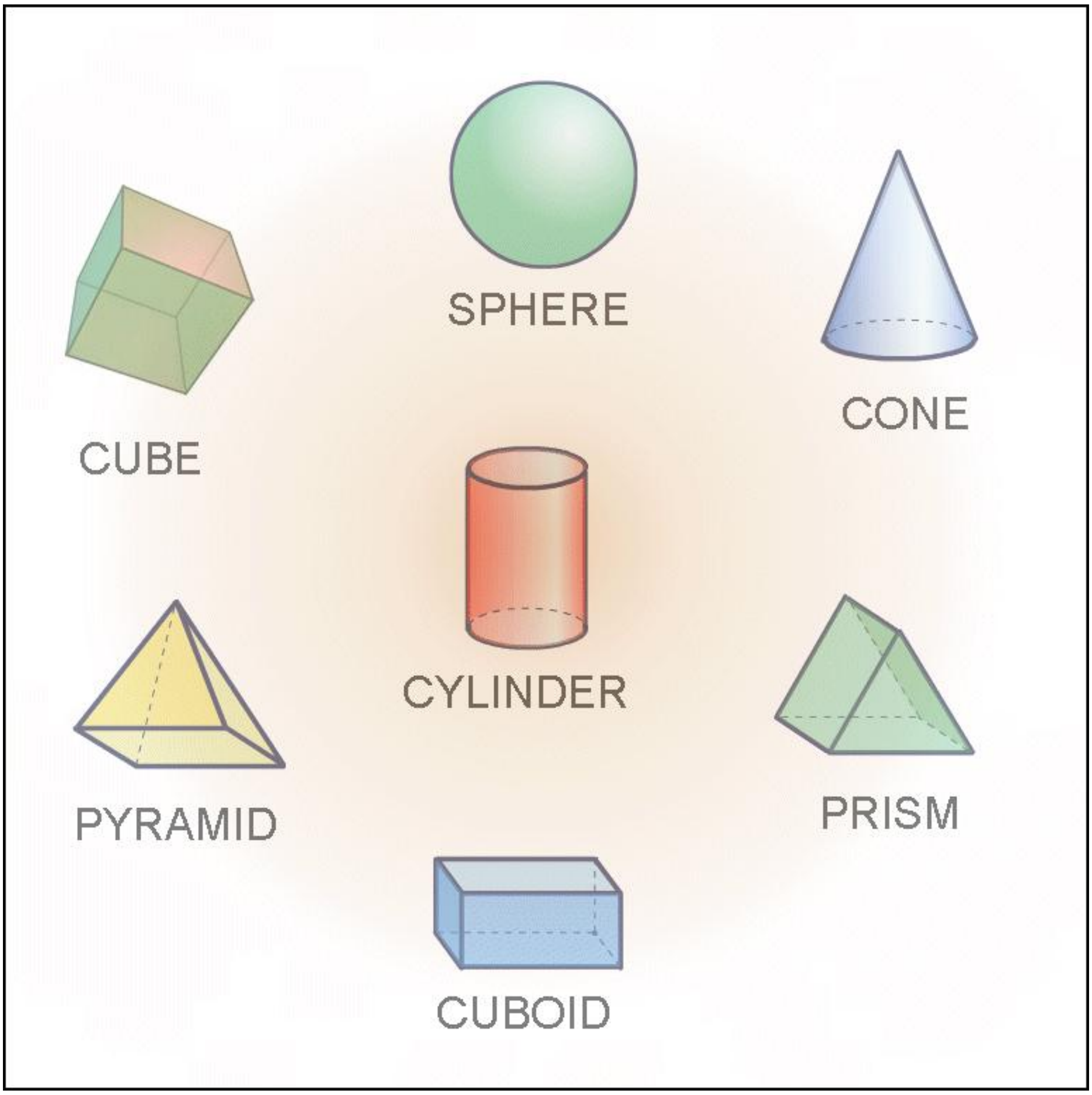
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

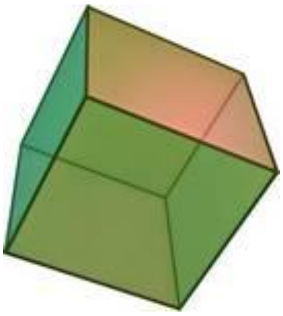
SOLID SHAPES

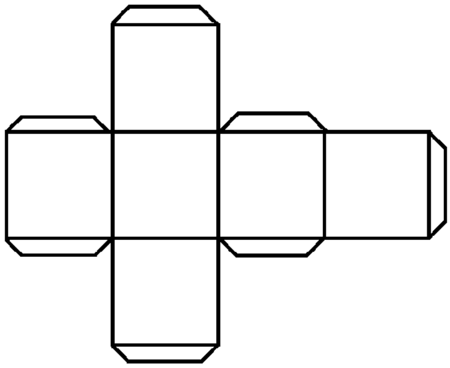


Version: 1.3 Date: 10-11-2015

SOLID SHAPES

The cube.



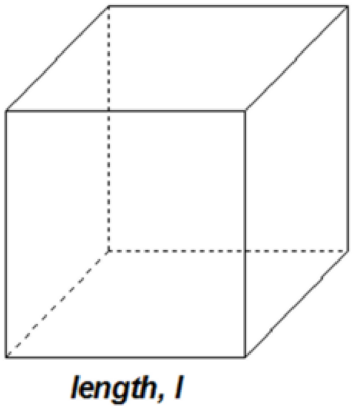


The cube is the most familiar of the 'regular' solids. It has 6 square sides or **faces**.

To the right of the figure of the cube is its **net**. The net of a solid can be formed by cutting along certain edges and unfolding the solid to give a flat 'map'.

To make a solid cube from the net, cut out the net, score the edges and cement the tabbed edges to the plain ones so that three faces meet at a vertex.

Because all the edges of a cube are equal in length and all the faces are squares, the formulae for its area and volume are especially simple:



Volume of cube = l^3 where l is the length of an edge.

Surface area of cube = $6l^2$, again where l is the length of an edge.

Example (1): Find the surface area and volume of a cube whose sides are 5 cm long.

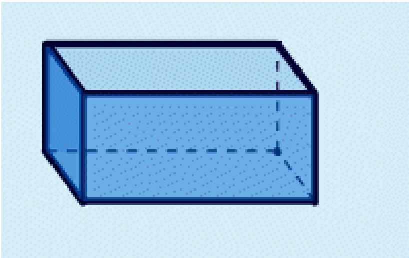
The volume of the cube is 5^3 cm^3 = 125 cm^3.
The surface area is 6×5^2 cm^2 = 150 cm^2.

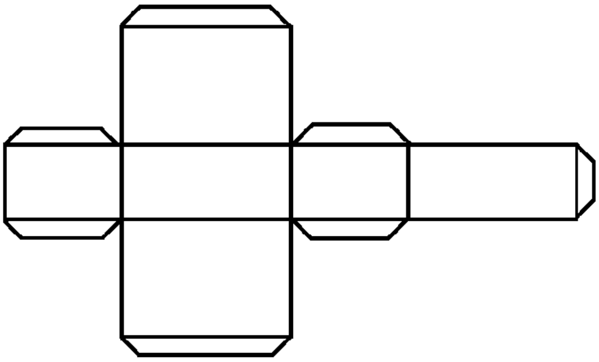
Example (2): An open cubic box (without lid) has an inside surface area of 3.2 m^2. Find its volume in litres, given that 1000 litres = 1 m^3.

Because we are told that the box has no lid, there are **five** faces making up the surface area.
One face therefore has a surface area of one-fifth of 3.2 m^2, or 0.64m^2, and therefore a side of ($\sqrt{0.64}$) metres or 0.8m.

The volume is therefore 0.8^3 m^3, or 0.512m^3, or 512 litres.

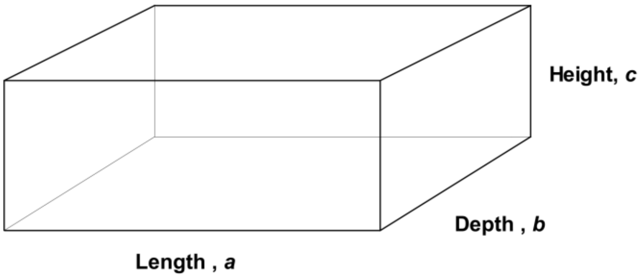
The cuboid.





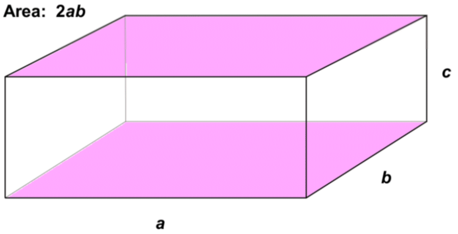
A **cuboid** is related to the cube, but is slightly less regular, with at least one pair of rectangles for faces rather than having six square faces like the cube.

Cuboids are very familiar in everyday life – most boxes are of that shape.

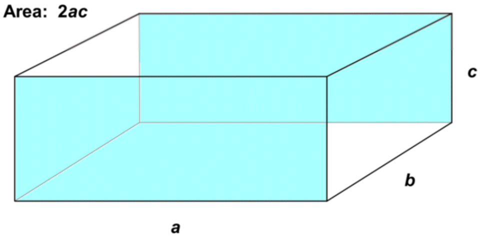


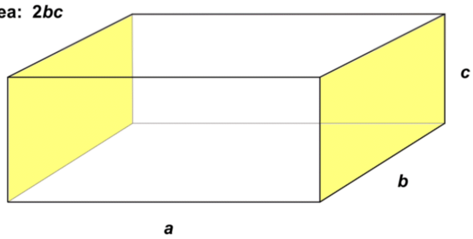
A cuboid has the same number of faces, vertices and edges as a cube, but now only opposite pairs of faces are equal.

One such pair has a total area of $2ab$, (length × depth).



There are two other pairs, one with combined area $2ac$, (length × height) and a third pair with combined area $2bc$, (depth × height).





The formulae for the cube can be adapted as follows:

Volume of cuboid = abc where a, b and c are the length, depth and height.

Surface area of cuboid = $2(ab + ac + bc)$; again a, b and c are the length, depth and height.

Example (3): Find the surface area and volume of a cuboid whose dimensions are 8 cm × 5 cm × 4 cm.

The volume of the cuboid is 8 × 5 × 4 cm^3 = 160 cm^3.
The surface area is 2((8 × 5) + (8 × 4) + (5 × 4)) cm^2 = 2 × (40 + 32 + 20) cm^2 = 184 cm^2.

Example (4): A cuboidal room is 5 metres long, 4 metres wide and 2.5 metres high.
The ceiling and walls require two coats of paint, with a covering power of 12.5 m^2 per litre.
Work out the amount of paint needed after deducting 7 m^2 for doorways and windows.

Area of ceiling = length × width = 5 × 4 m^2 = 20 m^2.

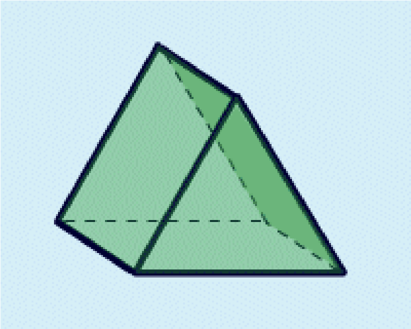
Areas of walls = (perimeter × height) =
2 × ((length × height) + (width × height)) = 2 × ((5 × 2.5) + (4 × 2.5)) = 45 m^2.

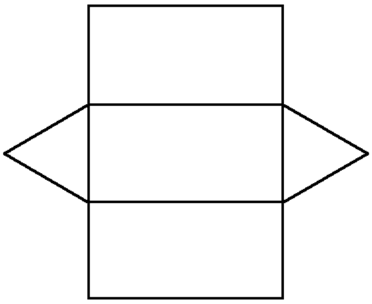
This gives a total area of 65 m^2, but we must deduct 7m^2 for doorways and windows, giving the total area to be painted as 58 m^2.

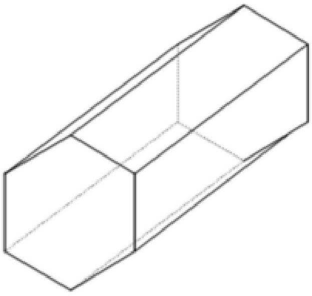
This area requires painting twice, so we need paint to cover 116 m^2. Since 1 litre of paint covers 12.5 m2, we will need $\frac{116}{12.5}$ litres, or 9.28 litres.

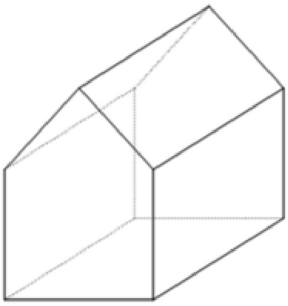
Two 5-litre cans will therefore be enough !

The prism.









A **prism** is any solid of constant cross-section.

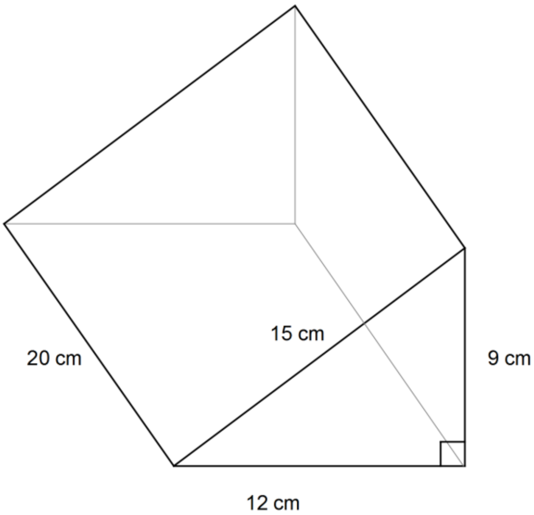
The side faces are all rectangles (they can be squares) and the end faces are identical.

(The net above is of a triangular prism, seen in the physics lab and a certain brand of Swiss chocolate !)
The hexagonal prism is found in the honeycomb, and in an unused pencil.

To find the volume of a prism, we multiply the cross-sectional surface area by the length.

Volume of prism = surface area of end face × length.

Example (5a): A triangular prism has two right-angled triangles for its end faces, and is 20 cm long. The sides of the end faces are 9 cm, 12 cm and 15 cm. Find the volume of the prism.



Because the triangular end faces are right-angled, with a base of 12 cm and a height of 9 cm, the surface area of a single face = $\frac{1}{2} \times 12 \times 9$ cm^2 = 54 cm^2.

Since the length of the prism is 20 cm, its volume is therefore 54×20 cm^3 = **1080 cm^3.**

Example (5b): Find the total surface area of the triangular prism in example (5a).

We have already calculated the area of a single triangular end face as 54 cm^2. The two of them thus have a total area of 108 cm^2.

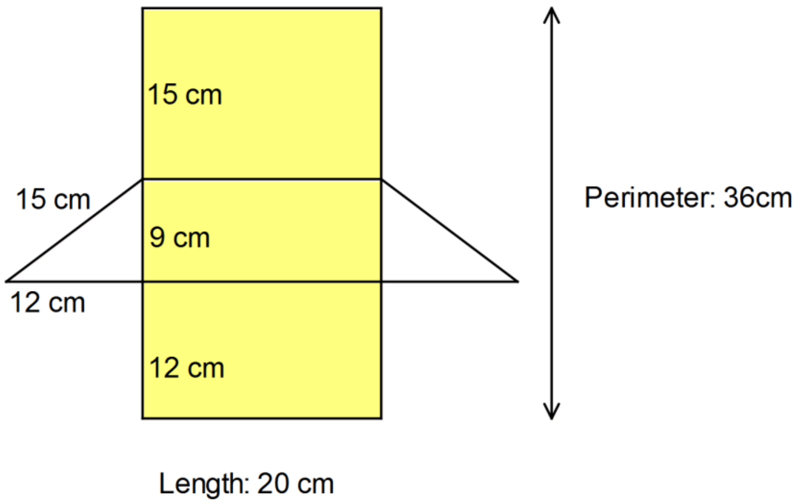
We now calculate the areas of the rectangular side faces.
The base has an area of 12×20 cm^2 = 240 cm^2.
The vertical face has an area of 9×20 cm^2 = 180 cm^2.
The sloping face has an area of 15×20 cm^2 = 300 cm^2.

The three side faces have a combined area of (240 + 180 + 300) cm^2 = 720 cm^2.
Adding the area of the end faces gives the total surface area of the prism as 720 + 108, or **828 cm^2.**

Some of the calculations could have been simplified by adding the side lengths of the triangular end faces to give the perimeter, and then multiplying the perimeter by the length of the prism.

The three side faces have a combined area of $(12 + 9 + 15) \times 20$ cm^2 = 720 cm^2.

The net of the prism shows how we could treat the three rectangular side faces as if they were a single rectangle, of dimensions (length $\times$ perimeter of ends).

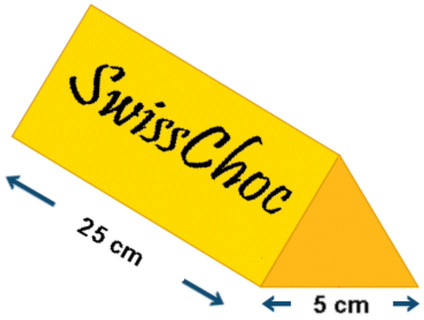


Hence: Surface area of prism = (2 $\times$ surface area of ends) + (length $\times$ perimeter of ends).

(The perimeter of the end face on the net is equal to the combined widths of the rectangular side faces).

In the case of the “circular prism”, otherwise known as the cylinder, the perimeter is the circumference.

Example (6) : SwissChoc bars are sold in cardboard boxes as shown on the right. The box is a prism whose end faces are equilateral triangles.



Given that the surface area of each triangular end face is 11 cm^2,

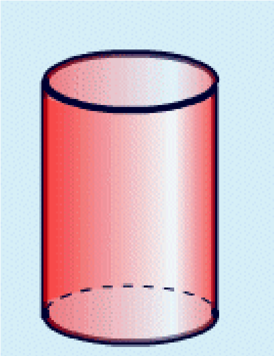
i) calculate the volume of the box to the nearest cm^3;
ii) calculate its total surface area to the nearest cm^2.

i) Since the volume of a prism = surface area of end face × length,
the volume of the container is (11 × 25) cm^3 = **275 cm^3**.

ii) The two end faces have a total area of 2 × 11 cm^2 = 22 cm^2.
The three side faces have a combined area of (3 × 5) × 25 cm^2 = 375 cm^2.

Therefore the total surface area of the prism is (22 + 375) cm^2 = **397 cm^2**.

The cylinder.



A **cylinder** is a special case of a prism, where the end faces are circles.

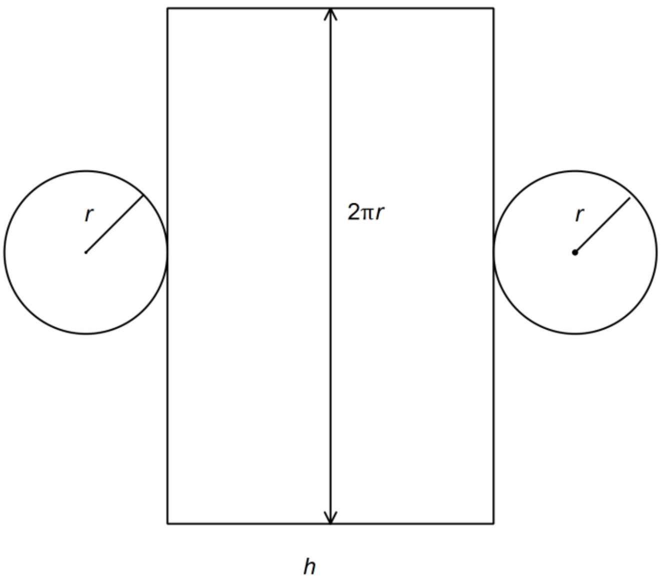
Since the formula for the area of a circle is $A = \pi r^2$, the volume of a cylinder is given by the formula $V = \pi r^2 h$, where h is the height and r is the radius of each circular end face.

A cylinder has two end faces, and a single continuous curved "face", like the label around a can. If we were to cut such a label, its length would be equal to the perimeter, or circumference, of the circle and its height would be the same as that of the cylinder.

See the net of a cylinder of radius r and height h shown right. Remember that the circumference is given by the formula $C = 2\pi r$.

Thus, the total surface area of cylinder = $2\pi rh + 2\pi r^2$ or $2\pi r(r + h)$.

The "$2\pi rh$" refers to the curved 'side' and the "$2\pi r^2$" refers to the circular ends.



Example (7): A cylinder has a height of 12 cm, and each end face has a radius of 5 cm. Calculate the volume and total surface area, leaving the results in terms of π.

The volume is given by the formula $V = \pi r^2 h$, where $r = 5$ and $h = 12$.
Hence the cylinder has a volume of $(\pi \times 5^2 \times 12)$ cm^3 or **300π cm^3**.

The two circular end faces have a total area of $2\pi r^2$ or $(2\pi \times 5^2)$ cm^2 or 50π cm^2.
The curved "side" has an area of $2\pi rh$ or $(2\pi \times 5 \times 12)$ cm^2 or 120π cm^2.

The total surface area of the cylinder is 50π + 120π cm^3 = **170π cm^2**.

Example (8): A cylindrical drum is 1.2m tall and has a diameter of 0.8m. Find its capacity in litres. (1000 litres = 1 cubic metre.)

The radius is half the diameter, i.e. 0.4m, and so the volume of the cylinder is given by
$\pi \times 0.4^2 \times 1.2$ m^3 = 0.603m^3 or **603 litres**.

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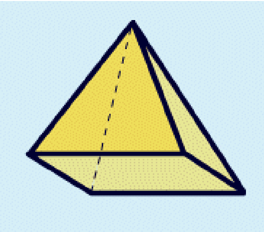
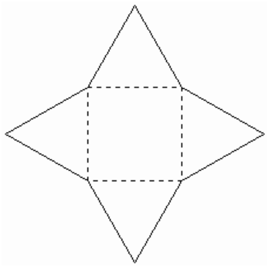
The pyramid.





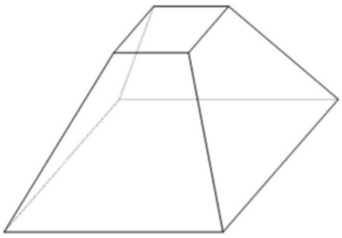
A **pyramid** is a solid with a polygonal base and sloping triangular faces, which all meet at an apex. The example above (with its net) is regular, with a square base and all its triangular faces equilateral, but this does not need generally need to be the case.

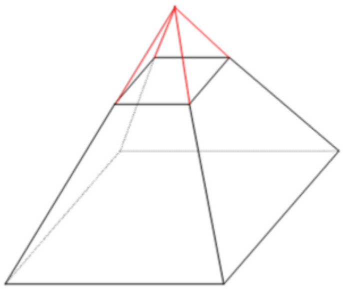
The formula for the volume of a pyramid is $V = \frac{1}{3}Ah$ where *A* is the area of the base and *h* is the **perpendicular** height (see diagram right).

This holds true whether the pyramid is upright or slanted.

If we were to make a cut parallel to the base of the pyramid, we will have two resulting solids, one of which will be a smaller, similar pyramid.

The other section, minus the smaller pyramid, is called a **frustum** (see left).





As the section removed is itself a pyramid similar to the main one , the formula for the volume of a frustum can be given as

$V = \frac{1}{3}AH - \frac{1}{3}ah$ where *A* and *H* are the base area and (perpendicular) height of the large pyramid, and *a* and *h* are the base area and height of the smaller missing section.

Example (9): Candles are made in the shape of a square-based pyramid.
Their bases are 10cm square and their heights are 18cm.

i) Find the volume of one candle.
ii) How many such candles can be made out of a cuboidal block of wax measuring 1m × 0.6m × 0.4m, assuming no wastage of wax ?

i) The base area of one candle is 10 × 10 = 100 cm^2, and its height is 18 cm, and so the volume of wax used per candle is $V = \frac{1}{3} \times 100 \times 18 = 600$ cm^3.

ii) The volume of the cuboidal block is 100 × 60 × 40 cm^3 = 240,000 cm^3 (Use consistent units)

The volume of one candle is 600 cm^3, and therefore the number of candles that can be made from the block is $\frac{240000}{600} = 400$.

Example (10): A square pyramid has a height of 60 cm and a base side length of 30 cm. A cut is made parallel to the base of the pyramid such that a smaller pyramid of base side length of 18 cm is removed.

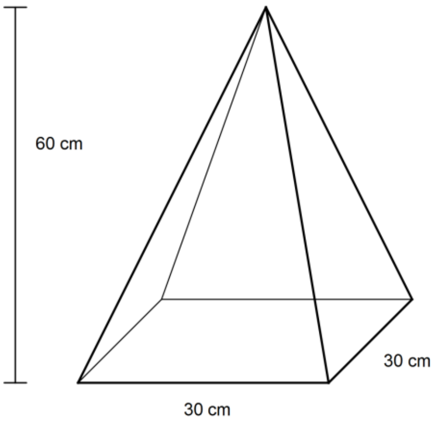
Find the volume of the remaining frustum in litres to 3 significant figures.

The base area A of the original pyramid is 30 × 30 = 900 cm^2, and its height H is 60 cm, so its volume is $V = \frac{1}{3} AH = \frac{1}{3} \times 900 \times 60 = 18{,}000$ cm^3, or 18 litres.

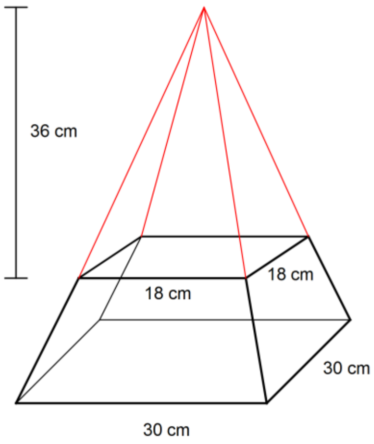
The smaller pyramid has a base of side 18 cm, which is $\frac{18}{30} = \frac{3}{5}$ of the base of the original pyramid.

Its height is thus $\frac{3}{5} \times 60 = 36$ cm, its base area is 18 × 18 = 324 cm^2 and thus its volume is

$v = \frac{1}{3} ah = \frac{1}{3} \times 324 \times 36 = 3{,}888$ cm^3, or 3.888 litres.



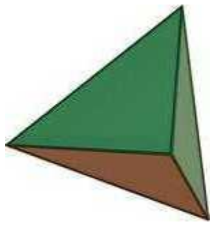
Original pyramid

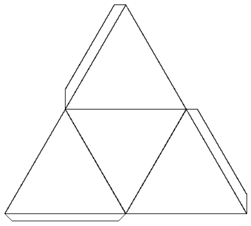


Resulting frustum

The volume of the frustum is therefore $V - v$ = (18 – 3.888) litres, or 14.112 litres = **14.1 litres** to 3 sf.

The tetrahedron.

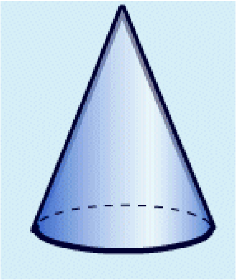




A tetrahedron is a special case of a pyramid with a triangular base.

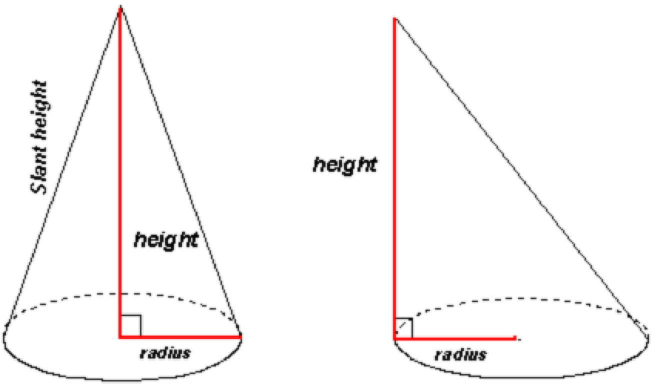
A regular tetrahedron (shown above with its net) has 4 equilateral triangles for its faces.

The cone.



The cone is another special case of a pyramid, this time with a circular base.

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where *r* is the radius of the base and *h* is the **perpendicular** height. Again, this holds whether the cone is upright or slanted.



The curved surface area of an **upright** cone is given by $A = \pi rl$ where *r* is the radius and *l* is the slant height.

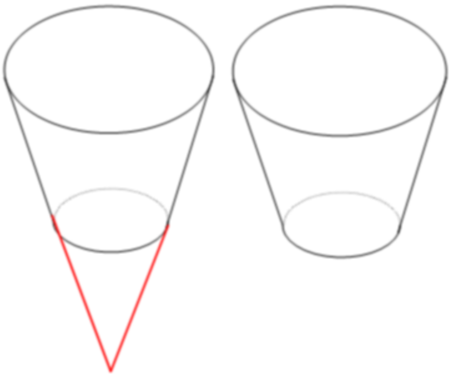
(This formula does not hold for slanted cones.)

If a question asks for the total surface area, do not forget to include the circular base in your calculations. In this case, $A = \pi rl + \pi r^2$

Again, as with the case of the pyramid, we can have a **frustum** of a cone.

The section removed is itself a smaller similar cone, so the volume formula for a frustum is

$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$ where *R* and *H* are the base radius and height of the large cone, and *r* and *h* are the base radius and height of the smaller missing section.



Working is similar to that of the pyramid example (10).

Example (11): The candlemakers from Example (9) also manufacture a range of upright conical candles, with bases 14cm in diameter and height of 24cm.

How many such candles can be made out of a cylinder of wax of diameter 28 cm and height 40 cm assuming no wastage of wax ?

The volume of the wax cylinder is $V = \pi r^2 h$, where r = 14 (halve the diameter !) and h = 40.
Hence the cylinder has a volume of ($\pi \times 14^2 \times 40$) cm^3 or **7840 π cm^3**.

The base radius of one candle is 7 cm, hence the base area is 49π cm^2 and its height is 24 cm.
The volume of wax used per candle is thus $V = \frac{1}{3}\pi r^2 h$, or $V = \frac{1}{3} \times 49\pi \times 24 = 392\pi$ cm^3.

∴ The number of candles that can be made from the block is $\frac{7840\pi}{392\pi} = 20$.

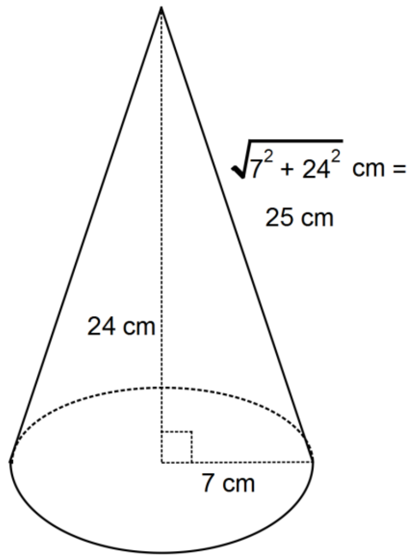
Notice also how all the working was left in terms of π rather than using numerical approximations, as in the end we were able to cancel π out anyway !

Example (12): The finished candles from Example (11) also require their curved surfaces coating with gold paint, though not their bases.

Calculate the total area that needs painting to 3 significant figures.

One candle has a surface area of $A = \pi rl$ where r is the radius and l is the slant height. We use Pythagoras to find l :

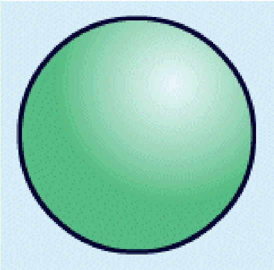
$$l = \sqrt{7^2 + 24^2} = 25$$



The curved surface area of one cone is therefore
$\pi \times 7 \times 25$ cm^2 = 175π cm^2, so the total curved surface area of all 20 of them is 20 × 175π cm^2 = 3500π cm^2 = 10996 cm^2.

∴ The total area to be painted = **11000 cm^2** to 3 significant figures

The sphere.



The sphere is the last of the solids based on the circle, and differs from the cylinder and the cone in one important respect: it cannot be modelled out of a flat piece of paper.

(The curved surface of the cylinder can be made by rolling up a rectangle, and a cone by rolling up a sector of a circle.)

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where *r* is its radius.

The surface area is given by $A = 4\pi r^2$.

Example (13): A world globe has a diameter of 40cm. Find its surface area and volume.

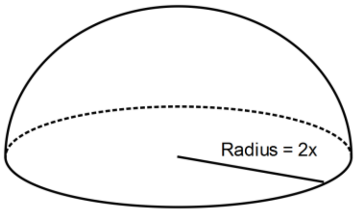
We substitute *r* = 20 (remember to use the radius !) into the formulae above to obtain

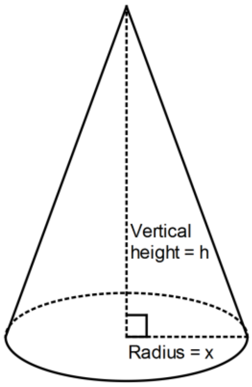
$V = \frac{4}{3} \times \pi \times 20^3$ cm^3 = 33,500 cm^3. $\therefore$ the volume of the sphere is 33.5 litres to 3 s.f.

The area of the sphere is $A = 4 \times \pi \times 20^2$ cm^2 = 5,030 cm^2.

Example (14): The hemisphere and cone shown below have the same volume, but the radius of the hemisphere is twice that of the cone.

Express the vertical height of the cone in terms of its radius.





The volume of the hemisphere will be half that of a sphere, i.e. $V = \frac{2}{3}\pi r^3$ where *r* is its radius.

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$ where *r* is its radius and *h* is its vertical height.

The volumes of the two solids are equal, so $\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3$

$\rightarrow \pi r^2 h = 2\pi r^3$ (multiplying by 3)

$\rightarrow r^2 h = 2r^3$ (dividing by π) $\rightarrow h = 2r$ $\therefore$ The height of the cone is twice its radius.

Example (15):

i) Show that the total surface area of a hemisphere can be given by the formula $A = 3\pi r^2$.
ii) Hence calculate the total surface area of a hemispherical paperweight of diameter 8 cm.

i) The curved surface of a hemisphere is half the total surface of a sphere, i.e. $2\pi r^2$.
We then add the area of the circular base, namely πr^2, to obtain the total of $2\pi r^2 + \pi r^2 = 3\pi r^2$.

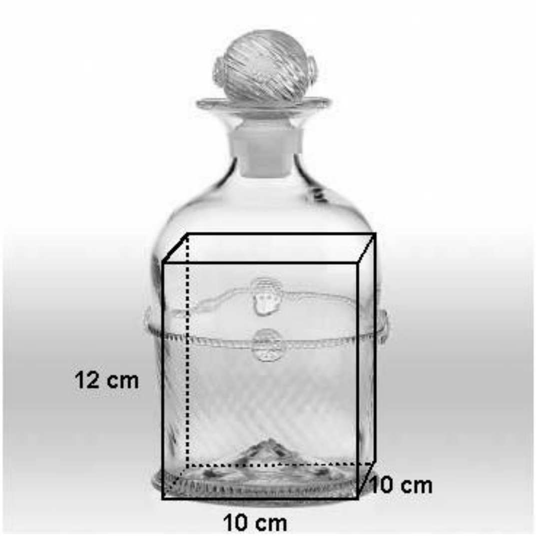
ii) Substituting $r = 4$ (halve the diameter !) into the formula $A = 3\pi r^2$, the total surface area of the paperweight is 48π cm^2 = **151 cm^2.**

Estimating volumes of irregular solid shapes.

In the section 'Measuring Shapes' we used a method of finding areas of irregular shapes by framing them inside rectangles.

A similar idea can be used to find volumes of irregular solids.

Example (15): The wine decanter below has the following dimensions. Estimate its capacity.



We can visualise the decanter contained in a cuboidal box as shown in the diagram on the right. The base and height have been slightly reduced to allow for 'packing space' and the fact that the outside base is wider than the inside base.

It can be seen that the decanter has roughly the same volume as a $12 \times 10 \times 10$cm box.
This gives an estimated capacity of 1200cm^3.

Example 15(a): Suggest a way of improving the estimate of the capacity of the decanter in Ex.(15).

We could have compared the decanter to a cylinder instead of a cuboid and used the formula for the volume of a cylinder to estimate the capacity.

The enclosing cylinder has a height of 13cm and a radius of 10cm, so its capacity is $\pi r^2 h$ cm^3, or $\pi \times 25 \times 13$ cm^3, or 1021cm^3.

As this is only an estimate, it is sufficient to say that the capacity is 1000 cm^3 or 1 litre.