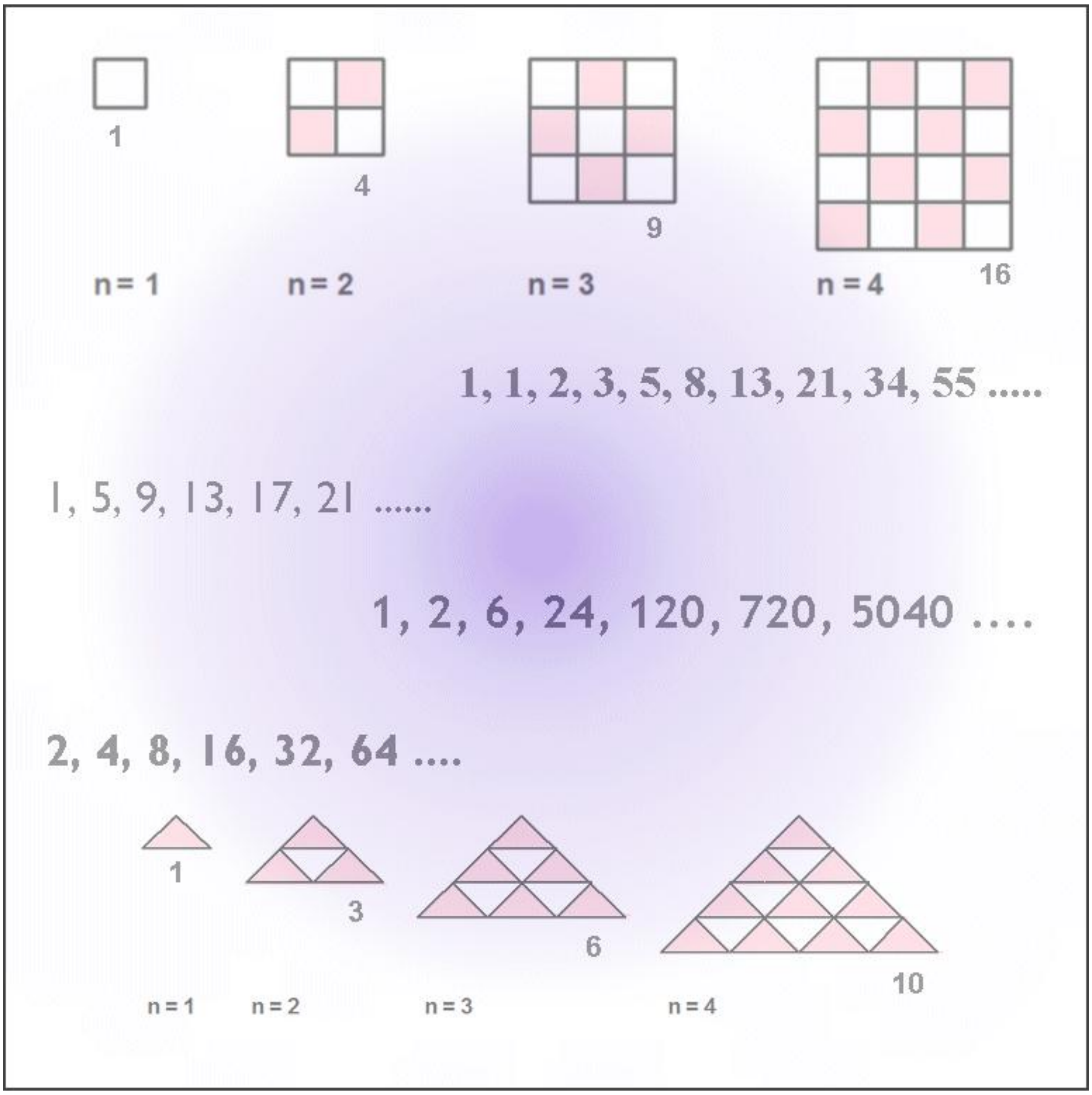
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

SEQUENCES



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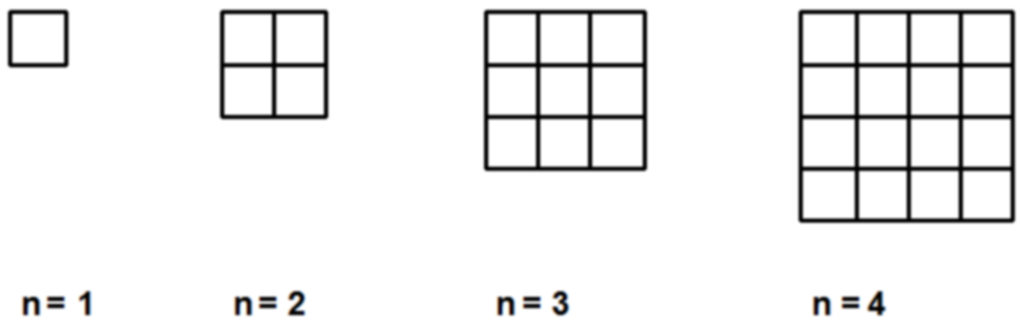
SEQUENCES

A **sequence** is a list of numbers following some rule for finding succeeding values.

Each number in a sequence is called a **term**.

A sequence can be defined by a formula for the n^{th} term, where n is the position of the term in the sequence.

Example (1): One example of a sequence is that of the square numbers, illustrated below.

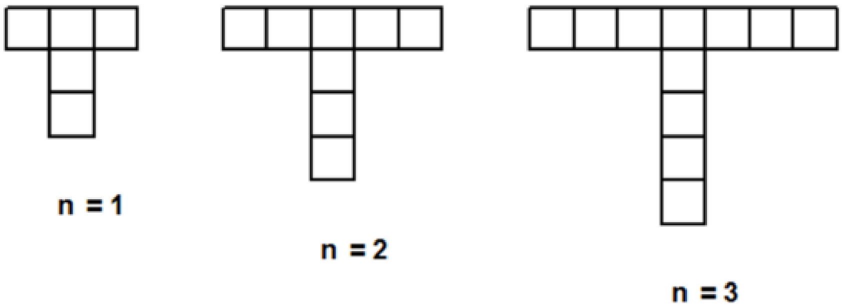


The first term of the sequence is 1, the second 4, the third 9 and the fourth, 16.
The number of squares in each of the diagrams is the same as the square of its position in the sequence.

The 5^{th} term in the sequence will therefore be 5^2 or 25, and we can also generalise by saying that the n^{th} term is equal to n^2.

The sequence of square numbers is thus 1, 4 , 9, 16, 25, 36, ...

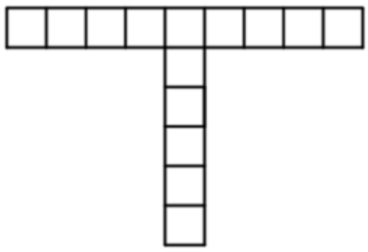
Example (2): Investigate the sequence of T-shapes below to draw the next T-shape in the sequence, and also to find a general formula for the total number of unit squares in the n^{th} T-shape.



Spotting the pattern here is not quite as easy as in the last example, but we can see that:

An extra unit square is added to the vertical bar of the T as *n* goes up by 1.
Two extra unit squares (one on each side) are added to the horizontal bar of the T as *n* goes up by 1.

The fourth T-shape in the sequence therefore looks like the diagram on the right.



The first T-shape is made up of 5 unit squares.
The second one is made up of 5 + (3 × 1) or 8 unit squares.
The third one is made up of 5 + (3 × 2) or 11 unit squares.
The fourth one is made up of 5 + (3 × 3) or 14 unit squares.

We can spot the pattern here: the n^{th} T-shape is made up of **5 + 3(*n*-1)** unit squares.

(An equally valid expression is **2 + 3*n***).

The first two examples were shape-based and intuitive, but it is also possible to determine the general term of a sequence by listing enough of its elements to spot a pattern.

Sequences with a common difference.

These sequences are generated by **adding a constant value** to each term as we go along.

We can see that the next two terms in the sequence 7, 14, 21, 28 .... are 35 and 42, the terms being the first multiples of 7. We can also see that the n^{th} term is 7*n*.

Since each term can be obtained from the previous one by adding 7, the common difference is 7.

The common difference can be found by taking any term (other than the first one), and subtracting the one before it.

Taking the sequence 5, 12, 19, 26, 33 .... we can see that the common difference is 7, and therefore the sequence resembles that of the multiples of 7 (7, 14, 21 ..), whose general term is 7*n*. However, the first term, where *n* =1, is 5, which is 7-2, and all the other terms are 2 less than the multiple of 7.
The general term is therefore 7*n* -2 where *n* is its position in the sequence.

Another way of reckoning is to say that the general expression for the n^{th} term is $an + b$ where *a* is the common difference and *b* is a constant to be added. This constant is a hypothetical "0^{th} term" derived by subtracting the common difference from the first term.

Examples (3a): Find the next two terms of the following sequences, their 100^{th} terms, and a general expression for the n^{th} term:

i) 1, 5, 9, 13, 17, ...
ii) 44, 39, 34, 29, 24, ...

In i) we see that the first term is equal to 1 and the common difference *d* (e.g.5-1) is 4.
The next two terms are 21 and 25.

The expression for the general term is therefore $4n + b$.
The first term is 1, so the "0th term" is 1 – 4 or -3, giving a general formula of **4*n* – 3.**
The 100^{th} term is therefore (4×100) – 3 or 397.

In ii) we can see that successive terms decrease in value, so the common difference will be negative.
The first term = 44 and the common difference is –5.
The next two terms are 19 and 14.
This time the expression for the general term is $-5n + b$.

The first term is 44, so the "0th term" is 44 – (-5) or 49, giving a general formula of **49 - 5*n*.**
The 100^{th} term is therefore 49 - (5×100) or -451.

Example (3b): For the sequence 1, 5, 9, 13, 17, ...

i) Find the position of the number 61.
ii) Show that the number 75 is not a term of the sequence.

i) Because the expression 4*n* -3 can be used to find the *n*th term in the sequence,
we solve the equation $4n - 3 = 61$, and the solution is $n = 16$.
The number 61 is therefore the 16^{th} term in the sequence.

ii) When we solve $4n - 3 = 75$ as in i), we have the result $n = 19.5$, which is not a positive whole number.
The number 75 is therefore not a member of the sequence.

Sequences with a common ratio.

These sequences are generated by **multiplying** each term by **a constant value** as we go along.

Since each term can be obtained from the previous one by multiplying by 3, the common ratio is 3.

The common ratio can be found by taking a term after the first, and dividing it by the one before it.

We can see that the next two terms in the sequence 3, 9, 27, 81 .... are 243 and 729, the terms being the first powers of 3. We can also see that the n^{th} term is 3^n.

(This type of sequence is also called a **geometric progression**.)

Examples (4): Find the next two terms of the following sequences.

i) 3, 6, 12, 24, 48, ...
ii) 800, 80, 8, 0.8, 0.08,............

In i) we see that the first term, a , is equal to 3 and the common ratio r (e.g.$6 \div 3$) is 2. (The numbers follow a doubling pattern).

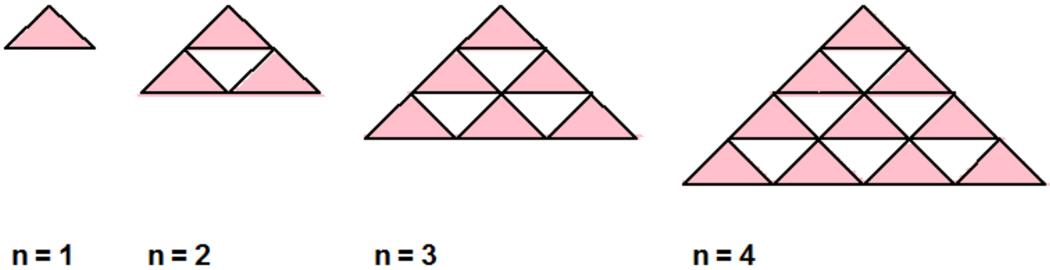
The next two terms are 96 and 192.

In ii) the successive terms decrease in value, and so the common ratio will be less than 1.
The first term a = 800 and the common ratio is 0.1 (numbers decrease by factor of 10).
The next two terms are 0.008 and 0.0008.

Sequences with a changing difference.

These are trickier than the last two, because the difference itself changes between each pair of numbers.

Example (5): Take the sequence of triangular numbers.



Find the next three terms of the sequence.

The triangular numbers are represented here by the shaded triangles.

The first triangular number is 1.
The second triangular number is 1 + 2 or 3.
The third triangular number is 1 + 2 + 3 or 6.
The fourth triangular number is 1 + 2 + 3 + 4 or 10.

The sequence of triangular numbers therefore goes 1, 3, 6, 10, ...

The difference between the first term and the second is 1.
The difference between the second term and the third is 2.
The difference between the third term and the fourth is 3.

The differences between the terms increase by 1 as we progress along the sequence.

The sequence of triangular numbers goes 1, 3, 6, 10, **15** (add 5 to previous), **21** (add 6) and so on.

Or, we can tabulate the pattern of differences as follows :

Sequence terms	1		3		6		10
1^{st} differences		*2*		*3*		*4*	
2^{nd} differences			*1*		*1*		

We can therefore put two more instances of 1 in the "2^{nd} difference" row.
Then we can put 4 + 1 = 5 and 5 + 1 = 6 into the "1^{st} differences" row above.
Finally, we can put 10 + 15 = 15 and 15 + 6 = 21 into the top row to get the next triangular numbers.

Sequence terms	1		3		6		10		**15**		**21**
1^{st} differences		*2*		*3*		*4*		***5***		***6***	
2^{nd} differences			*1*		*1*		***1***		***1***		***1***

Example (6):

Find the next two terms of the sequence 4, 7, 16, 31, 52......

Sequence terms	4		7		16		31		52		**79**		**112**
1st differences		*3*		*9*		*15*		*21*		***27***		***33***	
2nd differences			*6*		*6*		*6*		***6***		***6***		

The first differences for the sequence are 7-4 or 3, 16-7 or 9, 31-16 or 15, and 52-31 = 21.... and the second differences are 9-3 or 6, 15-9 or 6 and 21-15 or 6 – i.e. they are constant at 6.

We can therefore put two more instances of 6 in the "2nd difference" row.
Then we can put 21 + 6 = 27 and 27 + 6 = 33 into the "1st differences" row above.
Finally, we put 52 + 27 = 79 and 79 + 33 = 112 into the uppermost row.

The next two terms of the sequence are therefore 79 and 112.

Generating sequences by rules.

Another way of generating a sequence is to follow some rule relating the terms.
Thus the sequence 1, 5, 9 ,13, 17... can be generated by the rules:

- Start with 1.
- Generate the next term by adding 4 to the term before it.

Similarly the sequence 3, 6, 12, 24, 48, ... can be generated by the rules:

- Start with 3.
- Generate the next term by doubling the term before it

One disadvantage of this definition is that it is not always easy to deduce a formula for the n^{th} term.

Example (7):

Write out the first ten terms of the sequence generated by the following rule:

- Set the first two terms to 1 and 1.
- Generate each further term by adding together the two previous ones.

(Do not attempt to find a general formula for the n^{th} term !)

The third term is 1 + 1 or 2, the fourth one is 1 + 2 or 3 and the fifth one is 2 + 3 or 5.

The sequence thus goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ....

Aside: This is the 'Fibonacci' sequence, named after a 13th – Century mathematician, Leonardo of Pisa, 'Fibonacci – son of Bonaccio' who first investigated it.

Another curiosity: the seventh term of the sequence is 13, and the sum of the first ten terms is

1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143.

The sum of the ten terms, namely 143, is 11 times the seventh entry, 13.

This might seem unremarkable, but if we generate a similar sequence with any starting numbers other than 1 and 1, the sum of the first ten terms is still 11 times the seventh number !

Take the sequence 1, 3, 4, 7, 11, 18, 29, 47, 76, 123. The seventh term is 29, and the sum of the first ten terms is 11 times 29, or 319 !