.

All relationships leading to inverse proportion lead to a curved graph related to that of $y = \frac{1}{x}$, with a general equation of $y = \frac{k}{x}$ where k is any positive number.



The next example is a little more tricky, as it combines both direct and inverse proportion in the same question. Such problems are best tackled in stages.

Example (17): Given that 12 checkout staff at a supermarket can serve 48 customers in 20 minutes,
i) how long would it take 15 checkout staff to serve 84 customers ?
ii) how many checkouts need to be open to serve 240 customers in one hour ?

If we increase the number of checkout staff, we increase the number of customers being served in the same time interval. Hence the checkout staffing level and the number of customers are in direct proportion when time is unchanged.

More checkouts = more customers (in same time)

Also, an increase in the number of checkout staff would lead to a decrease in the time taken to serve the same number of customers. This time, the checkout staffing level and the time taken to serve the customers are in inverse proportion when the number of customers is unchanged.

More checkouts = less time (for same number of customers)

Finally, if the number of checkout staff remains constant, an increase in the number of customers would lead to an increase in the time taken to serve them. In other words, the number of customers and the time taken to serve them are in direct proportion when the number of checkout staff is unchanged.

More customers = more time (for same number of checkouts)

i) We can see that the number of checkout staff has increased from 12 to 15, or in the ratio 15 : 12, simplifying to 5 : 4. Therefore 15 checkout staff can serve $\frac{5}{4} \times 48$, or 60 customers, in 20 minutes.
We now have the correct number of checkout staff, but we still need to increase the number of customers to 84, or in the ratio 84 : 60, simplifying to 7 : 5.
Since customer numbers and the time are in direct proportion, the time taken is $\frac{7}{5} \times 20$, **or 28 minutes**.

	12 checkouts;	48 customers;	20 minutes	Original problem
→	**15 checkouts;**	**60 customers;**	20 minutes	Customers / checkouts in direct proportion, so increase both in ratio 5 : 4 (i.e. multiply by $\frac{5}{4}$)
→	15 checkouts;	**84 customers;**	**28 minutes**	Customers / time in direct proportion, so increase both in ratio 7 : 5 (i.e. multiply by $\frac{7}{5}$)

ii) This time, the number of customers has increased from 48 to 240, a ratio of 5 : 1, and so the time taken will increase by the same ratio, from 20 minutes to 5×20 or 100 minutes.

The question however asks us to work out the checkouts per hour, and so we have to decrease the time from 100 to 60 minutes, or multiply it by $\frac{60}{100}$ or $\frac{3}{5}$.

Because the time taken and the number of checkouts are inversely proportional, we have to **multiply** the number of checkouts by $\frac{5}{3}$, giving us $\frac{5}{3} \times 12$, **or 20 checkouts**.

	12 checkouts;	48 customers;	20 minutes	Original problem
→	12 checkouts;	**240 customers;**	**100 minutes**	Customers / time in **direct** proportion, so increase both in ratio 5 : 1
→	**20 checkouts;**	240 customers;	**60 minutes**	Checkouts / time in **inverse** proportion, so **decrease** time in ratio 5 : 3 (i.e. multiply by $\frac{3}{5}$)and **increase** checkouts in ratio 5 : 3 (i.e. multiply by $\frac{5}{3}$)