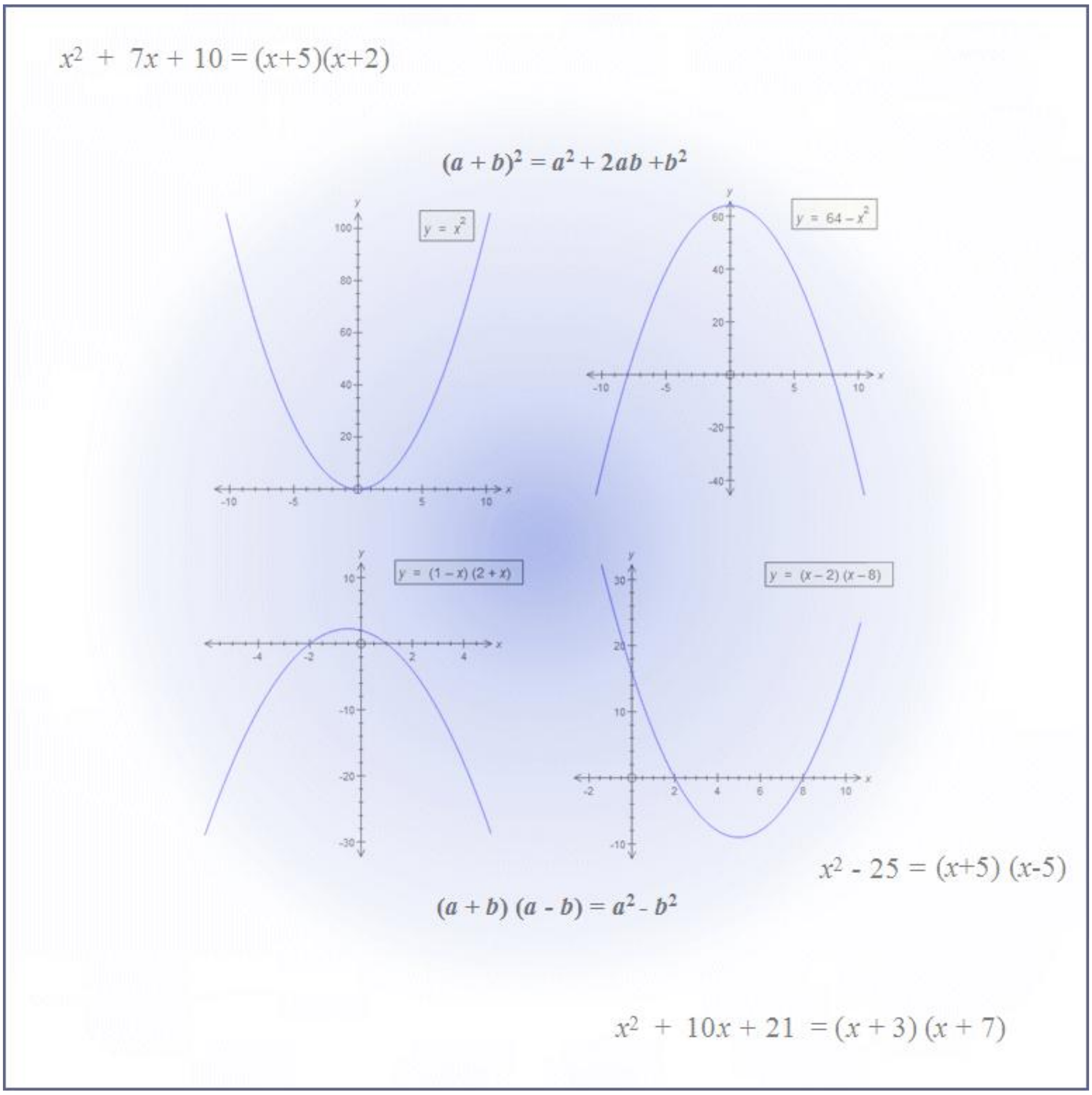


M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Foundation Tier

QUADRATIC EQUATIONS



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QUADRATIC EQUATIONS

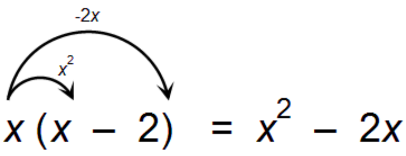
A **quadratic** expression is one of the form $ax^2 + bx + c$ where a, b and c are constants, and a is not zero. The highest power of x is 2 (the square of x).

Recall the technique of expansion of brackets.

Examples (1): Expand i) $x(x-2)$; ii) $(x+3)(x+5)$

i) $x(x-2) = x^2 - 2x$

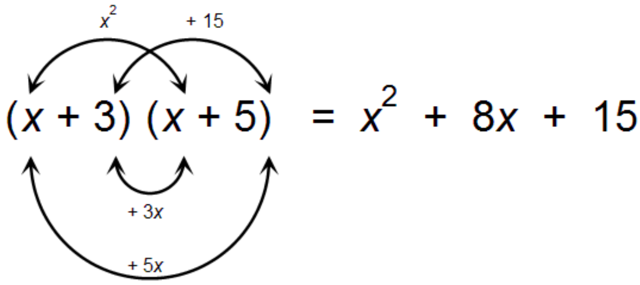
This is a single bracket expansion. Notice that there is no constant term here.



ii) $(x+3)(x+5) = x^2 + 8x + 15$.

There are two terms in x in the expansion, $3x$ and $5x$. They can be collected to give $8x$.

Alternatively, we could work in one line as follows :



$(x+3)(x+5) = x(x + 5) + 3(x + 5)$

$= x^2 + 5x + 3x + 15$ (expand)
$= x^2 + 8x + 15$ (collect)

Each term in the first bracket is multiplied by each term in the second bracket, and like terms collected. Note that this time we have three terms, including a constant.

The following special results should also be noted:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

This is the 'difference of two squares' result.

Quadratic Equations.

Equations of the form $ax^2 + bx + c = 0$, where a is non-zero, are called quadratic equations.

Such equations can be solved in various ways, but at Foundation Tier we are only concerned with factorisation of equations of the form $x^2 + bx + c = 0$, where the multiple of x^2 is unity, i.e. 1.

Solving Quadratics by Factorisation.

Example (2). Solve the equation $x^2 = 7x$.

We might be tempted to divide both sides by x to give $x = 7$, but that does not give the complete result. True, $7^2 = 7 \times 7 = 49$, but $x = 0$ is also a solution, since $0^2 = 0 \times 0 = 0$. By dividing by x, we have 'lost' a possible solution of the equation.

The correct procedure is to re-express the equation as $x^2 - 7x = 0$.

Example (2a): Factorise $x^2 - 7x$ and hence solve $x^2 - 7x = 0$.

$x^2 - 7x = 0$ can be rewritten as $x(x - 7) = 0$.
$x^2 - 7x$ is equal to 0 if and only if $x = 0$, or $x - 7 = 0$.
The solutions of the quadratic are thus $x = 0$ and $x = 7$.
They are also known as the **roots** of the equation.

The last case was easy enough, as we were able to factor out x.

In general, any quadratic equation of the form $x^2 + bx = 0$ (i.e. having no constant number term) can be solved by factoring out x to give $x(x + b) = 0$, and hence giving us the solutions of $x = 0$ and $x = -b$.

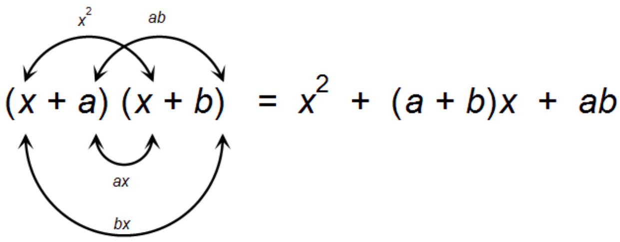
Example (3a): **Case where all the numbers are positive**

Factorise $x^2 + 7x + 10$ and hence find the roots of $x^2 + 7x + 10 = 0$.

To begin to factorise such an expression, let's look at the general case:

When factorised, the expression would be of the form $(x+a)(x+b)$ where a and b must be found.

Looking at the expansion, we can see that the number in front of the x is the sum of a and b, and the constant number is the product of a and b.



Therefore to factorise $x^2 + 7x + 10$, we must find two numbers a and b that add up to the number in front of the x (here 7) and multiply to give the number term (here 10).

Such a pair of numbers is $a = 5$, $b = 2$.

$x^2 + 7x + 10 = (x+5)(x+2)$.

The solutions of the equation are the values of x which make one of the bracketed terms equal to zero. In this case they are $x = -5$ and $x = -2$.

Example (3b): Factorise $x^2 + 9x + 20$ and hence find the solutions of $x^2 + 9x + 20 = 0$.
Two numbers with a product of 20 and a sum of 9 are 5 and 4, so

$x^2 + 9x + 20 = (x+5)(x+4)$.

The corresponding solutions of $x^2 + 9x + 20 = 0$ are hence $x = -5$ and $x = -4$.

The above method can be used to try and factorise any expression of the form $x^2 + bx + c = 0$, and also works when b and c are negative.

Example (4a): **Case where the term in *x* is negative, but the number term is positive**

Factorise $x^2 - 13x + 40$ and hence solve $x^2 - 13x + 40 = 0$.

Two negative numbers have a positive product but a negative sum, and therefore the factorised form of this expression will be of the form $(x – a)(x – b)$ where a and b are to be determined.

We still look for two numbers whose sum is 13 and whose product is 40.
Such a pair of numbers is $a = 5$ and $b = 8$, but the factorised form of $x^2 - 13x + 40$ is not now $(x+5)(x+8)$, but $(x-5)(x-8)$.

(Note that the sum of -5 and -8 is -13, and their product is 40).

So, $x^2 - 13x + 40 = (x-5)(x-8)$, and hence the roots of $x^2 - 13x + 40 = 0$ are $x = 5$ and $x = 8$.

Example (4b): Factorise $x^2 - 7x + 12$ and hence solve $x^2 - 7x + 12 = 0$.

Two numbers with a product of 12 and a sum of 7 are 3 and 4, so

$x^2 - 7x + 12 = (x - 3)(x - 4)$.

The corresponding solutions of $x^2 - 7x + 12 = 0$ are hence $x = 3$ and $x = 4$.

Example (5a): **Case where the number term is negative**

Factorise $x^2 - 2x - 15$ and hence solve $x^2 - 2x - 8 = 7$.

Note how the constant term in the expression $x^2 - 2x - 15$ is now negative. The factorised version will therefore be of the form $(x + a)(x – b)$ with one positive solution and one negative one. This is because two numbers of different signs give a negative product.

When looking for the number-pair here, we are now looking for a **difference** of 2 and a product of 15. Two numbers with a difference of 2 are 5 and 3, but we also need to know which one of them goes in the $(x + a)$ term, and which one in the $(x – b)$ term.

If we were to expand $(x + 5)(x - 3)$, then the expansion would be $x^2 + 2x – 15$, not $x^2 - 2x – 15$. We look at the number in front of the x in the expression $x^2 - 2x – 15$ and see that it is negative. Therefore, the larger number (5) goes with the term with the (-) sign and the smaller number (3) goes with the term with the (+) sign.

Hence $x^2 - 2x - 15 = (x + 3)(x - 5)$.
(Note that $3 + (-5) = -2$ and $3 \times (-5) = -15$).

The equation $x^2 - 2x - 8 = 7$ might look unrelated to the earlier part of the question at first.

What we must not do is to try and factorise the left-hand expression as it stands as $(x + 2)(x – 4)$ and solve $x + 2 = 7$ or $x - 4 = 7$. This is completely wrong !

The correct thing to do is to subtract 7 from both sides, to obtain the equation $x^2 - 2x – 15 = 0$, with the important zero on the right-hand side.

From the earlier factorisation, the solutions of $x^2 - 2x - 15 = 0$ are $x = 5$ and $x = -3$.

Example (5b): Factorise $x^2 + 3x - 28$ and hence solve $x^2 + 3x - 28 = 0$.

Two numbers with a product of 28 and a difference of 3 are 7 and 4.
The number in front of the x is positive, so the expression factorises to $(x + 7)(x - 4)$
(The larger number of 7 goes with the + sign in the brackets)

The corresponding solutions of $x^2 + 3x - 28 = 0$ are $x = -7$ and $x = 4$.
Again, the numbers -7 and 4 have a sum of 3 and a product of -28.

Example (6): Factorise $x^2 + 12x + 36$ and hence solve $x^2 + 12x + 36 = 0$.

Two numbers with a product of 36 and a sum of 12 are 6 and 6, so
$x^2 + 12x + 36 = (x+6)(x+6) = (x+6)^2$.

This time, there is only one solution – that of $x = -6$.

The expression $x^2 + 12x + 36$ is in fact a perfect square of the form $a^2 + 2ab + b^2$ where $a = x$ and $b = 6$.

The square of 6 (36) in the constant term and twice 6 (12) in the x-term can be seen by inspection.

Example (7): Factorise $x^2 - 18x + 81$ and hence solve $x^2 - 18x + 81 = 0$.

We have another perfect square here, this time of the form $a^2 - 2ab + b^2$ where $a = x$ and $b = 9$.
The square of 9 (81) and its multiple of 2 (18) can be seen by inspection.

$x^2 - 18x + 81$ is therefore the same as $(x - 9)^2$, and so the root of $x^2 - 18x + 81 = 0$ is $x = 9$; again, this is a unique solution.

Example (8): Factorise $x^2 – 25$ and hence solve $x^2 - 25 = 0$.

Here we can recognise the ‘difference of two squares’ form :

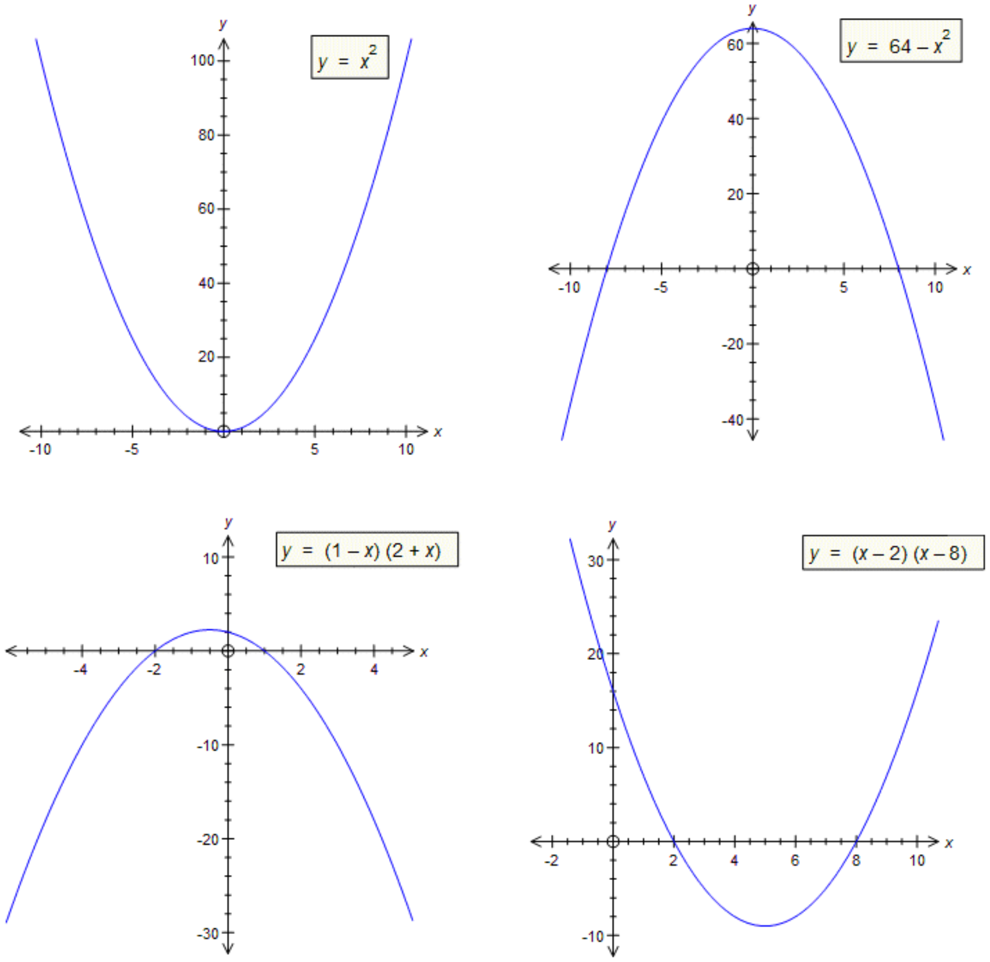
$a^2 - b^2 = (a + b)(a - b)$ where $a = x$ and $b = 5$.
Factorising gives $(x + 5)(x - 5) = 0$, with solutions of $x = 5$ and $x = -5$.

Note how adding 5 and -5 gives 0, hence the absence of the term in x in the original expression.

For brevity, we could say the solutions are $x = \pm 5$.

Quadratic graphs.

These graphs are of functions of the form $y = ax^2 + bx + c$ where a, b and c are constants, and a is not zero. The highest power of x is 2 (the square of x). The basic graph of $y = x^2$ is shown upper left.



These graphs are parabolic or 'bucket-shaped'.

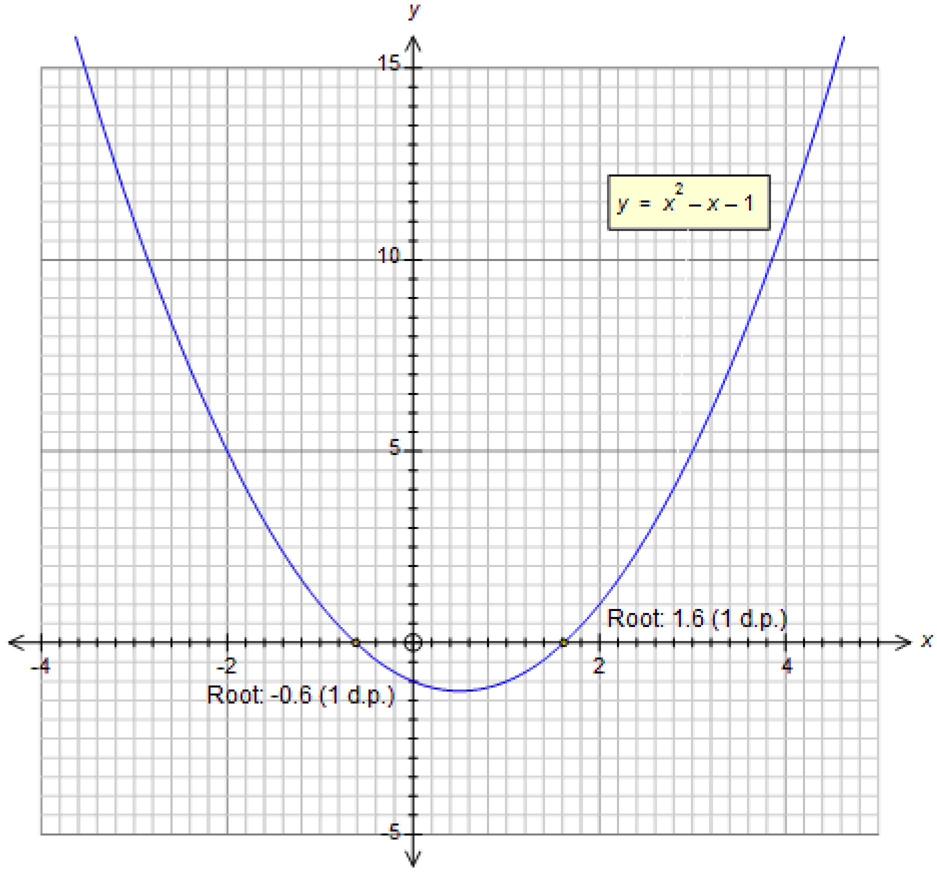
When the x^2 term is positive, the graphs point downwards at a trough and the function takes a minimum value. The expansion of $y = (x - 2)(x - 8)$ is $y = x^2 - 10x + 16$.

On the other hand, they point upwards at a crest and have a maximum value when the x^2 term is negative. The expansion of $y = (1 - x)(2 + x)$ is $y = 2 - x - x^2$.

The 'depth' of a parabolic graph can vary, but this is as dependent on the scaling of the graph axes as on the actual function.

Graphical methods of solving quadratic equations.

Sometimes an exam question might include a quadratic graph and ask for approximate solutions.



We can read off approximate solutions (or *roots*) of the equation $x^2 - x - 1 = 0$ by examining where the graph cuts the x-axis. It can be seen that the solutions of the equation are about –0.6 and 1.6.

The solutions in the original example could be thought of being the x-coordinates of the points where the curve $y = x^2 - x - 1$ and the line $y = 0$ meet.