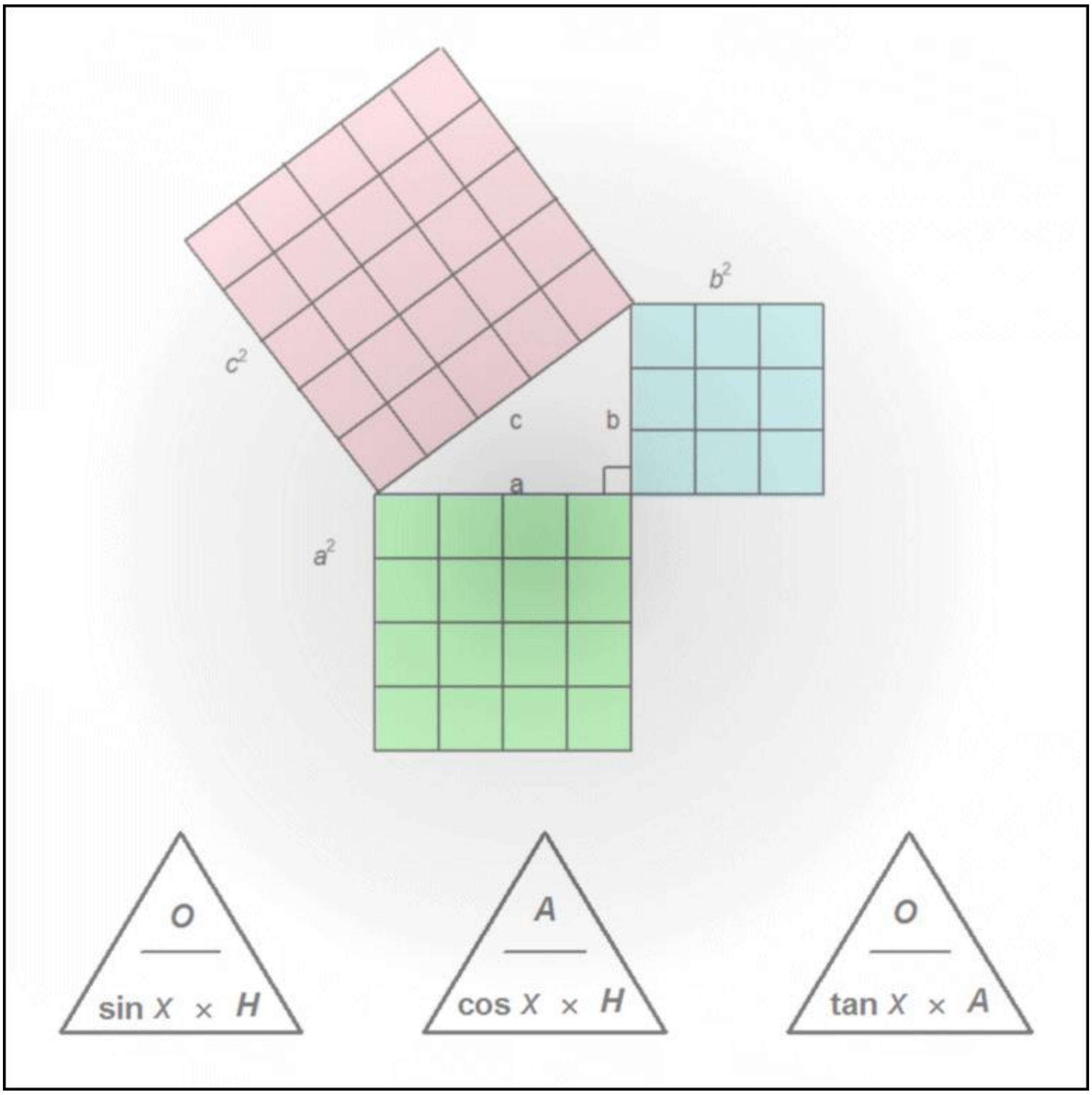
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

BASICS OF TRIGONOMETRY



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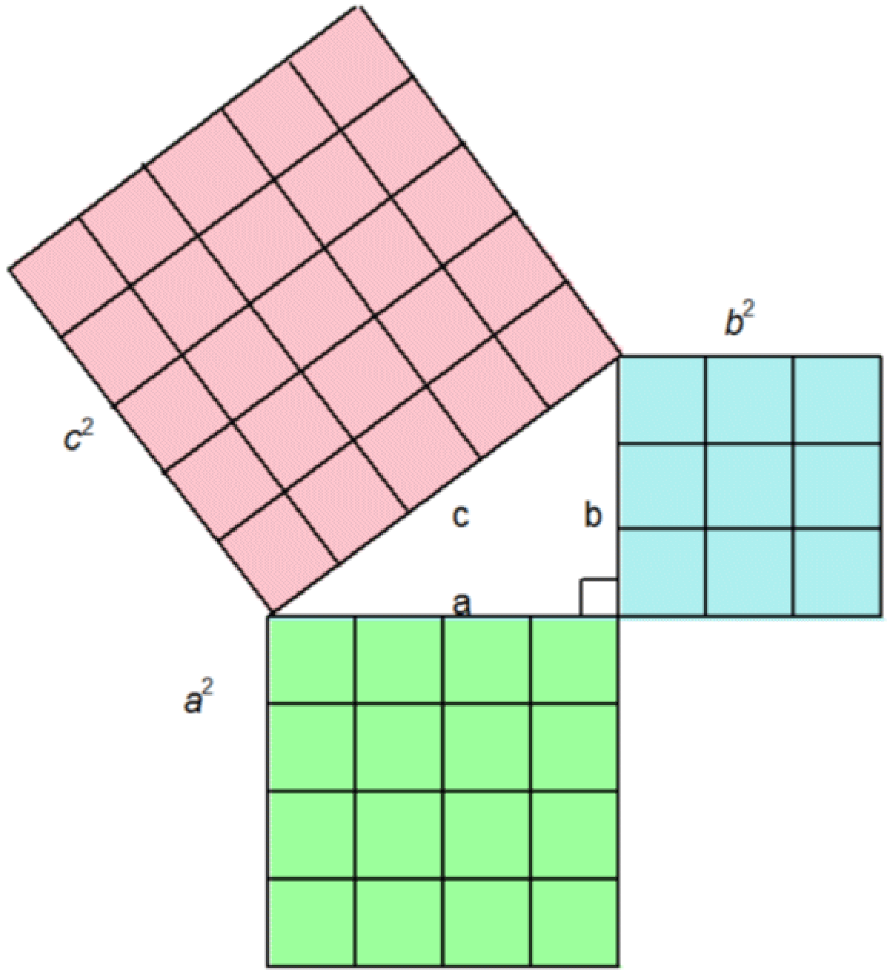
Basics of Trigonometry - Pythagoras' Theorem.



Pythagoras' theorem states that for any right-angled triangle, **the square on the hypotenuse is equal to the sum of the squares of the other two sides.**

The hypotenuse is the side opposite the right angle, and is always the longest side.

In the diagram above, it thus follows that $c^2 = a^2 + b^2$,
or $c = \sqrt{a^2 + b^2}$, where *c* is the hypotenuse.

The above form is used when the hypotenuse is unknown, but it can be adapted to find an unknown side when the hypotenuse is known.

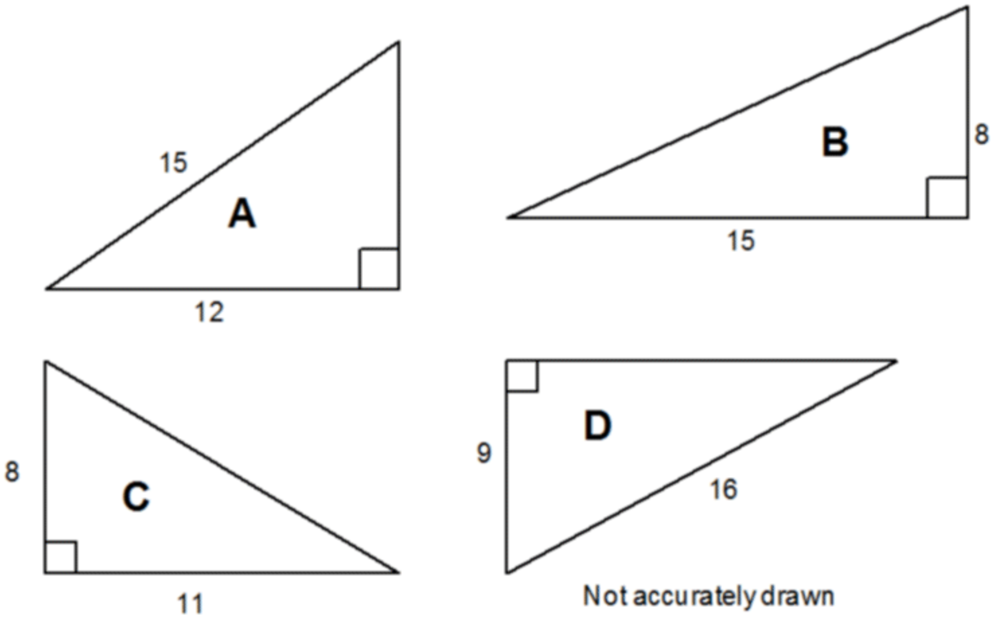
Then, $a^2 = c^2 - b^2$ or $a = \sqrt{c^2 - b^2}$ if *a* is unknown, and similarly

$b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ if *b* is unknown.

If the hypotenuse is unknown, we **add** the squares of the two other sides and take the square root.

If a shorter side is unknown, we **subtract** the smaller of the squares of the other two sides from the larger one and take the square root.

Example 1. Find the length of the missing side in each of the right-angled triangles below.



In triangle **A**, the hypotenuse is known, therefore the missing side has a length of

$\sqrt{15^2 - 12^2} = \sqrt{225 - 144} = \sqrt{81} = 9$ units

In triangle **B**, the hypotenuse is unknown, therefore its length is

$\sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$ units

In triangle **C**, the hypotenuse is again unknown, therefore its length is

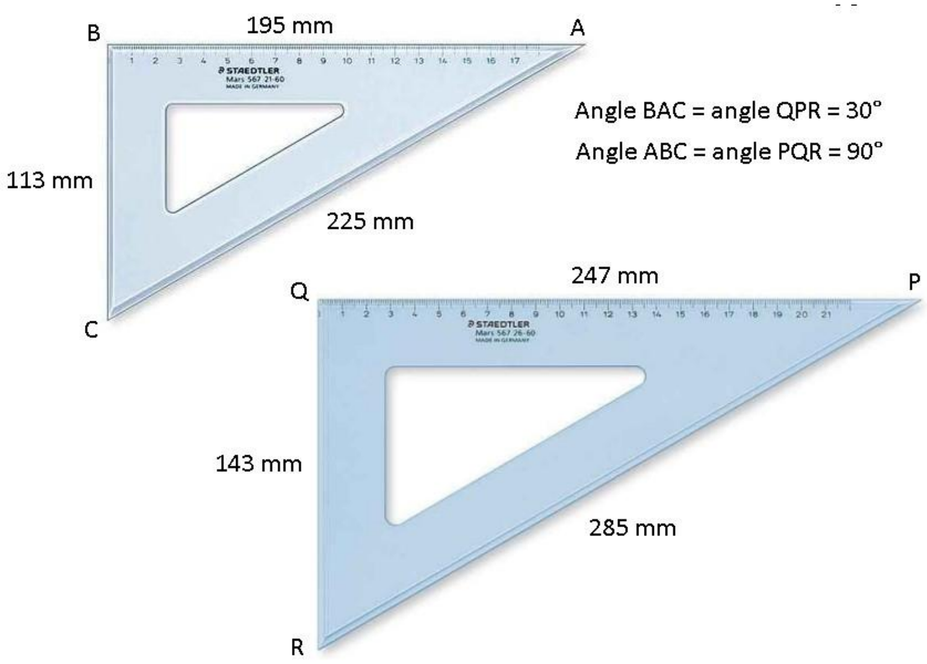
$\sqrt{11^2 + 8^2} = \sqrt{121 + 64} = \sqrt{185} = 13.6$ units, to 1 decimal place.

In triangle **D**, the hypotenuse is known, therefore the missing side has a length of

$\sqrt{16^2 - 9^2} = \sqrt{256 - 81} = \sqrt{175} = 13.2$ units to 1 decimal place.

More on right-angled triangles – introduction to the sine, cosine and tangent.

We have used Pythagoras' theorem to find the third side of a right-angled triangle, whilst given the other two. This next section shows how to find other sides and angles of a right-angled triangle, also called "solving" the triangle.



Two right-angled triangles are shown above, and are similar because all the angles of one are equal to all the angles in the other.
(These are also known as set-squares because they contain a right angle.)

As an activity, the lengths all the sides in each triangle were measured to the nearest millimetre.

Sides *AB* and *PQ* correspond, as do sides *BC* and *QR*, and sides *AC* and *PR*.

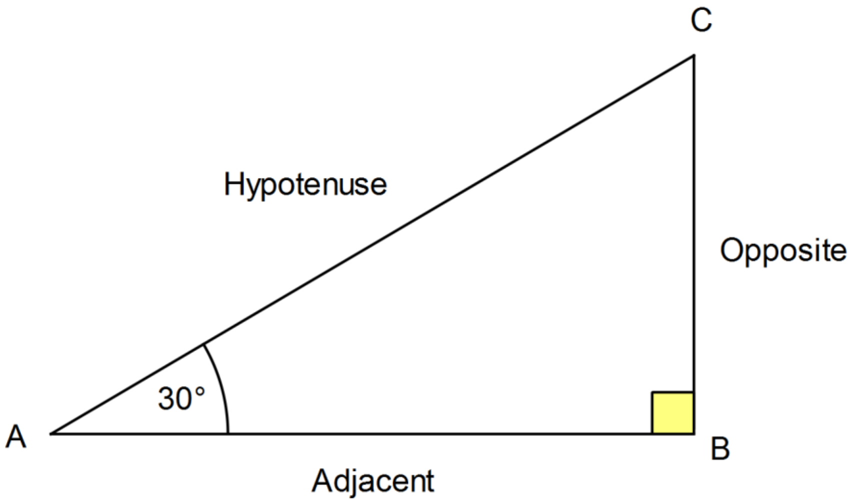
The next activity was to find the ratios between corresponding pairs of sides in each setsquare, and the following results obtained :

Ratio	Fraction	Decimal
BC / AC	113/225	0.50
AB / AC	195/225	0.87
BC / AB	113/195	0.58

Ratio	Fraction	Decimal
QR / PR	143/285	0.50
PQ / PR	247/285	0.87
QR / PQ	143/247	0.58

This last result shows that the ratios between the lengths of each pair of sides are not dependent on the lengths of the sides, but only on the angles of the triangles, as would be expected from similar triangles.

It is now time to label the sides of the triangle as follows :



As mentioned earlier, the side opposite the right angle is called the **hypotenuse**. In this case it is the side *AC* opposite angle *B*.

The side opposite the other labelled angle *A* is side *BC* and is termed the **opposite**.

The remaining side *AB*, containing the right angle and the angle *A*, is termed the **adjacent.**

The angle *C* is clearly 60° by subtraction, but had *that* angle been labelled instead of *A*, then side *AB* would be the opposite and side *BC* the adjacent.

Returning to the earlier results, we now have the following ratios relating to the 30° angle:

Ratio		Value (2dp)	Name
BC / AC	Opposite / Hypotenuse	0.50	sin 30°
AB / AC	Adjacent / Hypotenuse	0.87	cos 30°
BC / AB	Opposite / Adjacent	0.58	tan 30°

The ratios are called the **sine**, **cosine** and **tangent**, abbreviated to **sin**, **cos** and **tan**.

So for any angle *x*,

The sine of angle *x* (sin *x*) is the ratio: opposite ÷ hypotenuse

The cosine of angle *x* (cos *x*) is the ratio: adjacent ÷ hypotenuse

The tangent of angle *x* (tan *x*) is the ratio: opposite ÷ adjacent

In brief, **S** = $\frac{O}{H}$, **C =** $\frac{A}{H}$ **and T =** $\frac{O}{A}$ or **'SOHCAHTOA'.**

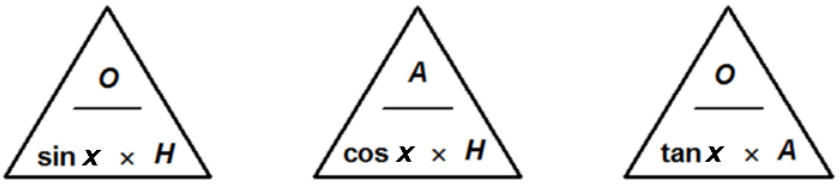
The formulae above can also be rearranged to find missing sides by changing the subject.

Hypotenuse = opposite ÷ sin *x* or adjacent ÷ cos *x*

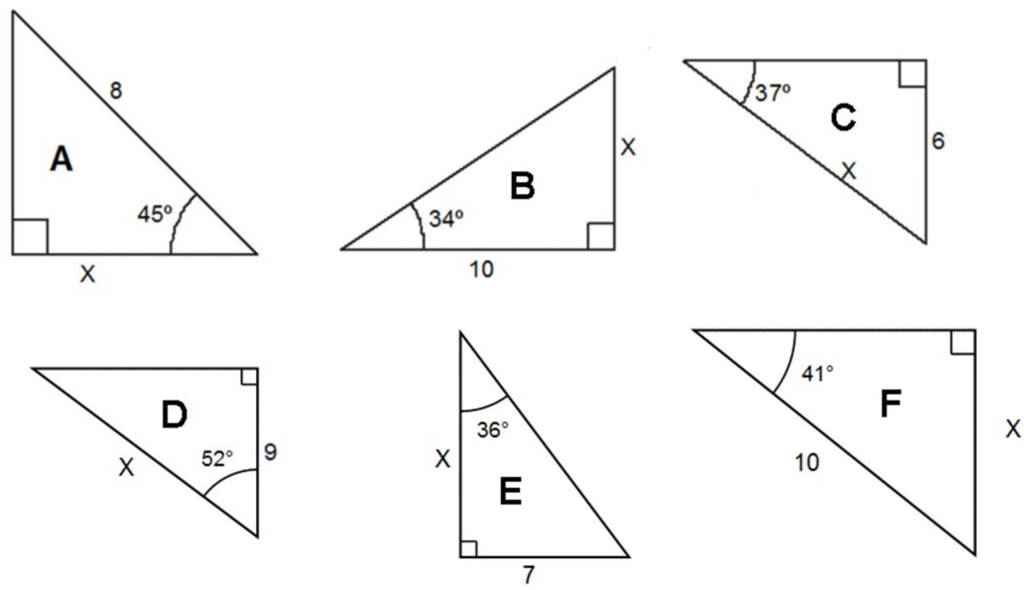
Opposite = hypotenuse × sin *x* or adjacent × tan *x*

Adjacent = hypotenuse × cos *x* or opposite ÷ tan *x*

The relationship can also be shown by means of formula triangles:



Example (2): Find the sides labelled X in the triangles below. All lengths are in centimetres. Give results to 2 decimal places.



Triangles not accurately drawn !

In triangle **A**, we are given the angle of 45° and the hypotenuse of 8 cm. Side X is not opposite the 45° angle, so it is the adjacent.

The length of X (the adjacent) is therefore (hypotenuse × cos 45°), or 8 cos 45°, or 5.66 cm.

We have the angle and the adjacent in triangle **B**, but we need to find the opposite. Use the formula opposite = (adjacent × tan 34°), giving the length of X as 10 tan 34° or 6.75 cm.

In triangle **C**, we are given the angle (37°) and the opposite (6 cm), but we are required to work out the hypotenuse X.

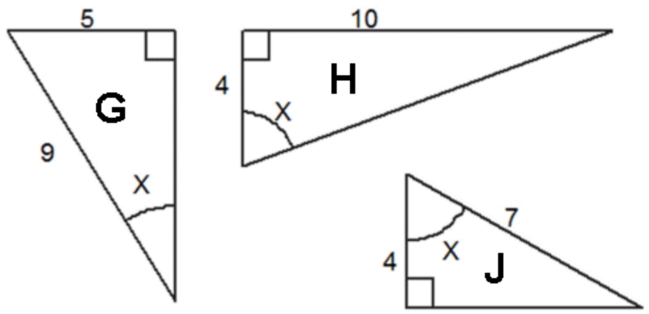
Hence the hypotenuse = $\frac{6}{\sin 37°}$, so X = 9.97 cm.

In triangle **D**, we are given the angle of 52° and the adjacent (9 cm),
so X = hypotenuse = $\frac{9}{\cos 52°}$ = 14.62 cm.

Triangle **E** has the opposite (7 cm) and the angle (36°) given,
so X = adjacent = $\frac{7}{\tan 36°}$ = 9.63 cm.

Finally, triangle **F** has the angle (41°) and the hypotenuse (10 cm) given,
so X = opposite = 10 sin 41°= 6.56 cm.

Example (3): Find the angles labelled X in the triangles below. All lengths are in centimetres. Give results to 1 decimal place.



Triangles not accurately drawn !

This time we are looking for angles, not sides, so we make use of **inverse trig functions**.

These functions are used to find an unknown angle given one of its ratios.
They are $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. For example, $\sin^{-1}(0.5) = 30°$ and $\tan^{-1}(1) = 45°$.

The input to $\sin x$, $\cos x$ and $\tan x$ is in each case an angle, and the output a number.
With the inverse functions, the input is a number and the output is an angle.

Triangle **G** has the opposite (5 cm) and hypotenuse (9cm) known, therefore $\sin X = \frac{5}{9}$,
in other words $X = \sin^{-1}\left(\frac{5}{9}\right) = 33.7°$.

The opposite (10cm) and adjacent (4cm) are known in triangle **H**, so $\tan X = \frac{10}{4}$,
or $X = \tan^{-1}\left(\frac{10}{4}\right) = 68.2°$.

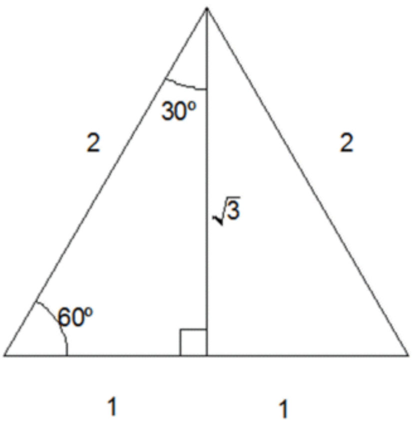
The adjacent and hypotenuse are known in triangle **J**, so $\cos X = \frac{4}{7}$,
i.e. $X = \cos^{-1}\left(\frac{4}{7}\right) = 55.2°$.

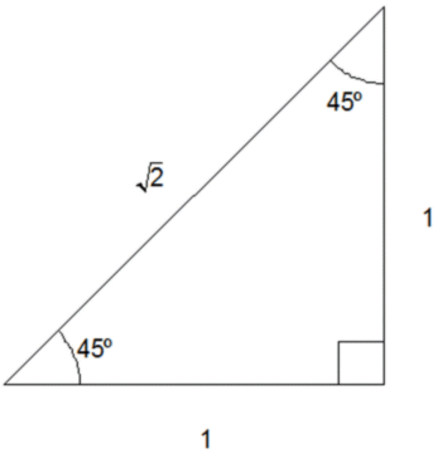
Trigonometric ratios of special angles.

The trigonometric ratios of certain angles can be deduced by using Pythagoras' theorem.

Notice how the equilateral triangle has been split into two congruent right-angled triangles, each with acute angles of 60° and 30°.

You will find both of these right-angled triangles in a standard geometry set. They are also known as set-squares.





Look at one of the right-angled triangles making up half of the equilateral triangle on the left.
Its hypotenuse is 2 units long and the base is one unit long. The vertical side is the missing short side, so its length is $\sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$ units.

In the case of the isosceles right-angled triangle shown right, the length of the hypotenuse is

$\sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$ units.

The standard results, which can be verified by SOHCAHTOA methods, are summarised as follows.

$\sin 30° = \cos 60° = \frac{1}{2}$

$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\tan 60° = \sqrt{3}$

$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

$\sin 90° = \cos 0° = 1$

$\cos 90° = \sin 0° = 0$

$\tan 0° = 0$

Note that tan 90° is undefined.