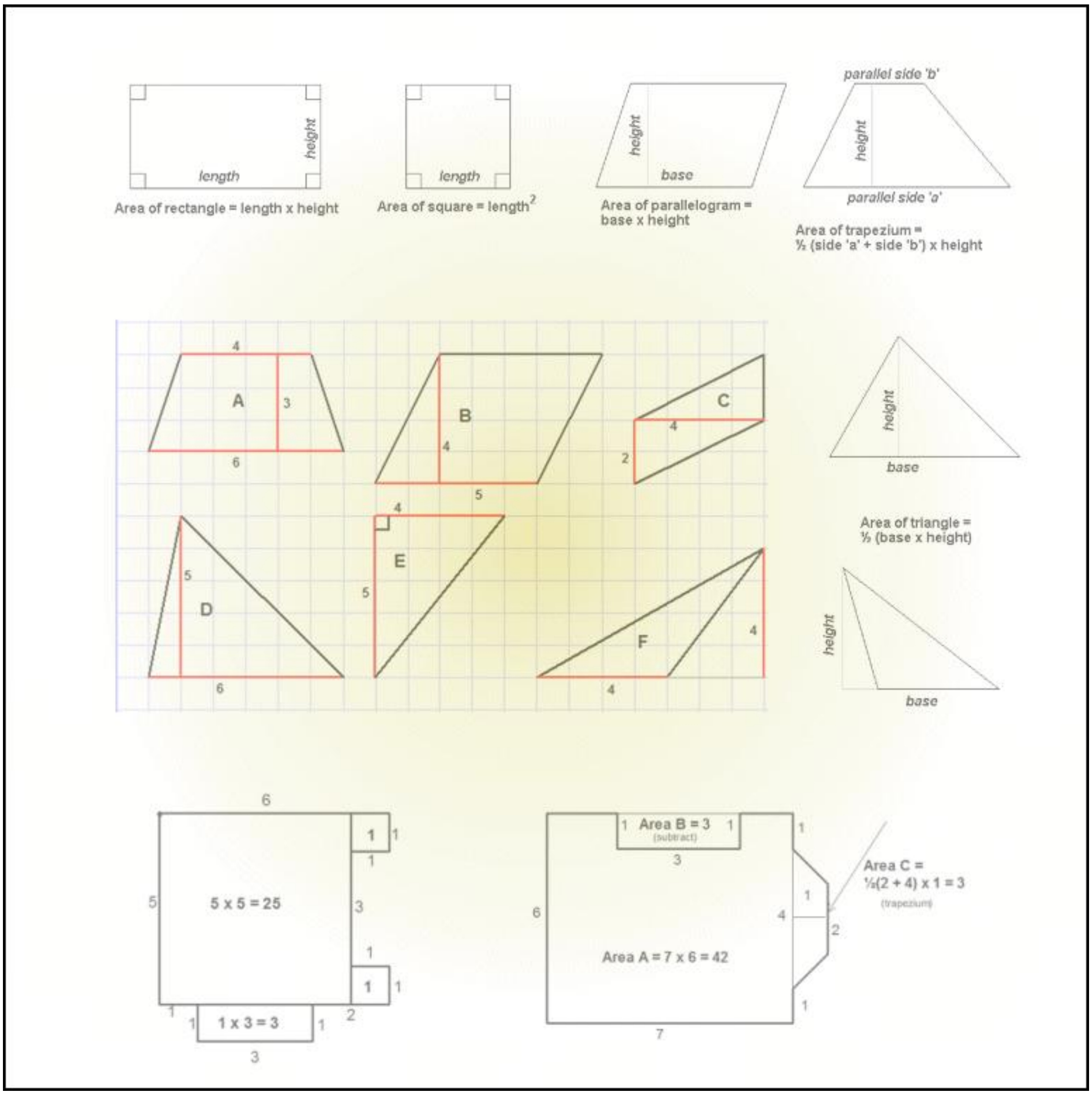
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

MEASURING SHAPES



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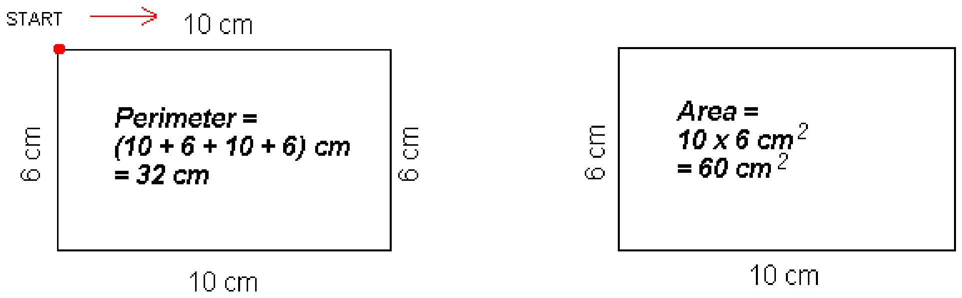
MEASURING SHAPES

Perimeter and area.

Those two terms are sometimes confused – the examples below will explain the difference.

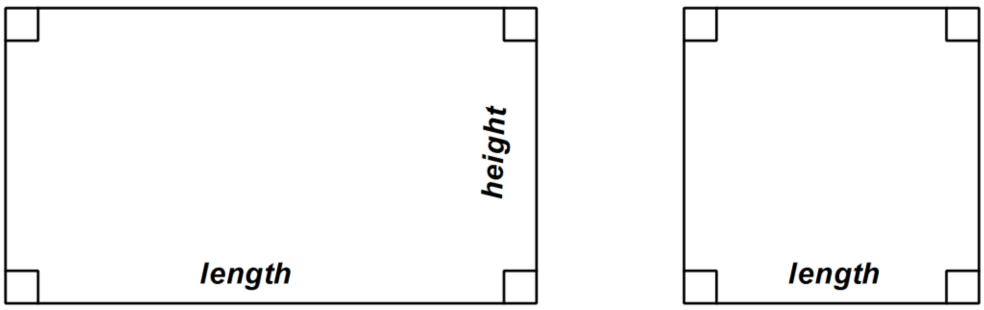


The **perimeter** of a shape is the distance around it, and is found by adding the side lengths together. Thus the perimeter of a rectangle measuring 10 cm by 6 cm is (10 + 6 + 10 + 6) cm or 32 cm. Since the opposite sides of a rectangle (or for that matter, any parallelogram) are equal in length, we can also say that the perimeter is $2 \times (10 + 6) = 32$ cm.

Notice that the perimeter is a **length** – it is measured in centimetres here.

The **area** of a shape is the amount of space it occupies in two dimensions – the 10 cm by 6 cm rectangle has an area of $10 \times 6 = 60$ cm^2. Notice that the area is measured in **square** centimetres.

The rectangle and square.



Area of rectangle = length x height **Area of square = length^2**

Example (1): Find the perimeter and area of i) a rectangle measuring 8 cm by 5 cm;
ii) a square of side 6 cm.

i) The rectangle has a perimeter of $2 \times (8 + 5) = 26$ cm, and an area of $8 \times 5 = 40$ cm^2.

ii) The square has four sides of 6 cm, so its perimeter is simply 6 cm $\times$ 4 , or 24 cm.
Its area is 6×6 cm^2, or 36 cm^2.

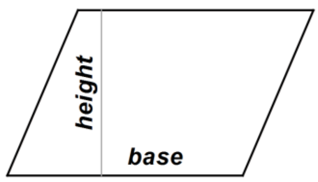
In general, the perimeter of a **regular** polygon is simply the side length multiplied by the number of sides. Thus a regular hexagon of side 5 cm has a perimeter of 30 cm.

(Areas of regular polygons other than squares are more complicated and not covered in the course.)

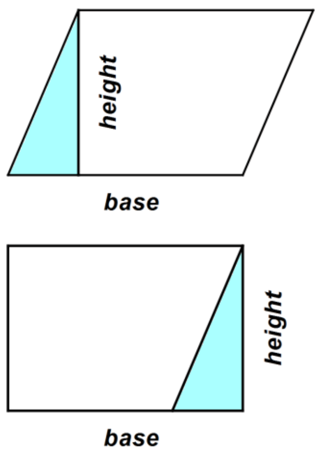
The parallelogram.

In the case of the parallelogram, the height is specifically the height **perpendicular** to the base, and not the length of the sloping side. The dissections in the right-hand diagrams demonstrate the formula.

(We remove a right-angled triangle from one side of the parallelogram and move it to the other).



Area of parallelogram = base x height

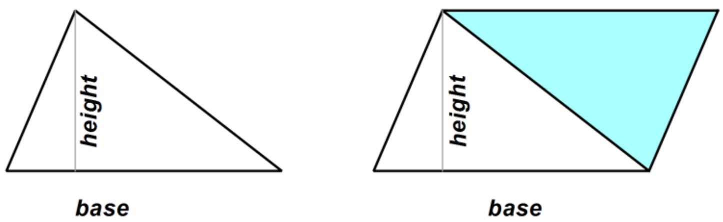


The triangle.

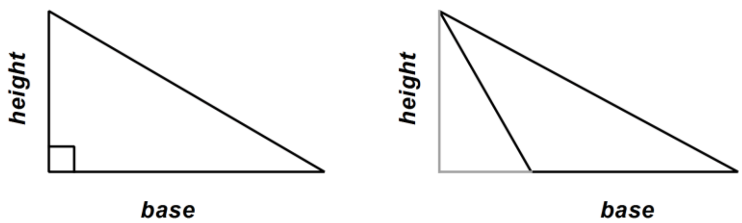
We again use the perpendicular height to measure triangles. The diagram upper right shows how we can derive the area formula from that of the parallelogram by "doubling up" the triangle.

The right-angled triangle is a special case, where the height and base correspond to the sides containing the right angle.

Note also how we need to extend the base of the obtuse-angled triangle to obtain the height.

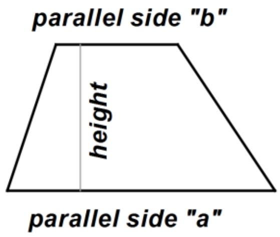


Area of triangle = ½ (base x height)



The trapezium.

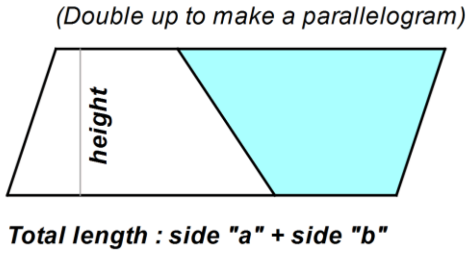
With the trapezium, we must take the mean (average) of the two parallel sides and not the others !
Again the height is perpendicular to the two parallel sides.



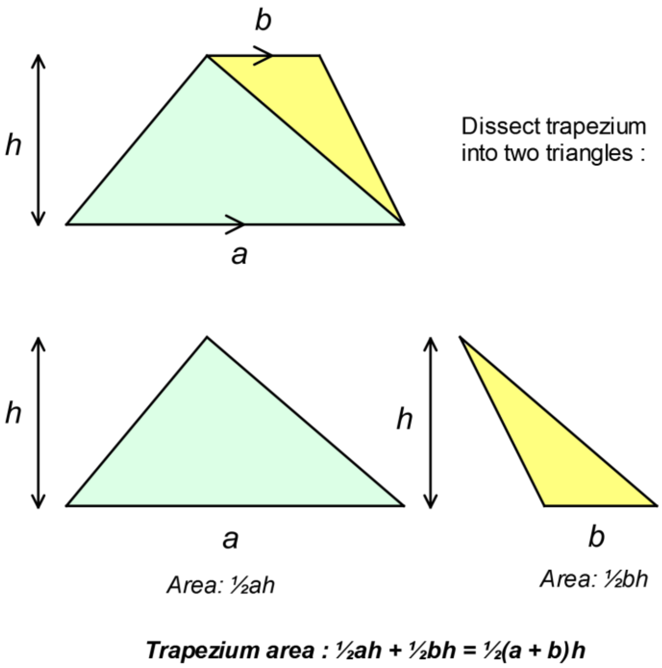
Area of trapezium =
½ (side "a" + side "b") x height

We can visualise this formula by joining together two identical trapezia to form a parallelogram. This resulting parallelogram has a base whose length is the sum of the two parallel sides and whose height is the same as that of the original trapezium.

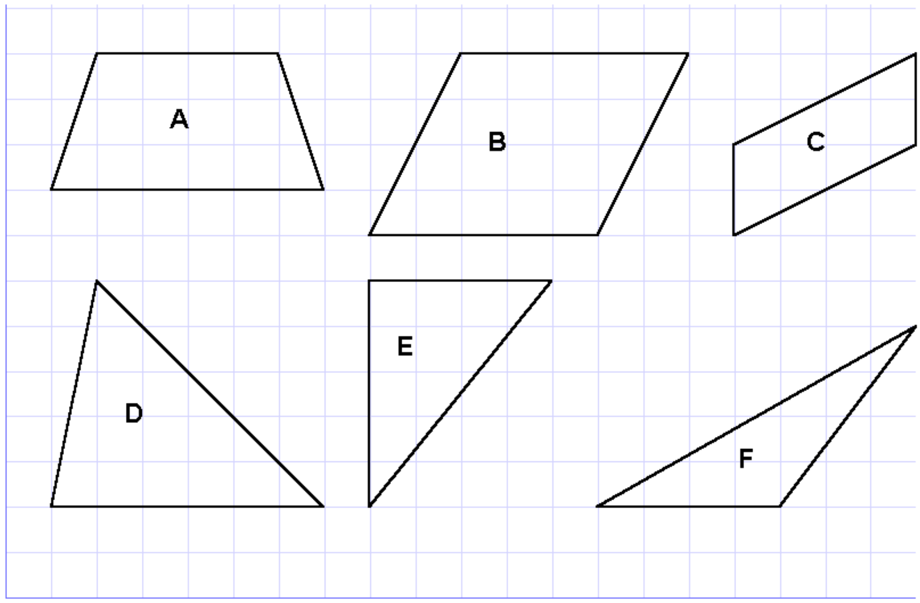
To find the area of the original trapezium, we have to halve the sum of the parallel sides before multiplying by the height.



Another dissection is shown below, where we use the triangle area formula.



Example (2): Find the areas of the shapes shown on the centimetre grid below.



Shape **A** is a trapezium whose parallel sides are 4cm and 6cm respectively, and whose height is 3 cm. Its area is therefore ½(4 + 6) × 3 cm^2 = 15 cm^2.

Shape **B** is a parallelogram of base 5 cm and height 4 cm, so its area is (5 × 4) cm^2 = 20 cm^2.

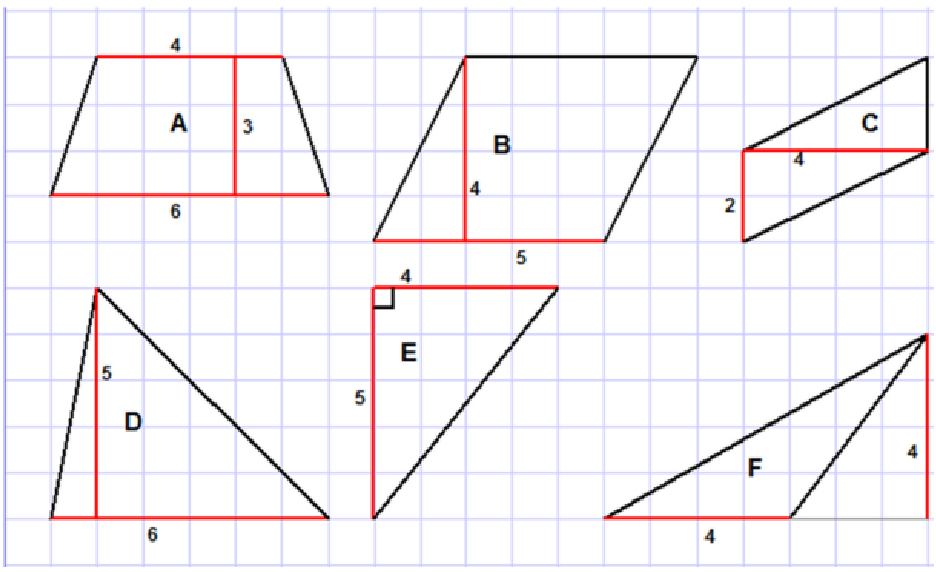
Shape **C** is also a parallelogram, but in this case it is better to treat the vertical as the base and the horizontal as the height. The base is 2 cm and the height 4 cm, so the area is 8 cm^2.

Shape **D** is a triangle of base 6 cm and height 5 cm, and so its area is ½(6 × 5) cm^2 = 15 cm^2.

Shape **E** is a right-angled triangle of base 4 cm and height 5 cm, and so its area is ½(4 × 5) cm^2 = 10 cm^2.

Shape **F** is a triangle of base 4 cm and height 4 cm, and so its area is ½(4 × 4) cm^2 = 8 cm^2.
(We need to extend the base here to find the height.)

See the diagram below for working.



Example (3): Find the areas of: i) a rectangle measuring 8 cm × 9 cm; ii) a square of side 7 cm; iii) a triangle of base 10 cm and perpendicular height 12 cm; iv) a trapezium of height 6 cm, and whose parallel sides are 7 cm and 15 cm long.

i) The area of the rectangle is (8 × 9) cm^2 = 72 cm^2.
ii) The area of the square is 7^2 cm^2, or 49 cm^2.
iii) The area of the triangle is ½(10 × 12) cm^2 = 60 cm^2.
iv) The area of the trapezium is ½(7 + 15) × 6 cm^2 = 66 cm^2.

Sometimes a question might quote the area or perimeter of a figure, and ask for one of the other measurements.

Example (4): Find the following:
i) the short side of a rectangle of area 42 cm^2 and long side of 7 cm
ii) the base of a parallelogram of area 36 cm^2 and perpendicular height 4 cm
iii) the perpendicular height of a triangle of area 40 cm^2 and base 8 cm
iv) the base of a triangle of area 48 cm^2 and perpendicular height 6 cm
v) the side of a square of area 100 cm^2

i) The short side of the rectangle is $\frac{42}{7}$ cm or 6 cm. (Divide area by long side.)

ii) The base of the parallelogram is $\frac{36}{4}$ cm or 9 cm. (Divide area by perpendicular height.)

iii) Half of the base of the triangle is 4 cm, so the perpendicular height is $\frac{40}{4}$ cm or 10 cm. (Divide area by one-half of the base.)

iv) Half of the base of the triangle is $\frac{48}{6}$ cm or 8 cm, so the base is 16 cm. (Divide area by perpendicular height and double.)

v) By inspection, the square root of 100 is 10, so the sides of the square are 10 cm long.

Example (5): The perimeter of a rectangle is 26 cm and its area is 40 cm^2. Find the lengths of the sides, given that they are a whole number of centimetres.

We are looking for two numbers which give 40 when multiplied together, and half of 26, or 13, when added together. Such a pair of numbers is 8 and 5, so the rectangle measures 8 cm × 5 cm.

(Check : area = 8 × 5 = 40; perimeter = 2 × (8 + 5) = 26.)

Example (6): A trapezium has an area of 72 cm^2 and a height of 8 cm. The longer of the two parallel sides is 11 cm long. Find the length of the shorter side.

The area of a trapezium is the mean of the two parallel sides multiplied by the height, so the mean of the parallel sides here is (area ÷ height) , or $\frac{72}{8}$ cm, i.e. 9 cm.

If the mean of the two parallel sides is 9 cm, their sum must be twice that, or 18 cm.
The longer parallel side is given as 11 cm, and so the shorter one must be 7 cm.

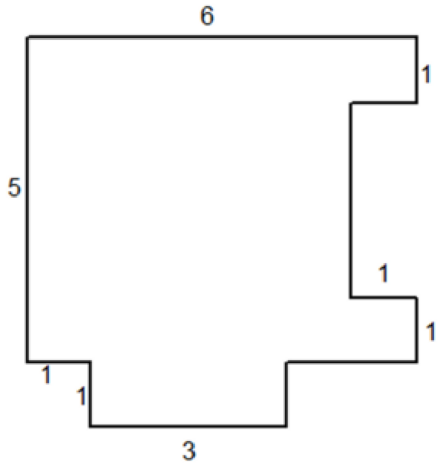
Finding perimeters and areas of compound shapes.

Many complex shapes can be broken up into simpler ones, such as rectangles and triangles, which makes area calculations easier.

Example (7): Find the perimeter and area of the room whose plan is shown on the right. All angles are right angles, and lengths are quoted in metres.

The first apparent problem here is that four of the lengths are missing.

We therefore label the corners on the diagram and use reasoning to find the missing sides.



To find the distance CD, we notice that it is parallel to EF and equal in length , ∴ **CD = 1 m.**

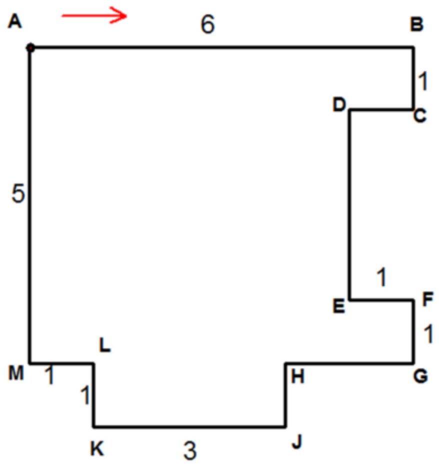
We see that AM = 5 m, so BC + DE + FG = 5 m
Since BC = 1 m and FG = 1m, **DE = 3 m.**

Also, we can see that AB = 6 m, and therefore
GH + JK + LM = 6 m. As JK + LM = 4 m , **GH = 2 m.**

Finally, HJ = KL = 1 m.

The perimeter of the room is therefore AB + BC + CD.... + MA, or

(6 + 1 + 1 + 3 + 1 + 1 + 2 + 1 + 3 + 1 + 1 + 5) metres, i.e. **26 metres**.



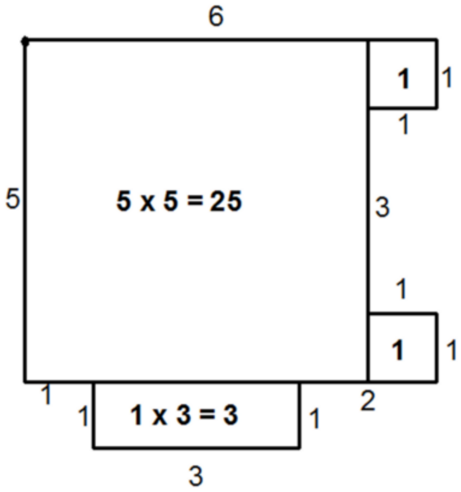
The area can be found by splitting the figure into rectangles.

The method shown right is one of many possible ones, and is probably the easiest at GCSE.

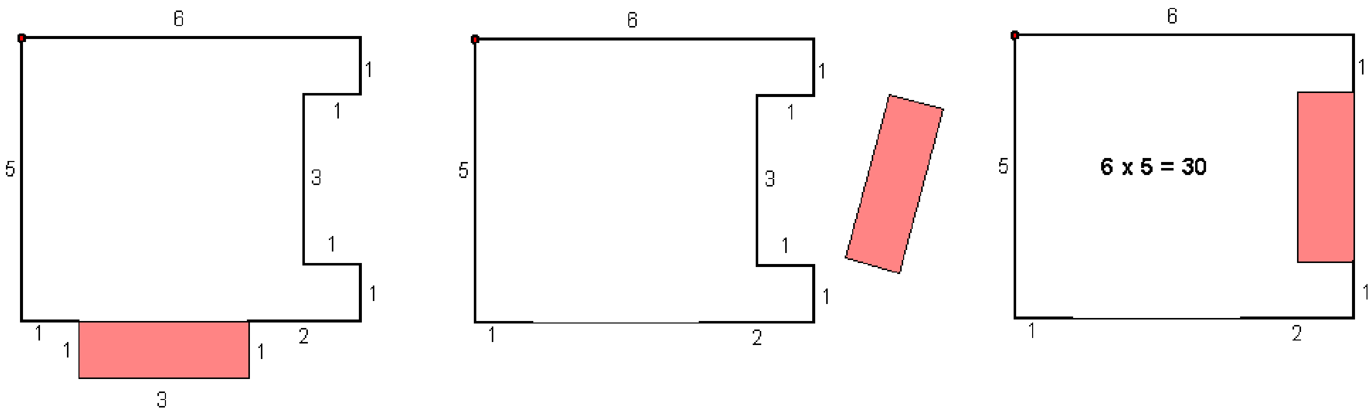
The largest section is a 5 × 5 m square, remembering that 1 m has been cut off the 6 m side.

The remaining sections are a 3 × 1 m rectangle and two 1 m squares.

The complete area of the room is 25 + 3 + 1 + 1 m^2 or **30 m^2.**



Another, more elegant, method is shown below:



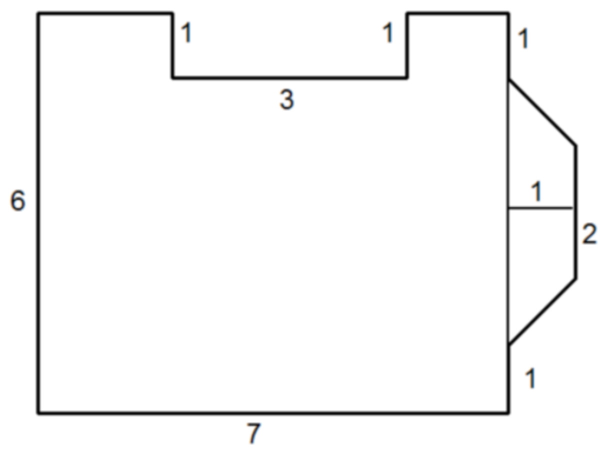
Example (8): Find the area of the bay-windowed room below (lengths in metres):

This time we have two unspecified lengths along the upper edge, the diagonal part of the bay, and the width of the bay itself.

The best way of looking at this shape is to visualise it as a 7×6 rectangle with a smaller 3×1 rectangle removed, plus a bay section.

The bay plus the two short 1 m sections must add up to the 6 m of the opposite wall, so the bay is 4 m at its maximum.

The bay is in fact a trapezium 1 m high and with parallel sides of 2 m and 4 m.

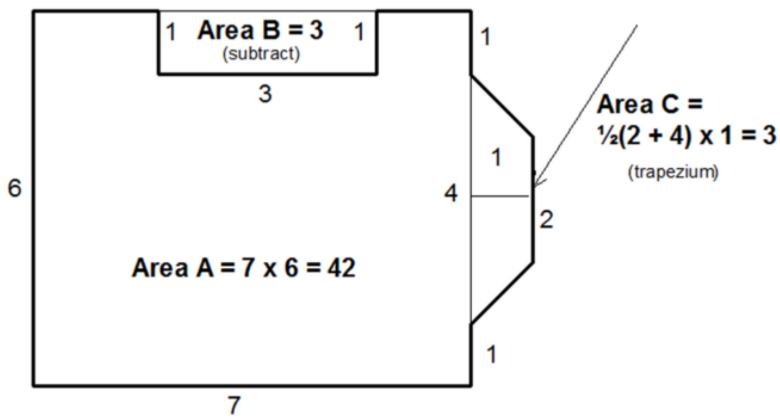


To work out the area, we first treat the main room as area A, namely a 7×6 m rectangle with area 42 m^2.

Next, we subtract the small area B, namely 3 m^2, and add back the bay area C, also 3 m^2.

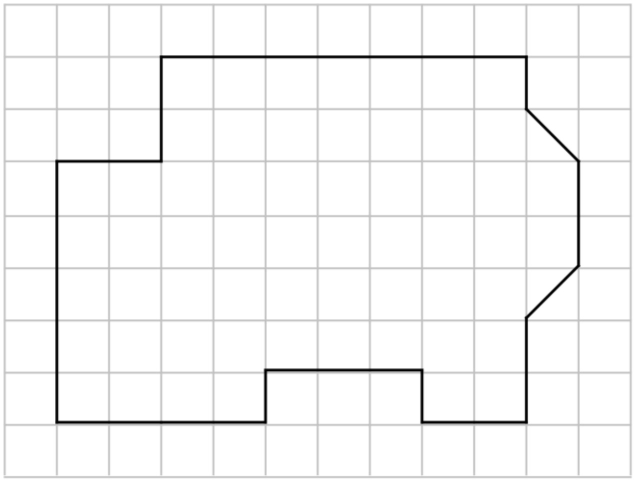
The total area of the room is (42 – 3 + 3) m^2 = 42 m^2.

Interestingly, the fact that the lengths along the upper edge are not specified does not prevent us from finding the area here !



Sometimes, exam questions will have the figure drawn accurately on a square grid.

Example (9): Find the area of the room shown in the plan on the right, on a square metre grid.

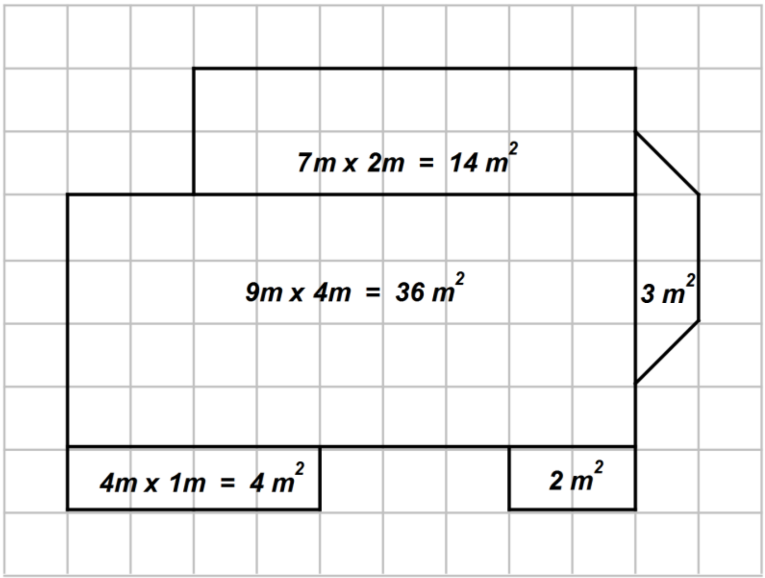


We can split the area up into rectangles (plus a trapezium) and then either count squares, or use actual lengths and apply area formulae.

By counting squares, we can see that there are 58 full squares and two half-squares.

Since we are using a square metre grid, the area of the room is 58 + (2 × 0.5) m^2 = **59 m^2**.

Alternatively we could have split the area into rectangles and a trapezium and used the standard area formulae. One way of dividing up the area is shown below.



The total area of the room is therefore (14 + 36 + 4 + 2 + 3) m^2 = **59 m^2**.

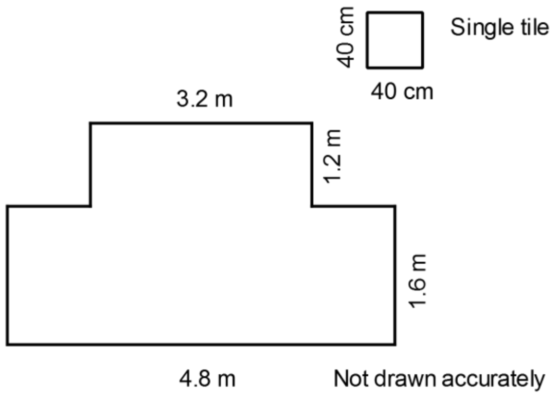
We use the trapezium area formula to find the area of the bay: it is ½(2 + 4) × 1 m^2 = 3 m^2 .

The method of dividing up into simpler shapes is necessary if the diagram has no grid, or is not drawn accurately.

Example (10): Julie wishes to tile her dining-room floor, as per the diagram on the right.

The tiles are square, measuring 40 cm × 40 cm.

Calculate how many tiles Julie needs to cover the floor.



The floor area can be divided up into two rectangles, a larger one of (480 × 160) cm and a smaller one of (320 × 120) cm. (We have converted metres to centimetres here.)

Since all the lengths in centimetres are whole-number multiples of 40 cm, we can divide each length by 40 to obtain the number of tiles for each rectangular area.

Since $\frac{480}{40} = 12$ and $\frac{160}{4} = 4$, the number of tiles needed to cover the larger rectangle is $12 \times 4 = 48$.

Also, as $\frac{320}{40} = 8$ and $\frac{120}{40} = 3$, the number of tiles needed to cover the smaller rectangle is $8 \times 3 = 24$.
Hence Julie needs 48 + 24, or **72**, tiles in total to cover the floor.

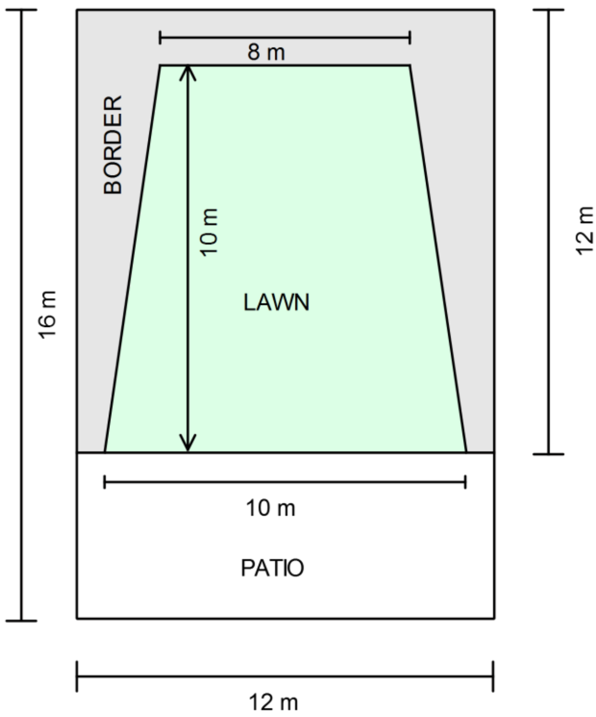
Example (11): The plan of a garden (not to scale) is shown on the right. The lawn is in the shape of a trapezium, and both the patio and the garden as a whole are rectangular.

Calculate the areas of the patio, the lawn and the border.

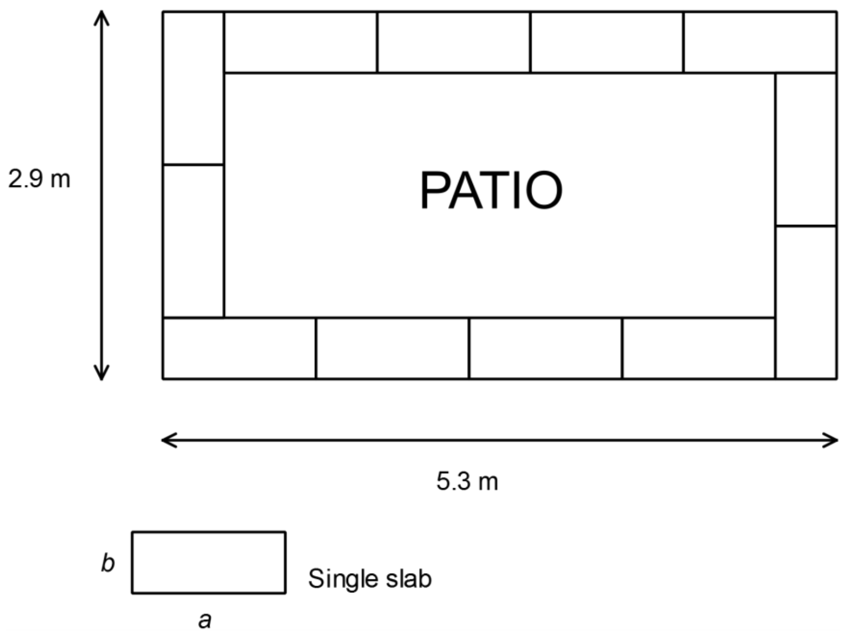
The area of the patio is $12 \times (16 - 12)$ m^2 = 48m^2.

The area of the lawn is $\frac{1}{2}(8 + 10) \times 10$ m^2 = 90 m^2.

The border and the lawn have a combined area of 12×12m^2, or 144 m^2, so we subtract the 90 m^2 area of the lawn to obtain the border's area of 54 m^2.



Example (12): Rakesh wants to border his rectangular patio with slabs all round, as illustrated on the plan below (not to scale), along with a single rectangular slab.



Find the lengths of the long and short sides of a single slab.

The long side of the bordered patio is 5.3 m in length, which is equivalent to four long sides '*a*' plus one short side '*b*' of a single slab. The length of the short side is 2.9 m, or two long sides '*a*' and one short side '*b*'.

From this information, we can set up simultaneous equations.

$4a + b = 5.3$ *A*
$2a + b = 2.9$ *B*

By subtracting equation *B* from equation *A* we can eliminate *b* :

$4a + b = 5.3$ *A*
$2a + b = 2.9$ *B*

$2a = 2.4$ *A-B* $\therefore$ $\mathbf{a = 1.2}$

The long side of the slab is 1.2 m long, so by substituting 1.2 for *a* in the first equation we have $2.4 + b = 2.9$, so $b = 0.5$.

The slabs therefore measure **1.2 m by 0.5 m.**

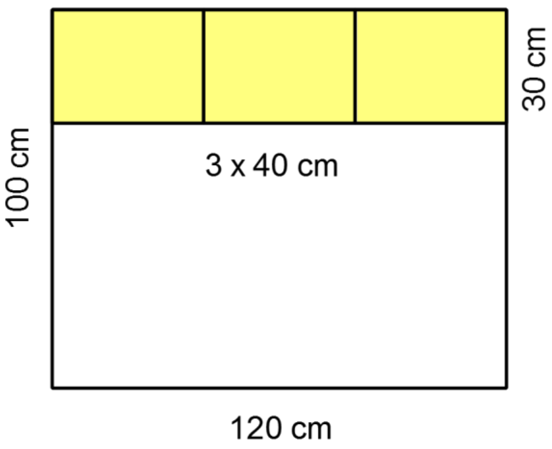
Example (13): A warehouse stores flat packs of carpet tiles measuring 40 cm × 30 cm. The packs are stored in layers on a pallet measuring 120 cm × 100 cm.

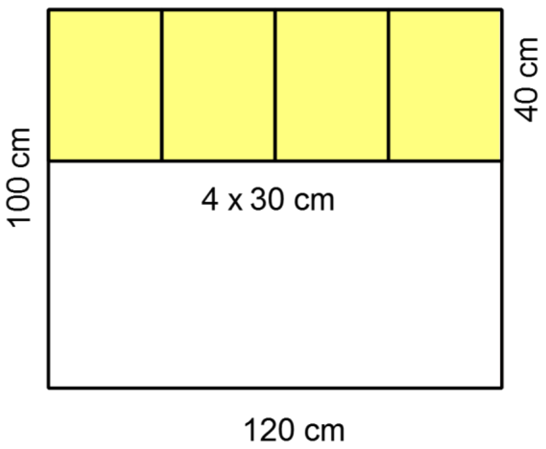
Show that ten such packs can completely cover the pallet to form a layer. You may use a diagram.

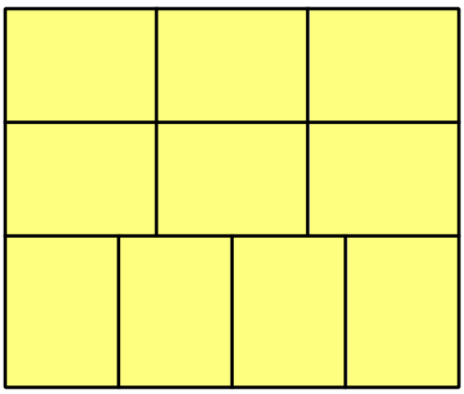
When we look at the dimensions of the pallet and the packs of tiles, we can see that both 30 and 40 are factors of 120, but that neither is a factor of 100.

This means that a long side of the pallet can have rectangular rows, but a short side cannot.

We can stack the cases to form rows on the pallet base in two distinct ways :





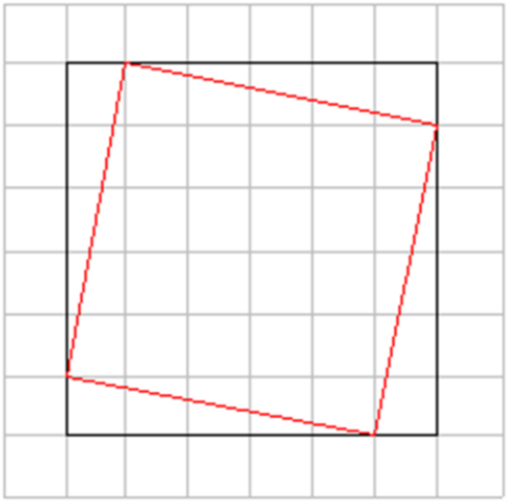


In the first case, we can place three packs to form a row measuring 120 cm by 30 cm; in the second case, the row measures 120 cm by 40 cm.

We need to find some combination of multiples of 30 and multiples of 40 that can add to 100 in order to find the required layer pattern. By trial and error, we find that $100 = (2 \times 30) + (1 \times 40)$, as the diagram on the right shows.

There are 3 packs in the first row from the top, 3 in the second and 4 in the third, making 10 packs per row in total.

Example (14): Find the area of the tilted square shown in the diagram on the right.
(Do not simply count squares).



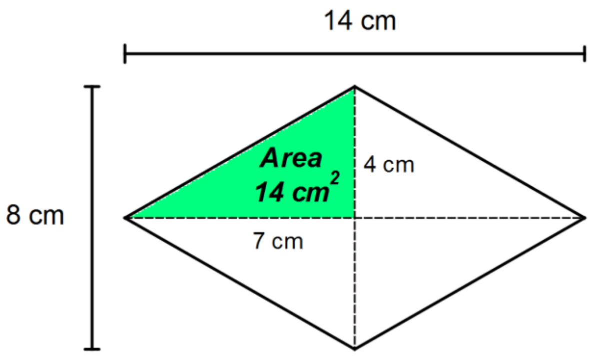
We can see that the tilted square is enclosed in a larger 6 × 6 square, and that there are 4 identical right-angled triangles making up the difference.

The height and base of each triangle are 1 unit and 5 units, so the area of each is ½ × 5 × 1 or 2½ square units.

The large square has an area of 36 square units, and the 4 triangles have a combined area of 10 square units, therefore the tilted square has an area of (36 – 10) or 26 square units.

In certain cases, the area of a plane figure can be determined from the lengths of the diagonals alone.

Example (15): Find the area of a rhombus whose diagonals are 8 cm and 14 cm long.



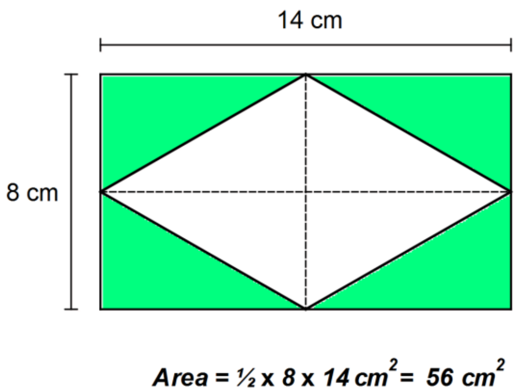
(Remember that the diagonals of a rhombus bisect each other at right angles.)

Although a rhombus is a type of parallelogram, the base × height formula cannot be used here since we are not given either.

The rhombus can be broken up into four right-angled triangles, each of which has a base of 7 cm and a height of 4 cm (i.e. half the diagonal length).

The area of a triangle is ½ (base × height), so here one small triangle has an area of 14 cm^2, and thus the entire rhombus has an area of **56 cm^2.**

This is also half the product of the diagonals, as can be shown below :



We can enclose the rhombus in a rectangle whose sides are parallel to the diagonals of the rhombus.

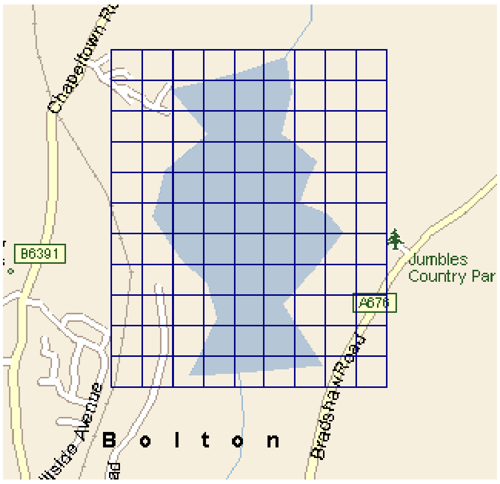
The sides of the rectangle are equal in length to the diagonals of the rhombus.

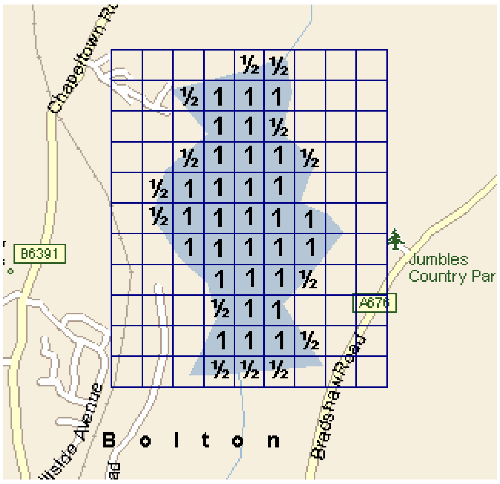
The unshaded triangles making up the rhombus have the same combined area as the shaded ones completing the rectangle.

The method of halving the product of the diagonals to find the area of a rhombus can also be used to find the area of a kite.

Estimating areas of irregular shapes.

Example (16): Estimate the area (in km^2) of Jumbles Reservoir using the map and grid below. One small square on the grid = 100m × 100m or 0.01 km^2.





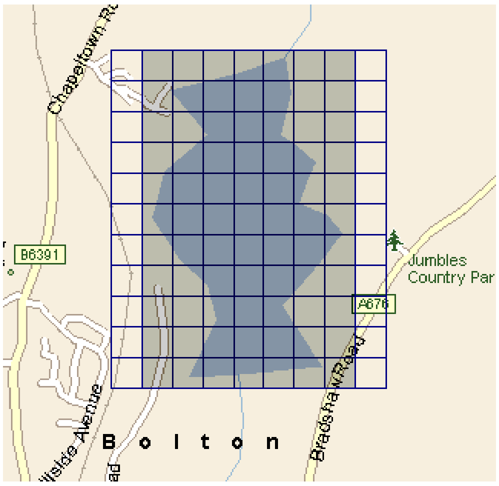
One method of estimating the area is shown upper right.

We simply count the number of complete grid squares – here one such square equals 0.01 km^2.
Next, we count the number of half-complete grid squares - here one such square equals 0.005 km^2.

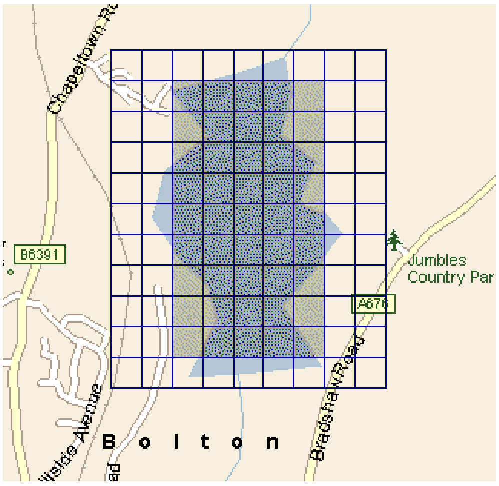
(Squares which are less than half-full are ignored – also squares which are nearly full are counted as full.)

There are 30 full squares and 14 half-squares marked in the right-hand diagram, so the approximate area of Jumbles Reservoir is (30 × 0.01 km^2)+ (14 × 0.005 km^2) or (0.3 + 0.07) km^2 , i.e 0.37 km^2 .

This method of finding the area of an irregular shape is a little long-winded despite its accuracy, and so a quicker one is to enclose the area in a rectangle.



Shaded area = 77 squares
equivalent to 0.77km^2



Shaded area = 45 squares
equivalent to 0.45km^2

In the diagram on the left, the rectangle has been chosen so that no part of the reservoir is omitted, giving an area of 0.77 km^2 . This figure is far too high, because there are many squares containing no part of the reservoir at all.

We must therefore select the rectangle in such a way that the area of reservoir outside the rectangle balances the area of dry land inside the rectangle.

The diagram on the right is an improvement, reducing the estimate to 0.45 km^2. This is probably the best we could do using whole squares, although there is still more dry land inside the rectangle than water outside it. We therefore reduce our estimate slightly to 0.4 km^2.

Example (17): Estimate the area of Heaton Park, Manchester, from the map below left.

This time there is no convenient grid to work off, so we draw a rectangle to enclose the area of Heaton Park as best we can.

In the diagram on the right, we have tried to equalise the areas of parkland outside the frame and non-parkland inside the frame.

By using the scales on the map, we can deduce that the area of Heaton Park is about 1.7×1.5 km^2, or 2.55km^2. In practice, such measurements are not that exact, so we only use 2 figures and say that the area is about 2.5km^2.