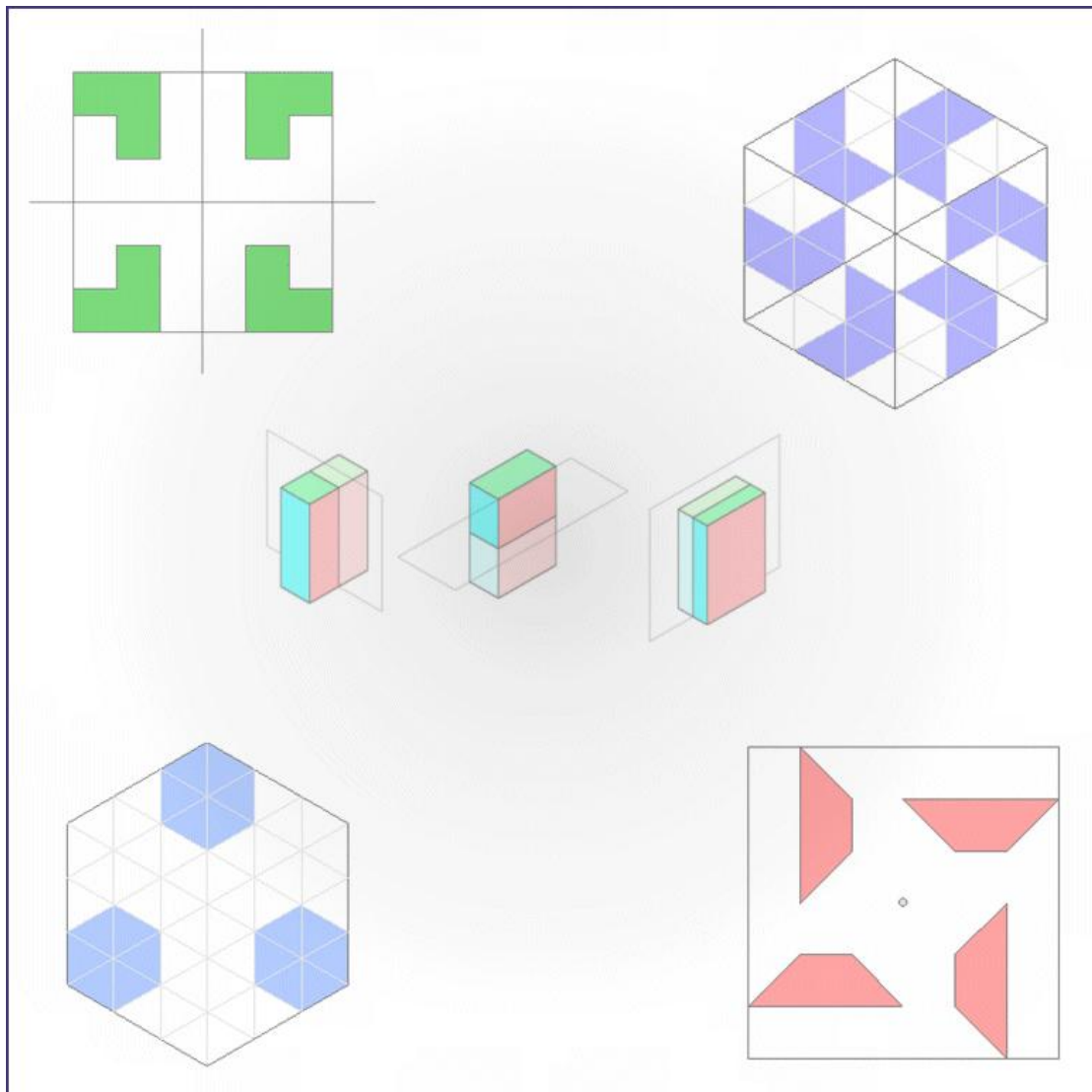# M.K. HOME TUITION

Mathematics Revision Guides

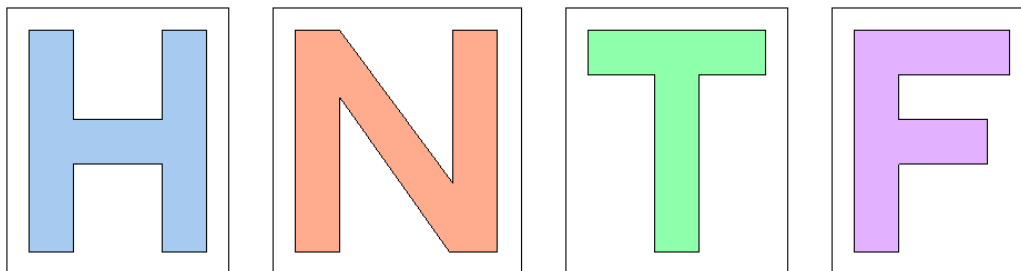Level: GCSE Foundation Tier

SYMMETRY

SYMMETRY

A figure is said to have **symmetry** if it has certain regular properties.

There are two types of symmetry with plane figures – reflective or 'mirror' symmetry, and rotational or 'turning' symmetry.

Examine the letters shown in the diagrams below.

Firstly, the letter F does not appear to have any symmetry at all.

The letter T can look unaltered by reflection in a mirror about its long bar.
It cannot, however, be turned through an angle of less than 360° and look unchanged along the way.

The letter N can look unchanged after a half-turn about its centre.
It cannot be reflected, as the original letter and its mirror image do not coincide.

The letter H can be reflected in a mirror both vertically and horizontally and appear unchanged.
It can also be rotated through 180° without altering its appearance.

Reflective symmetry.

A figure is said to have reflective symmetry if it looks unaltered by reflection in a mirror line.
This mirror line is a **line of symmetry.** See the example of the letter T below.

Line of symmetry of letter T
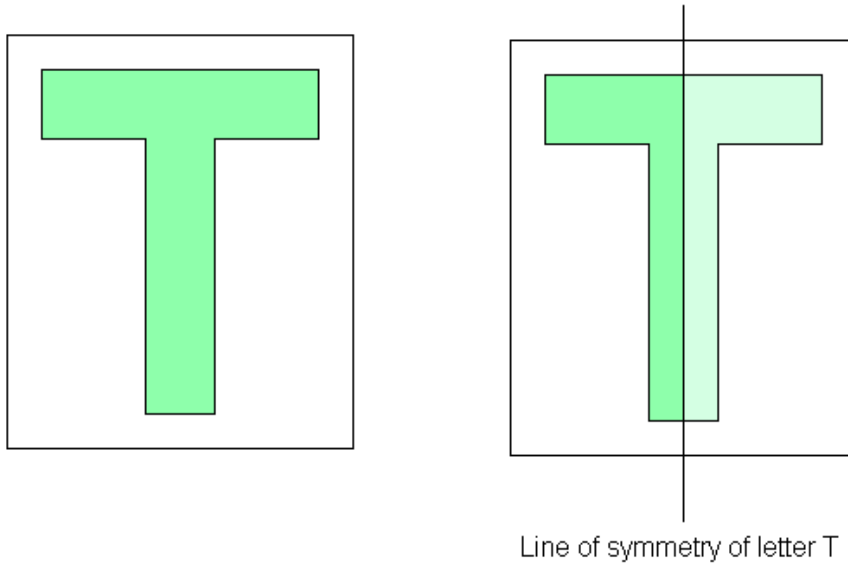
The letter H has two lines of symmetry, one vertical and the other horizontal:

Horizontal line
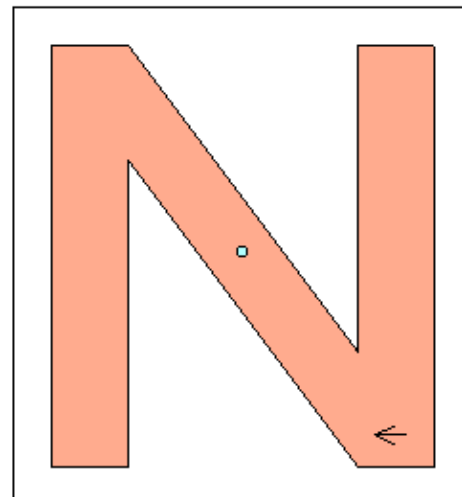of symmetry

Vertical line
of symmetry
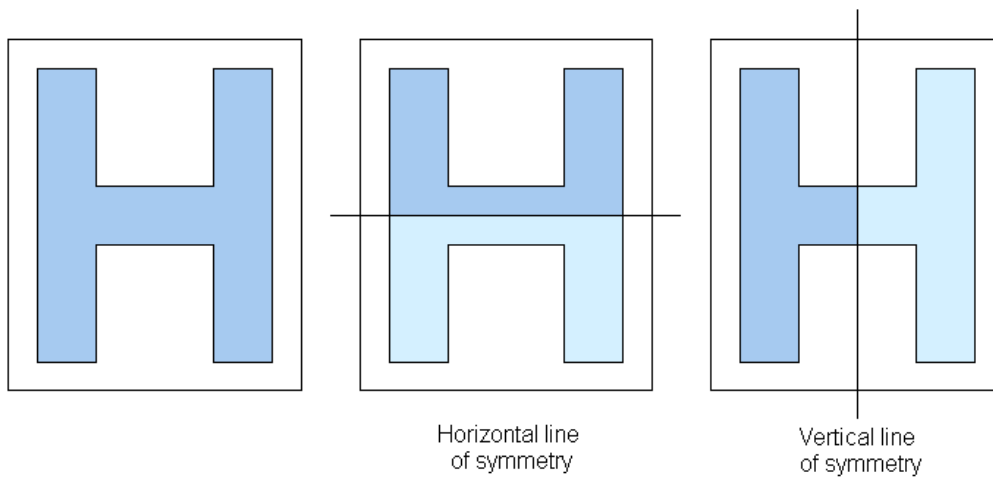
Rotational symmetry.

A figure is said to have rotational symmetry if it can be turned through an angle of less than 360° about a central point whilst looking unchanged.

See the example of the letter N below. The centre of rotation has been marked for reference, and an arrow mark placed on the letter.
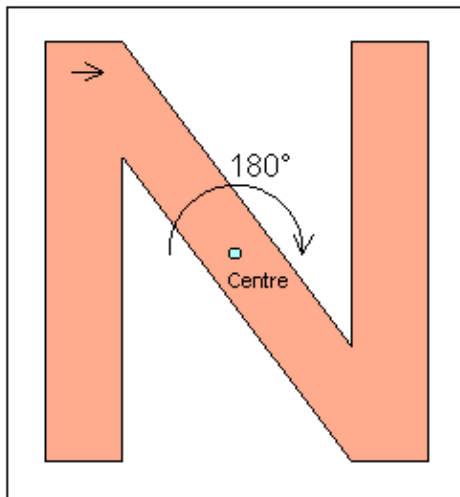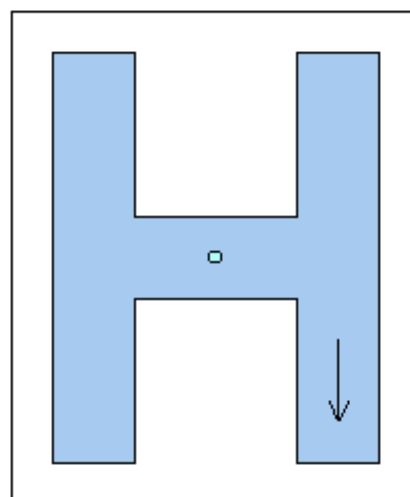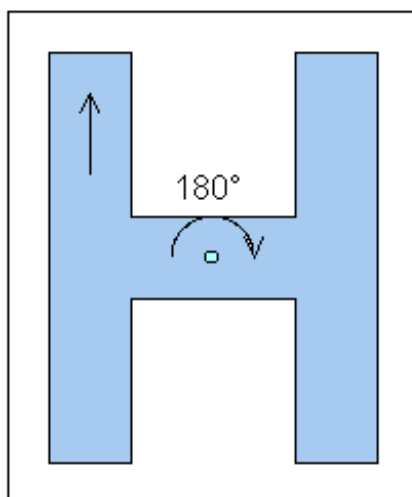

Letter N is rotated
through 180°

All points coincide, so letter N
has order-2 rotational symmetry

After rotating the letter N through 180° about the centre, all the points of the rotated letter coincide. Because we have rotated through half a circle, the order of rotational symmetry of the letter N is 2.
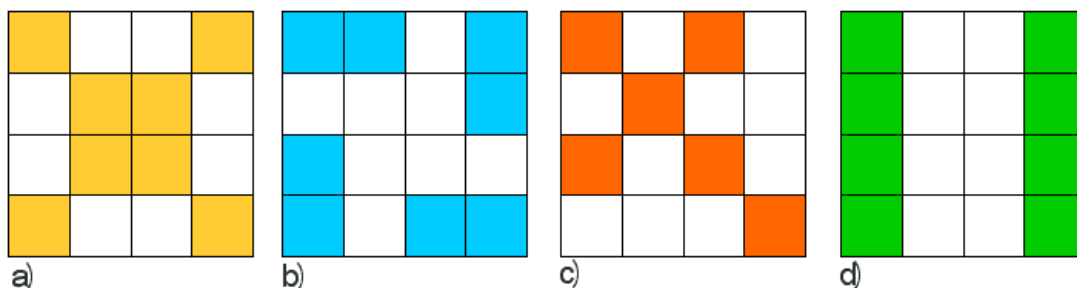
The letter H also has rotational symmetry of order 2.

By analogy, a figure has rotational symmetry of order 3 if it can be brought to coincidence three times in 360°, i.e. every 120° or one-third of a circle.

Likewise, if a figure can be brought to coincidence every 90° or quarter of a circle, it is said to have rotational symmetry of order 4.

Example (1): Describe the symmetries of the following patterns.

 Pattern (a) has rotational symmetry of order 4 in that it can be rotated through one, two or three quarter-turns and be brought to coincidence. It also has four lines of symmetry – horizontal, vertical and two diagonals.
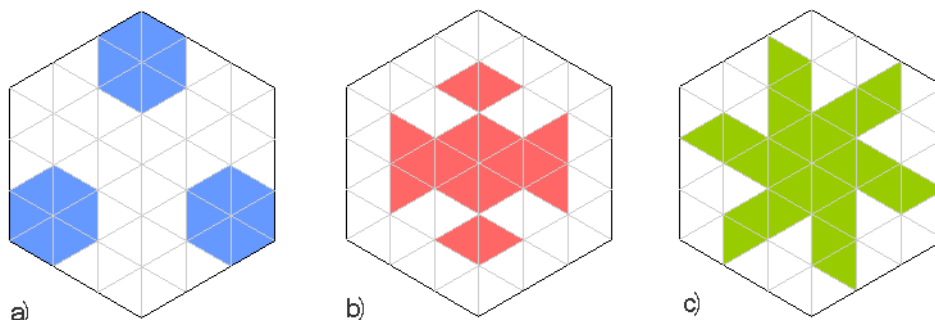
Pattern (b) likewise has rotational symmetry of order 4, but there is no line symmetry. The pattern and its mirror image (right) cannot be superimposed upon one another.
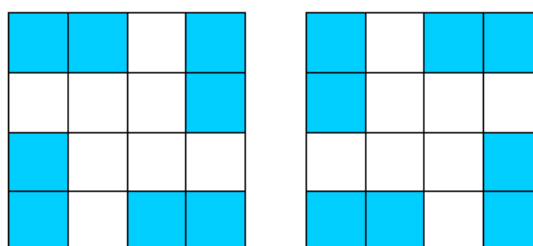
Pattern (c) has one line of symmetry , namely the main diagonal running top left to bottom right. There is no rotational symmetry though.

Pattern (d) has rotational symmetry of order 2 in that it can be rotated through one half-turn and be brought to coincidence. It also has two lines of symmetry – horizontal and vertical.

Example (2): Describe the symmetries of the following patterns.

Pattern (a) has rotational symmetry of order 3, and three lines of symmetry passing through opposite pairs of angles.

Pattern (b) has rotational symmetry of order 2, and two lines of symmetry – a vertical one through two angles and a horizontal one through the midpoints of the vertical sides.

Pattern (c) has rotational symmetry of order 6, but no lines of symmetry. Its mirror image cannot be superimposed upon the main pattern.

Symmetries of triangles, quadrilaterals and regular polygons are described in some detail in the document "Properties of Triangles and Quadrilaterals".

In general, a regular polygon has as many lines of symmetry, and also has rotational symmetry of the same order, as it has sides. A regular pentagon thus has 5 lines of symmetry, and also rotational symmetry of order 5.

Completing symmetrical patterns.

Many examination questions ask the student to complete a pattern so that the result has the requested symmetry.

Example (3): Shade in the figure on the right in such a way that the completed result has two lines of reflective symmetry (shown here) through the centre.
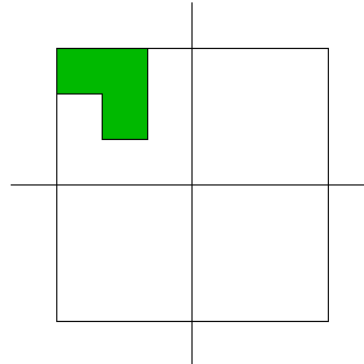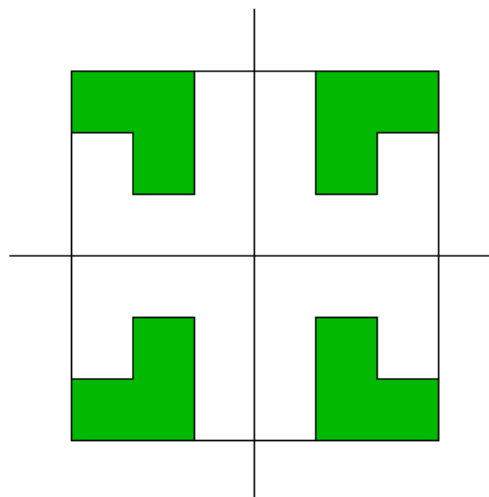
We begin by reflecting the shape in the vertical line. This can be done by drawing the figure and the line on tracing paper and turning the tracing paper over. We now have two L-shaped regions in the figure.

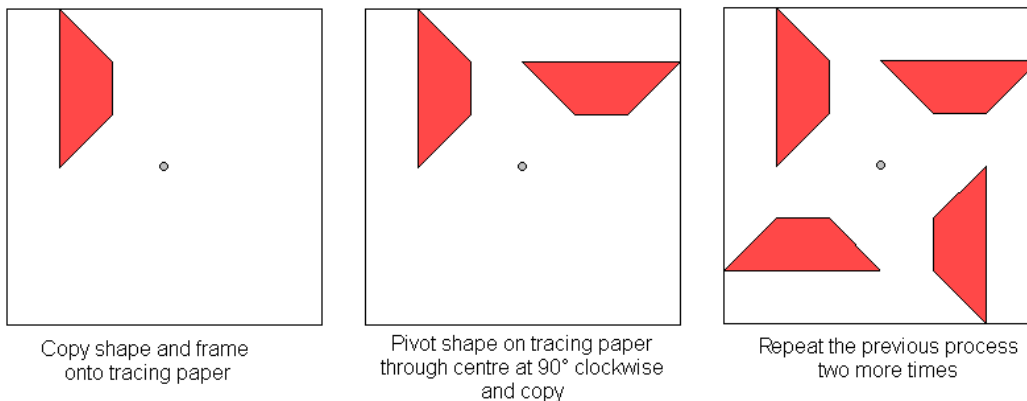Finally, we reflect the intermediate figure in the horizontal mirror line to obtain the result shown lower right.
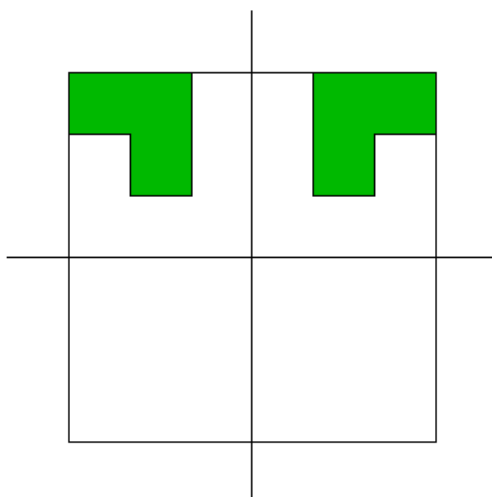
Reflect in vertical mirror line first

Finish by reflecting in horizontal mirror line

Example (4): Shade in the figure below left in such a way that the completed result has rotational symmetry of order 4 through the centre.

Copy shape and frame onto tracing paper

Pivot shape on tracing paper through centre at 90° clockwise and copy

Repeat the previous process two more times

Firstly we copy the shape and the centre point onto tracing paper. Because the completed result has rotational symmetry of order 4, we must pivot the shape on the tracing paper by a quarter-turn (90°) clockwise about the centre and copy the shape in its new position.

We then repeat the process two more times until we have the result shown in the right-hand diagram.

Example(5): Shade in the figure below left in such a way that the completed result has rotational symmetry of order 6 through the centre.

The work is similar to that in Example (3), but this time we pivot the traced shape six times around the centre, through an angle of one-sixth of a circle or 60° each time.


Original pattern

Completed result - rotational symmetry of order 6.
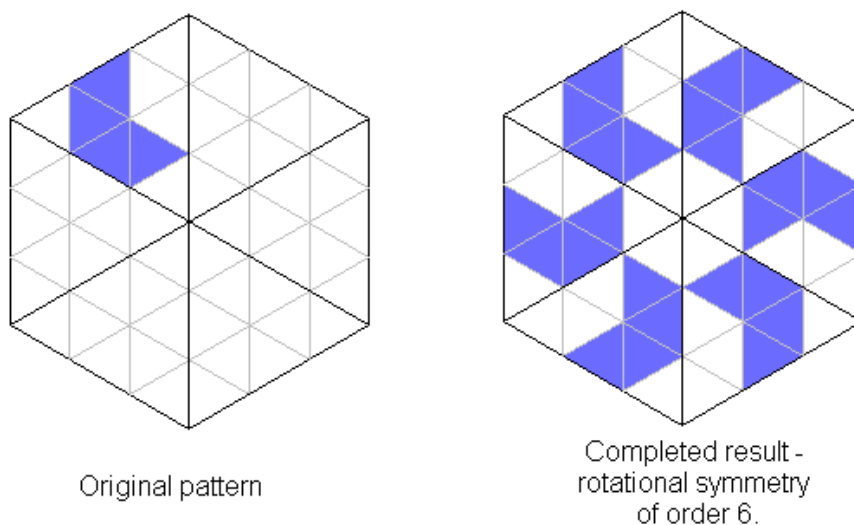
Example (6): Investigate the symmetries of the following symbols, logos and trade marks.
(Ignore lighting and shadow effects). Which two symbols have identical symmetry ?
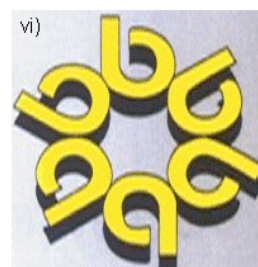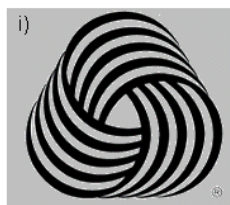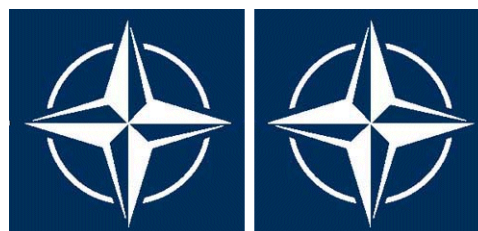

(All trademarks and logos are copyrighted to their respective owners.)

i) The Woolmark™ has rotational symmetry of
order 3, but no lines of symmetry. (Its mirror
image cannot be made to coincide with the
original.)
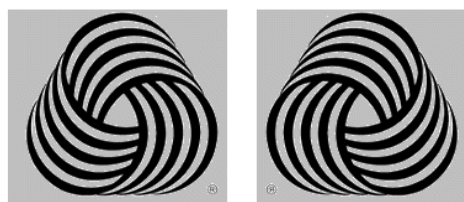
ii) The NATO badge has rotational symmetry of
order 4, but no line symmetry. Again, the original
figure and its mirror image cannot be superimposed.

iii) The Volkswagen™ badge has one vertical line of symmetry, but no rotational symmetry.

iv) The Mercedes-Benz™ badge has 3 lines of symmetry and also rotational symmetry of order 3.

v) The Manx symbol has rotational symmetry of order 3, but no lines of symmetry.

vi) The (altered) Bury Metro council logo has rotational symmetry of order 6, but no lines of
symmetry.

From the above results, the Woolmark™ and the Manx symbol have identical symmetry.

Symmetries of solid shapes.

The idea of lines of symmetry and rotational symmetry can be extended to include solid shapes.
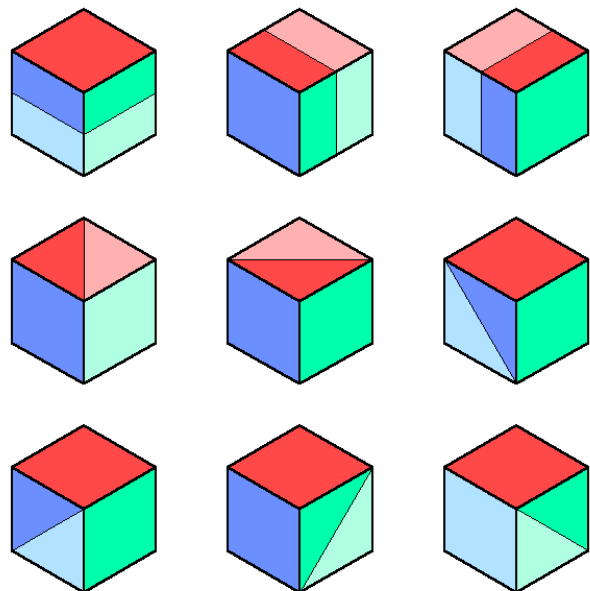
Planes of symmetry.

Just as a plane figure can have mirror lines of symmetry, so can a solid have mirror **planes** of symmetry dividing it into mirror-image halves.

Thus, a cuboid (shown below) has three planes of symmetry dividing it into two halves.
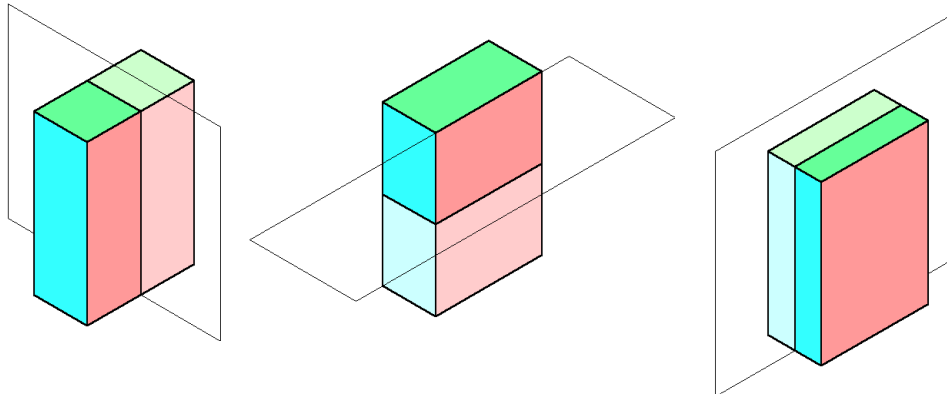
A **cube** has the same basic planes of symmetry as a cuboid, but since a square has symmetry about its diagonals as well, the cube has extra planes as per the diagram on the right.

There are now 9 planes of symmetry of a cube, if we include the six extra ones formed by the diagonals.

Example (7): Sketch one example of a plane of symmetry for: a) an equilateral triangular prism; b) a square-based pyramid; c) a cylinder; d) a cone.

a) All prisms have a plane of symmetry bisecting their side faces at right angles and parallel to their end faces, as on the left.

If the end faces also have any lines of symmetry, the prism also has planes of symmetry corresponding to those lines, as on the right.
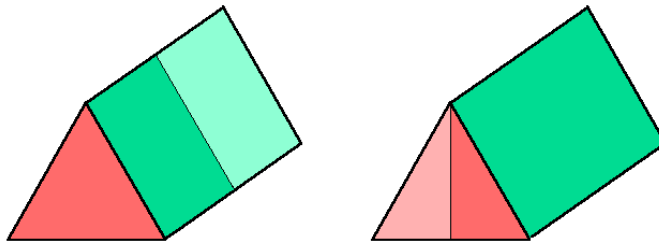
b) One plane of symmetry of a (shallow) square pyramid is shown here: it bisects a pair of opposite sides as per the shading.

Another one would pass through the vertex. and two edges.
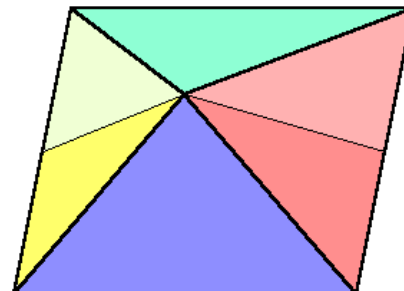
c) A cylinder is a special case of a prism with circular end faces. It has a plane of symmetry parallel to the end faces (see middle diagram). It also has an infinite number of planes passing through the diameters of the end faces (left).

d) A cone has one infinite set of planes of symmetry; it passes through the diameter of the base (not shown) and its vertex.
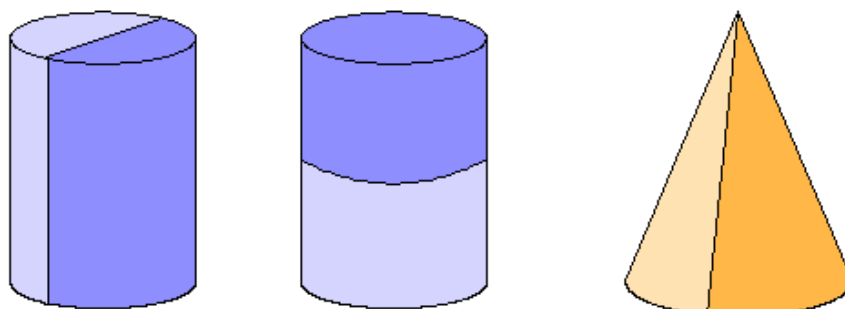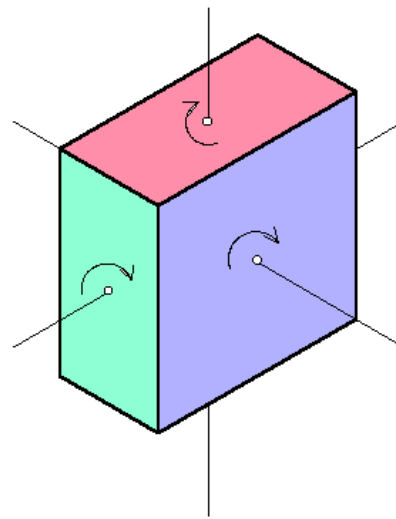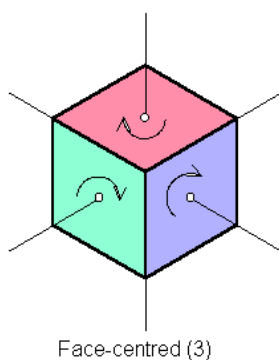
Axes of rotation.

Just as a plane figure can have rotational symmetry, so can a solid have **axes of rotation** whereby turning a solid through some fraction of a full turn would leave its appearance unchanged.

Taking the cuboid (shown right) we can see that there are three axes of rotation corresponding to the three planes.

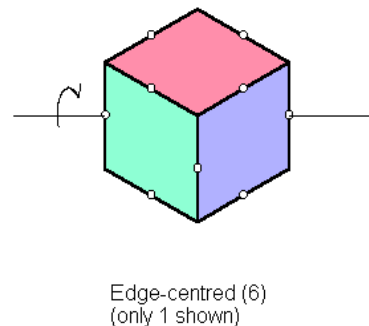We can rotate the cuboid through 180°about any of them and its orientation will remain unchanged.

The **cube** (below) has a larger number of axes of rotation than a cuboid because of its greater symmetry. In addition to the three face-centred axes of the cuboid, we have 4 axes passing through diagonally opposite pairs of vertices, and 6 axes passing through opposite edge midpoints.
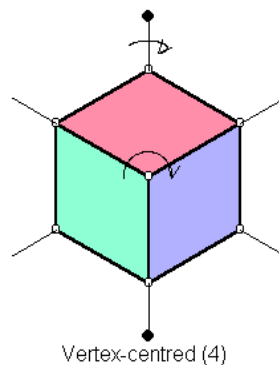
Face-centred (3)   Vertex-centred (4)   Edge-centred (6)
(only 1 shown)

Example (8): Sketch four solid shapes (other than a cube and cuboid). Include an axis of rotation in each case.

From left to right, we have a cylinder, a cone, an equilateral triangular prism and a square-based pyramid.

The prism has three possible rotations about its axis, and the square pyramid four.
The cylinder and cone can be said to have an infinite number of possible rotations.