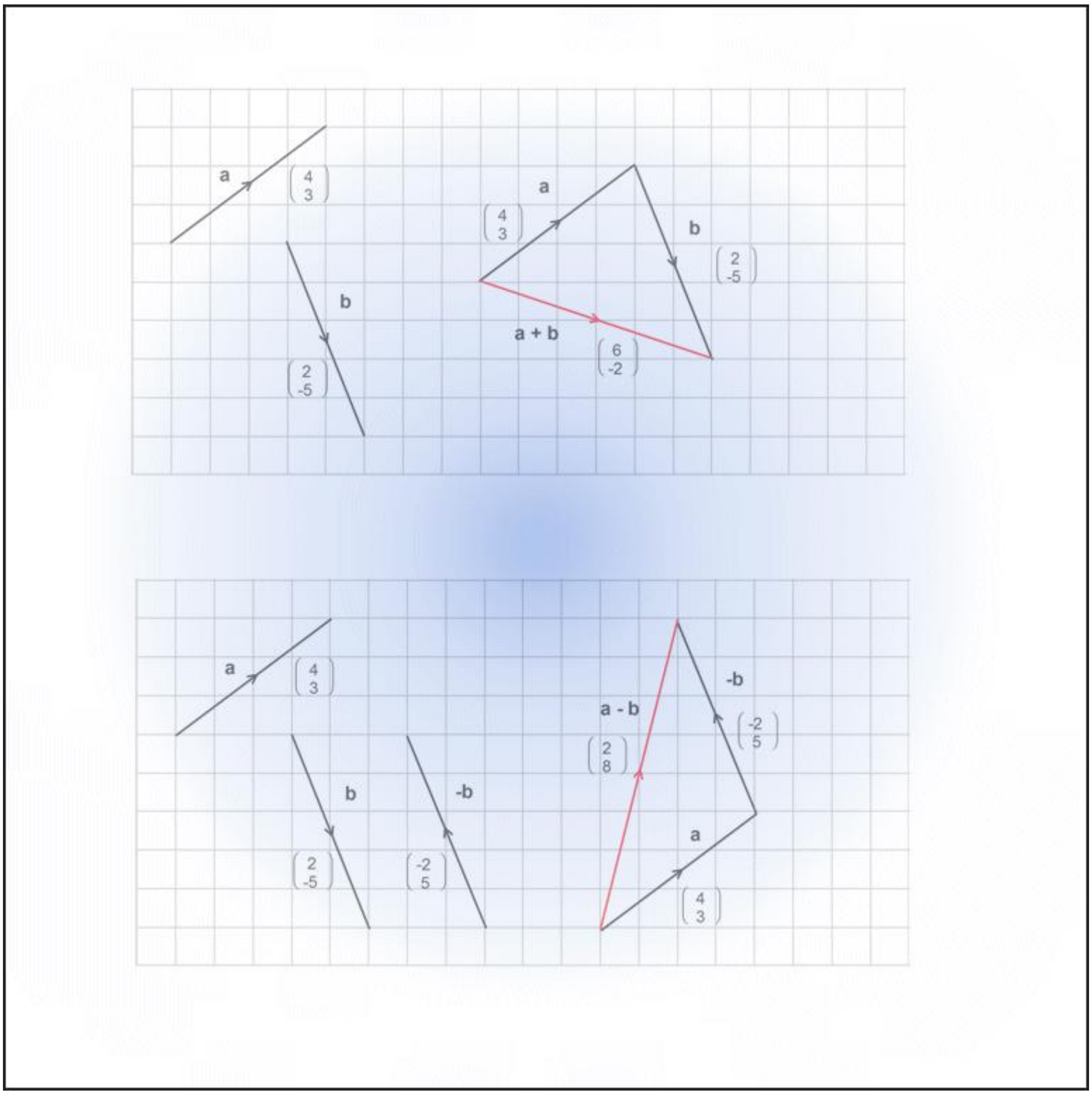
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Foundation Tier

VECTORS



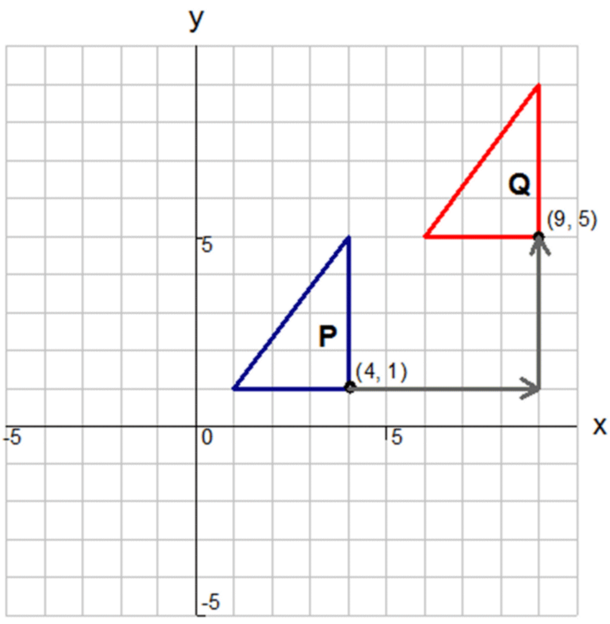
Version: 1.1 Date: 30-10-2015

VECTORS

Vectors are used in mathematics to illustrate quantities that have size (magnitude) and direction.
We have come across vectors when studying transformations.

Example (1):

Triangles **P** and **Q** are translations of each other. Note that the shape, size and orientation of each triangle remains unchanged.



TRANSLATION

If we were to take the point (4, 1) on triangle **P**, we find that it has moved to the point (9, 5) on triangle **Q**.

All other points on the same triangle have been moved 5 units right and 4 units up.
For example, the upper vertex of triangle **P** is at (4, 5) and the corresponding vertex of triangle **Q** is at (9, 9).

In other words, each point on triangle **P** has had its x-coordinate increased by 5 and its y-coordinate increased by 4 to map it to its corresponding point on triangle **Q**.

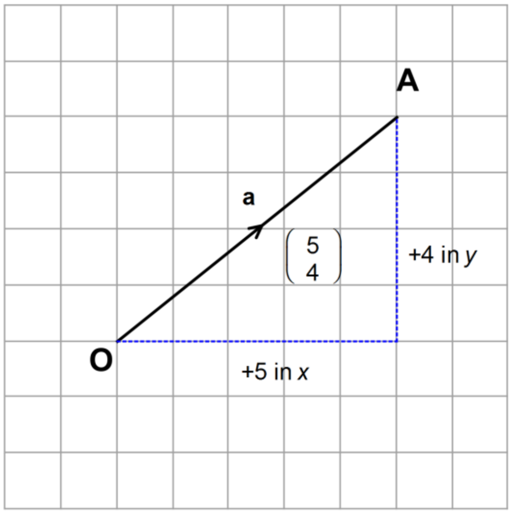
$\therefore$ Any point on **P** would map to (x+5, y+4) on **Q**.

This brings us to the idea of using **vectors** - quantities with size and direction. There are several ways of expressing those quantities, but the most convenient one is the column vector.

From the above example, triangle **Q** is a mapping of triangle **P** by the translation whose column vector representation is $\begin{pmatrix}5\\4\end{pmatrix}$. The upper figure is the change in x and the lower one the change in y.

A positive change in $x \rightarrow$ move figure to the right
A negative change in $x \rightarrow$ move figure to the left
A positive change in $y \rightarrow$ move figure up
A negative change in $y \rightarrow$ move figure down

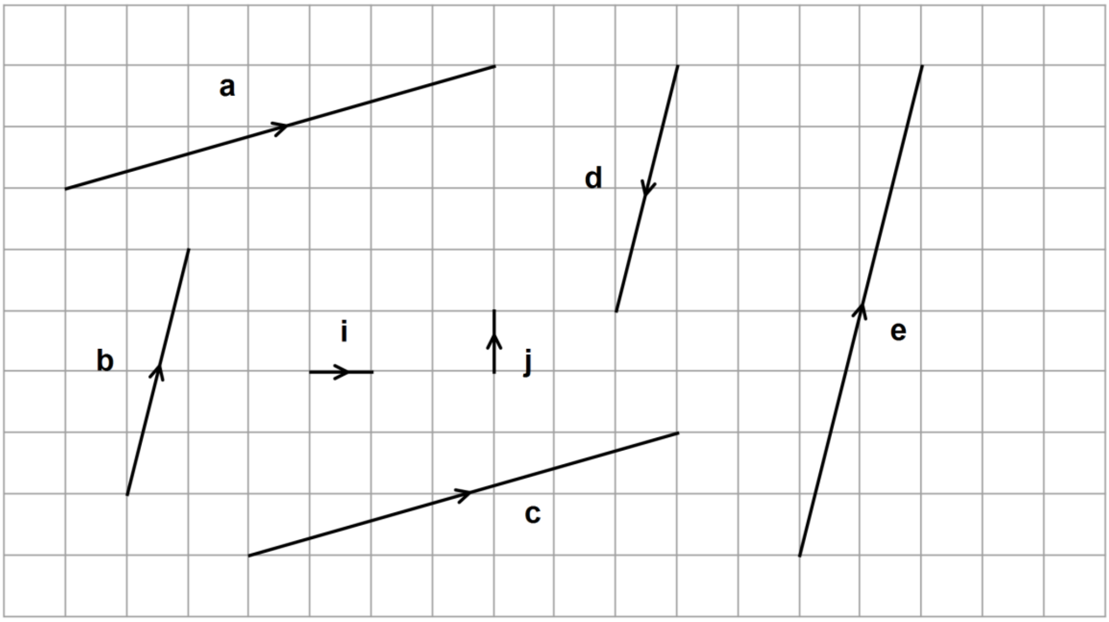
The point O is translated to point A by the vector $\begin{pmatrix}5\\4\end{pmatrix}$.



We can also denote this same vector as

$\mathbf{a} = \begin{pmatrix}5\\4\end{pmatrix}$, or we can place an arrow above the endpoints and say $\overrightarrow{OA} = \begin{pmatrix}5\\4\end{pmatrix}$.

Also, written work uses underlining where typing uses boldface, so we print **a** but write a̲.



Example (1): The diagram above shows a collection of vectors.

Describe the relationships between the following vector pairs :
i) **a** and **c** ; ii) **b** and **d** ; iii) **b** and **e ;** iv) **i** and **j**.

i) Vectors **a** and **c** are equal here; hence **a** = **c**.
Two vectors are equal if they have the same size and the same direction.

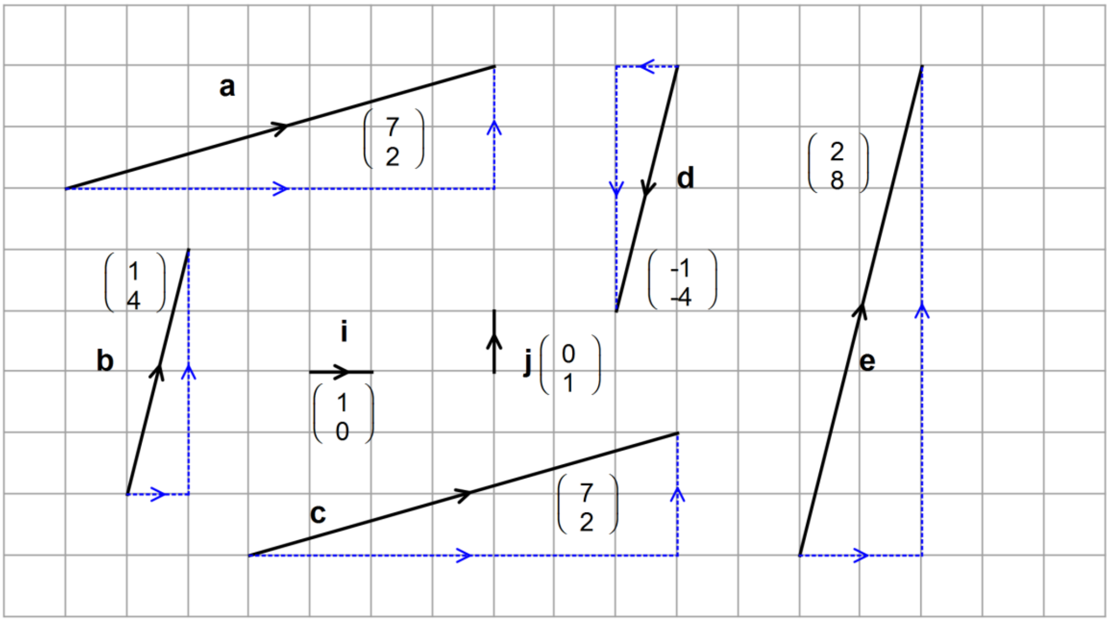
The fact that **a** and **c** have different start and end points is irrelevant.

ii) Vectors **b** and **d** have the same size, but opposite directions, therefore **d** = **-b**.
Two vectors are inverses of each other if they have the same size, but opposite directions.

iii) Vector **e** is exactly twice as long as vector **b**, so **e** = 2**b**.
The 2 is what is known as a **scalar** multiplier.
(A scalar multiplier of -1 signifies an inverse vector.)

iv) Vectors **i** and **j** are perpendicular to each other.

Example (2): Express the six vectors in the last example in column notation.



A movement in the direction of vector **a** corresponds to 7 units horizontally and 2 units vertically, as does that in the direction of vector **c**, given that the two vectors are equal .

Hence **a** = **c** = $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$

For vector **b**, the values are 1 horizontally and 4 vertically. Vector **d** is the inverse of vector **b**, so the movement is -1 unit horizontally and -4 units vertically.

Hence **b** = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and **d** = - $\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Vector **e** is twice vector **b** , so **e** = $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

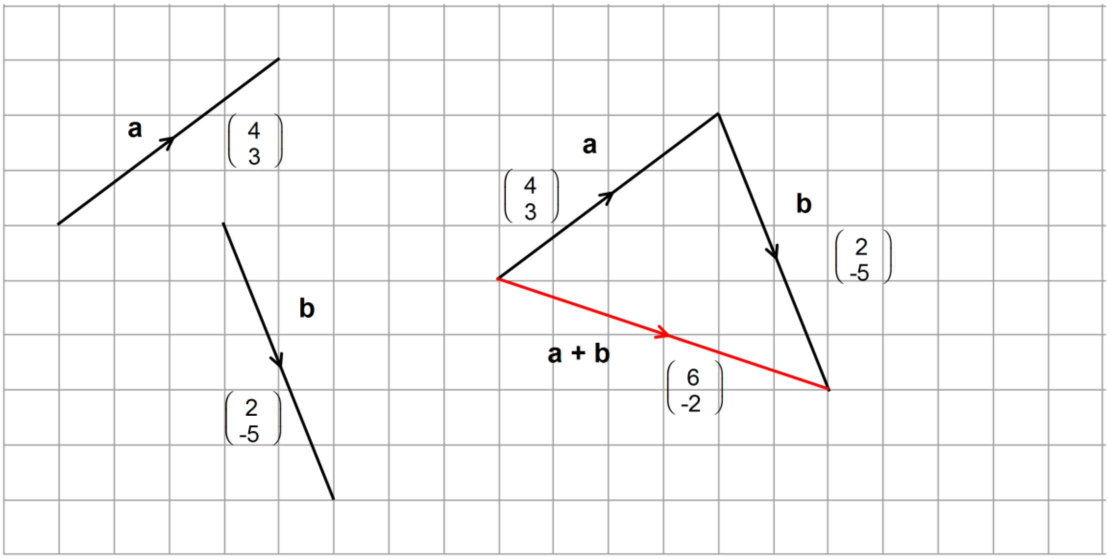
Finally the vector **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are termed the **standard unit vectors.**

A translation by **i** is by one unit in the positive *x*-direction; a translation by **j** is by one unit in the positive *y*-direction.

This leads to another way of expressing any vector in terms of these unit vectors.

For instance, **a =** 7**i** + 2**j** and **d =** –**i** - 4**j**.

Addition of vectors.



To add two vectors, join them "nose to tail" as in the diagram.

In column notation, $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

This result could also have been obtained without drawing the diagram.

$\mathbf{a}+\mathbf{b} = \begin{pmatrix} 4+2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

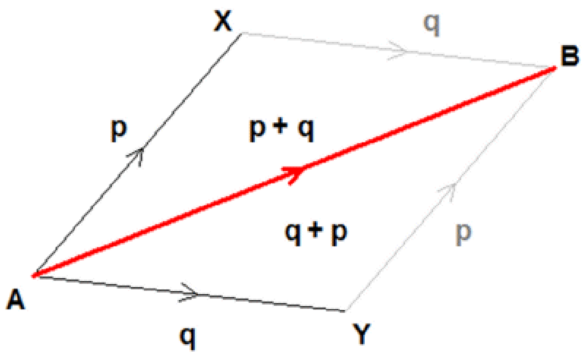
Addition of vectors can be carried out in any order.

Going from A to B via Y gives the same result as going from A to B via X.

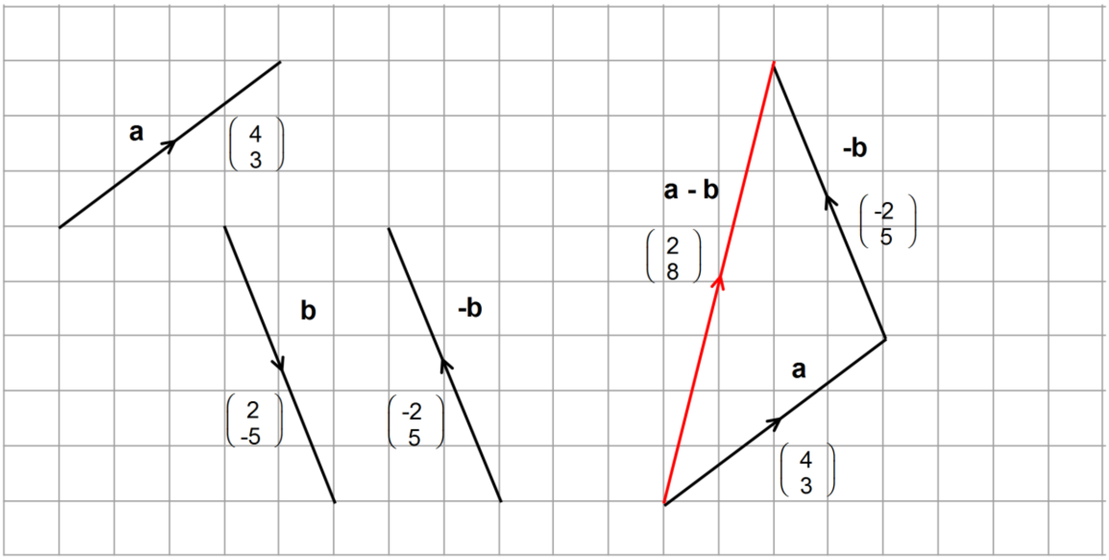
So, if $\overrightarrow{AX} = \overrightarrow{YB} = \mathbf{p}$ and $\overrightarrow{AY} = \overrightarrow{XB} = \mathbf{q}$,

$\overrightarrow{AX} + \overrightarrow{XB} = \overrightarrow{AY} + \overrightarrow{YB}$.

In other words, $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$.



Subtraction of vectors.



Subtracting vector **b** from **a** is identical to adding the inverse of vector **b** to **a**.

This time, we join **–b** to **a** "nose to tail".

In column notation, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, **-b** = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and **a-b =** **a** + **(-b)** = $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

This result could again have been obtained without drawing the diagram.

a-b = $\begin{pmatrix} 4-2 \\ 3+5 \end{pmatrix}$ = $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

Addition or subtraction of vectors in column form is very easy - just add or subtract the components !

Another special case is **a-a** = $\begin{pmatrix} 4-4 \\ 3-3 \end{pmatrix}$, where for example, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The result here is the **zero vector**, **0**. This is not the same as the number 0, which is a scalar.

Example (3).

Let vectors **p** = $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and **q** = $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Find i) **p** + 3**q** ; ii) 2**p** – **q** ; iii) **p** + 4**i** ; iv) **q** – 5**j** . Recall **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

i) **p** + 3**q** = $\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$.

ii) 2**p** - **q** = $2\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

iii) **p** + 4**i** = $\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

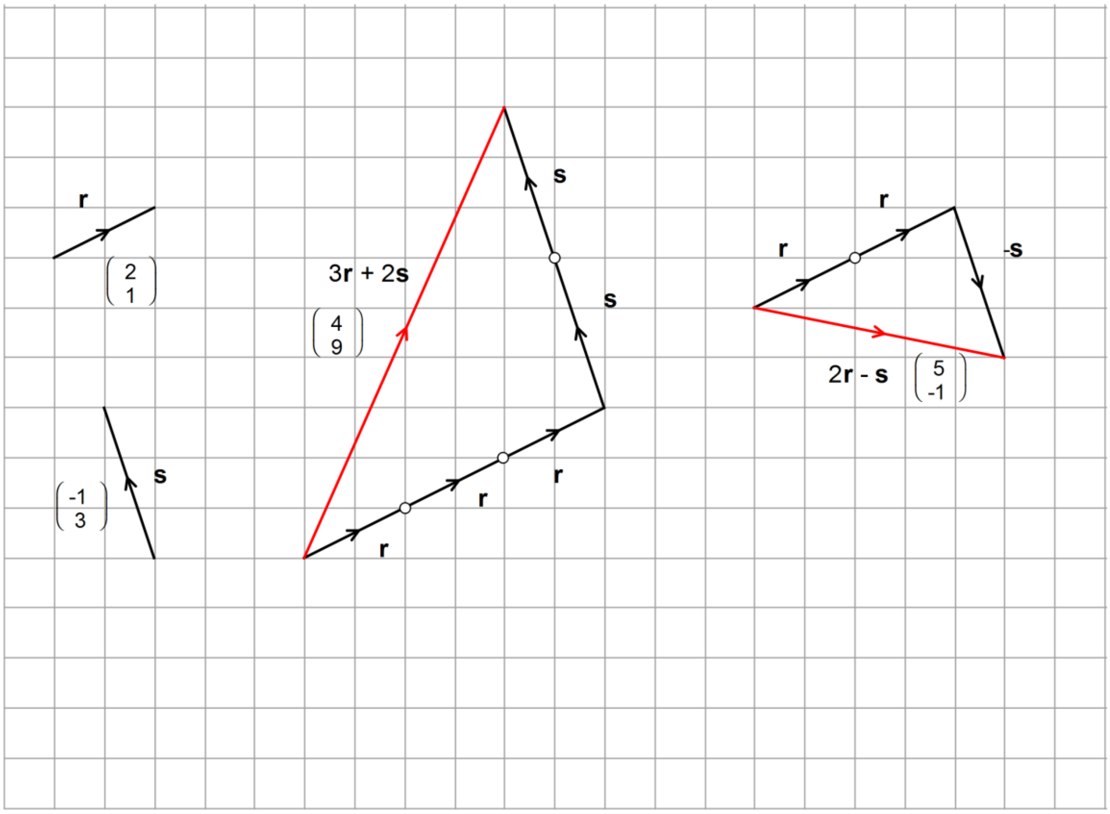
iv) **q** – 5**j** = $\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 5\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

Example (4).

Let vectors **r** = $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and **s** = $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Draw, on the same diagram, i) 3**r** + 2**s** ; ii) 2**r**– **s.**

Verify the resulting vectors using column vector arithmetic.



i) The result of the vector sum 3**r** + 2**s** is $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ as per the diagram.

By column vector addition, 3**r** + 2**s** = $3\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-2 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$.

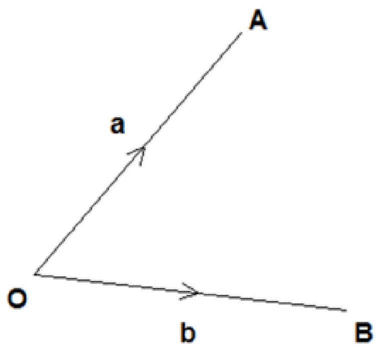
ii) The result of **2**r **–** s is $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ according to the diagram.

This can be verified by 2**r** – **s** = $2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4+1 \\ 2-6 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

Alternative Notation ("Two-letter").

Another way of denoting vectors is by stating their end points and writing an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.



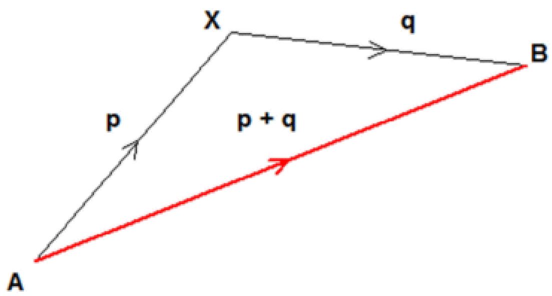
Therefore $\overrightarrow{OA}$ = **a** and $\overrightarrow{OB}$ = **b**.

The direction of the arrow is important here;
the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$.

Hence $\overrightarrow{AO}$ = - $\overrightarrow{OA}$ = -**a**.

Addition and subtraction of vectors in two-letter notation.

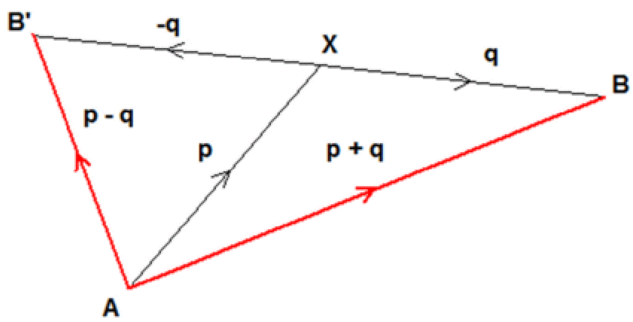
To add two vectors, apply the first, and then the second.



Thus $\overrightarrow{AB}$ = $\overrightarrow{AX}$ + $\overrightarrow{XB}$.

Here $\overrightarrow{AX}$ = **p** and $\overrightarrow{XB}$ = **q**.

Subtracting a vector is the same as adding its inverse, i.e. the parallel vector of the same magnitude but in the opposite direction.



Here, $\overrightarrow{XB'}$ = -**q**.

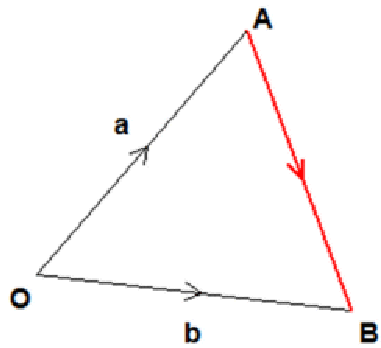
Thus $\overrightarrow{AB'}$ = $\overrightarrow{AX}$ + $\overrightarrow{XB'}$ = **p – q**.

Finding an unknown vector in terms of known ones.

Many problems and theorems in geometry can be analysed using vectors.

When asked to find an unknown vector between two points, just work it out as an alternative route made up of known vectors.

Example (6): Express the vector $\overrightarrow{AB}$ in terms of **a** and **b**.



We want to go from *A* to *B* directly, but we do not have the vector for it.

We therefore go via *O*, as in $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now $\overrightarrow{AO}$ is the same as **a** but in the reverse direction, whilst $\overrightarrow{OB}$ = **b**.

Hence $\overrightarrow{AB}$ = **-a** + **b** or **b – a**.

Example (6a): The coordinates of points *A* and *B* are (2, 4) and (6, 1) respectively. Find $\overrightarrow{AB}$ in column notation given that *O* is the origin.

Since *O* is the origin, vector **a** = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and **b** = $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Hence $\overrightarrow{AB}$ = **b – a** = $\begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Example (6b): Point *M* is the midpoint of *OA*. Find $\overrightarrow{OM}$ in column notation given that *O* is the origin.

Since $\overrightarrow{OM}$ is half of $\overrightarrow{OA}$, its vector is ½**a** = ½$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.