M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Foundation Tier

INTRODUCTION TO ALGEBRAIC PROOF

Term No.	Value
1	p
2	q
3	$p+q$
4	$p+2q$
5	$2p+3q$
6	$3p+5q$
7	$5p+8q$
8	$8p+13q$
9	$13p+21q$
10	$21p+34q$
Sum of 10	$\mathbf{55p+88q}$

$$(2k+1)^2 = 4k^2+4k+1$$
$$= 4(k^2+k)+1.$$

$$(n+5)^2-(n+1)^2$$
$$= (n^2+10n+25)-(n^2+2n+1)$$
$$= 8n+24 = 8(n+3).$$

x	x^2+x+11
1	13
2	17
3	23
4	31
5	41
6	53

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Introduction to Algebraic Proof.

This section is a brief introduction to on how to prove mathematical conjectures using algebra.

The following methods are the ones normally encountered at GCSE :

- Proof by algebraic reasoning
- Proof by exhaustion
- Disproof by counterexample

The important thing about proving a conjecture is that 'every step must be justified'.

Proof by algebraic reasoning.

This uses mathematical logic and uses well-established results to prove a conjecture or a theorem.

Example (1): The "Fibonacci" sequence is defined by the following rules:

- The first two terms are 1 and 1.
- Each subsequent term is generated by adding together the two previous ones.

The first terms of the sequence are 1, 1, 2, 3, 5, 8, 13 .....

A curious fact is that the fifth term of the sequence is 5, and the sum of the first six terms is

$1 + 1 + 2 + 3 + 5 + 8 = 20$, which is four times the fifth term.

This might seem unremarkable, but if we start with any numbers other than 1 and 1 and follow the same rules, the sum of the first six terms is still four times the fifth number !

Take the sequence 1, 3, 4, 7, 11, 18, 29 ....
The fifth term is 11, and the sum of the first six terms is 4 times 11, or 44 !

Prove that this rule holds true for all generalised Fibonacci sequences !

Call the first two terms of such a sequence p and q.

The table shows the first six terms, along with their total sum :

Term No.	Value
1	p
2	q
3	$p + q$
4	$p + 2q$
5	$2p + 3q$
6	$3p + 5q$
Sum of 6	$\mathbf{8p + 12q}$

The fifth term is $2p + 3q$ and the sum of the six terms is $8p + 12q = 4(2p + 3q)$.

Hence the sum of the first six terms of any generalised Fibonacci sequence is always four times the fifth term.

Example (2): By looking at the square numbers of 9, 25, 49 and 81, it can be seen that they all leave a remainder of 1 when divided by 4.

Prove that this fact holds true for all odd square numbers greater than 1.

Any odd number greater than 1 can be expressed as $2k + 1$ where k is a positive integer.
Squaring, we have $(2k + 1)^2 = 4k^2 + 4k + 1$.
Subtracting the remainder of 1 gives us $4k^2 + 4k$ which factorises into $4(k^2 + k)$

Hence $(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
This right-hand expression is a multiple of 4 plus a remainder of 1.

Example (3): Prove that $(n + 5)^2 – (n + 1)^2$ is a multiple of 8 for all integers n.

Here we must expand and simplify the expression, and then show that we can take out a factor of 8.

Proof:

$(n + 5)^2 – (n + 1)^2 = (n^2 + 10n + 25) – (n^2 + 2n + 1) = 8n + 24$
The last expression can have a factor of 8 taken out: $8n + 24 = 8(n + 3)$.

$\therefore$ $(n + 5)^2 – (n + 1)^2$ is a multiple of 8 for all integers n.

Proof by exhaustion.

This uses exhaustive testing when the set of results to be tested is finite. This is often used together with mathematical reasoning.

Example (4) (Harder): Prove that no square number ends in 2, 3, 7 or 8.

We begin by taking the squares of the integers from 0 to 9; they are 0, 1, 4, 9, 16, 25, 36, 49, 64 and 81.

Next, we can express any integer greater than 10 as $10m + n$ where m and n are integers, $m > 0$ and $0 \le n \le 9$.

Squaring $10m + n$ gives $100m^2 + 20mn + n^2 = 10m(10m + 2n) + n^2$.

The terms involving m are divisible by 10, and so the value of m will have no effect on the last ('units') digit in the square, i.e. $(10m + n)^2$ ends in the same digit as n^2.

(For example, the squares of 17, 27, 37,.... end in 9 because the square of 7 does so).

The squares of the integers from 0 to 9 have a 'units' digit of 0, 1, 4, 5, 6 and 9 – there are none ending in 2, 3, 7 or 8.

$\therefore$ no square number ends in 2, 3, 7 or 8.

Disproof by counterexample.

When we had to prove conjectures to be true in the earlier examples, we had to use exhaustive and rigorous methods. It is easier to disprove a statement, as a single counterexample suffices.

Example (5): Carl states : “The square of any number less than 1 is less than the number itself”.

Prove that Carl’s statement is incorrect.

At first, Carl’s reasoning seems sound enough. If we square one half, we obtain a quarter, and when we square 0.1, we have 0.01.

However, we run into difficulties when we consider zero or negative numbers:

The number -2 is less than 1, but its square is 4, which is greater than -2.
The square of 0 is 0.

Example (6): Prem has tabulated the values of $x^2 + x + 11$ for the first few positive integers:

x	$x^2 + x + 11$
1	13
2	17
3	23
4	31
5	41
6	53

He notices that all the calculated values are prime, and claims :
“The value of $x^2 + x + 11$ is prime for all positive x.”

Prove whether Prem’s claim is true or false

Substituting values x = 7, 8 and 9 gives $x^2 + x + 11$ = 67, 83, 101... , which are all prime.
Unfortunately if x = 10 , then $x^2 + x + 11$ = 121 which is not a prime, it being the square of 11.

Hence Prem’s conjecture is false.