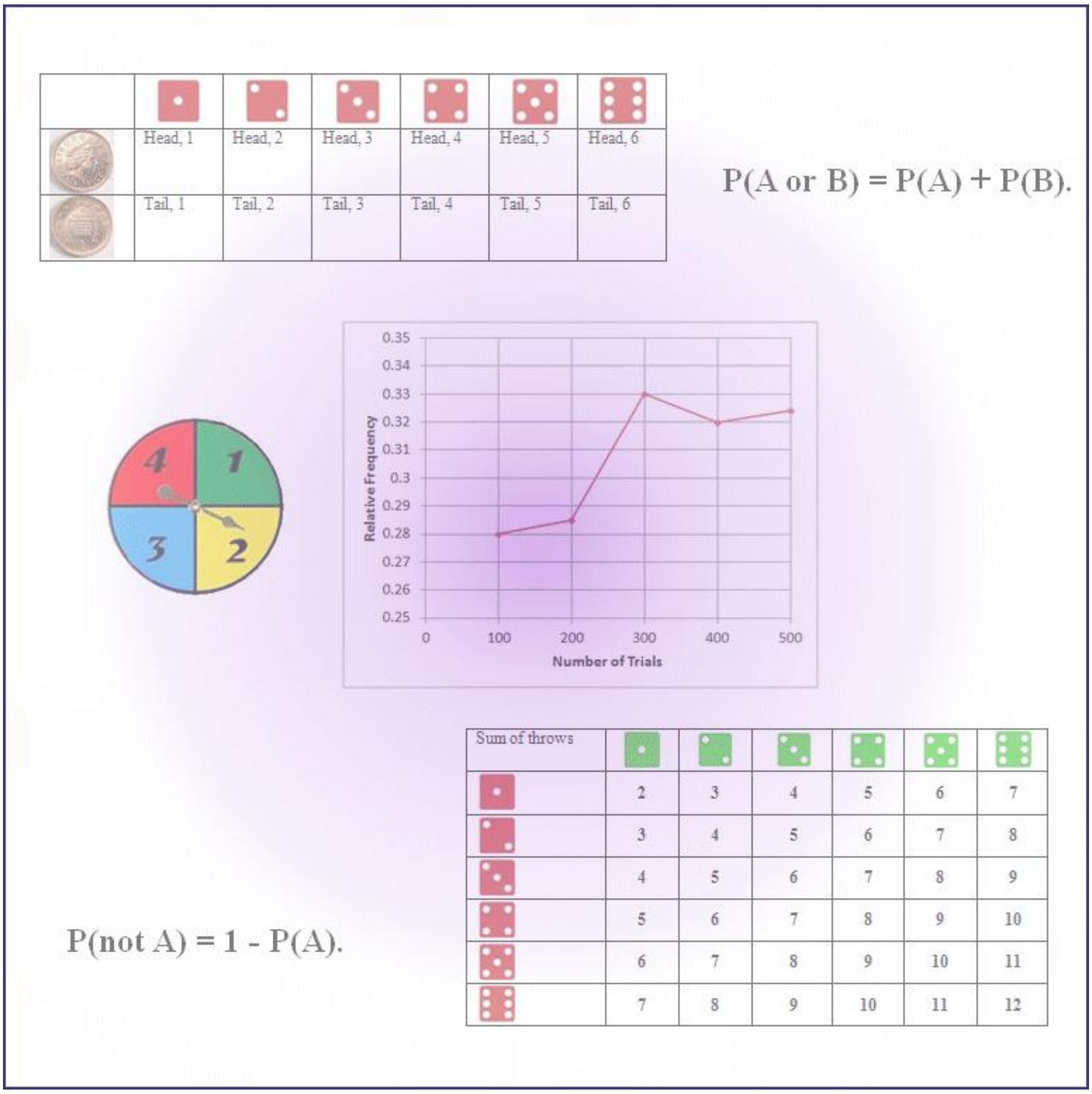
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Foundation Tier

PROBABILITY



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PROBABILITY

The probability of the outcome of an event is a fraction between 0 and 1.

A probability of zero means the event cannot occur at all.
A probability of 1 means the event is 100% certain to happen.

Examples of events are tossing a coin, throwing dice, drawing a card from a pack.

The toss of a fair (unbiased) coin has two possible outcomes; ‘heads’ or ‘tails’.
The probability of ‘heads’ turning up is $\frac{1}{2}$, as is that of ‘tails’.
The sum of the two outcomes is 1, and this would also hold good for a biased coin.
If a coin was biased so that ‘heads’ had a $\frac{3}{5}$ probability of turning up, then ‘tails’ would have a probability of $\frac{2}{5}$.

Example (1): A fair dice has 6 sides. What are the probabilities of throwing i) a 6, ii) an odd number under 5, and iii) any number except a 3 ?

A dice throw has six possible outcomes, namely 1 to 6, and all are equally likely with a probability of $\frac{1}{6}$. Those outcomes are **mutually exclusive** since it is impossible to throw more than one number at one time.

The probability of throwing a 6, which can be shortened to P(6) is therefore $\frac{1}{6}$.

There are two odd numbers less than 5 on a dice, 1 and 3. Each has a probability of $\frac{1}{6}$ occurring, and so the probability of either occurring is equal to the sum of each, i.e. $\frac{1}{6} + \frac{1}{6}$ or $\frac{1}{3}$.

P(1 or 3) = P(1) + P(3) = $\frac{1}{6} + \frac{1}{6}$ or $\frac{1}{3}$.

In general, if A and B are mutually exclusive events, then the probability of either event occurring can be worked out by adding their individual probabilities together.

P(A or B) = P(A) + P(B).

The probability of throwing a 3, which can be shortened to P(3) is $\frac{1}{6}$.
The result of the throw will either be 3 or ‘not 3’, and the two probabilities must combine to give 1.
The probability of not throwing a 3 is therefore 1 - $\frac{1}{6}$, or $\frac{5}{6}$.

In general, if A is an event , then the probability of A *not* occurring is the probability of A occurring subtracted from 1.

P(not A) = 1 - P(A).

Biases in probability.

In most of the examples in this section, we will assume objects like coins, spinners or dice to be ‘fair’. By this we mean that the actual probabilities come close to the theoretical ones.

Testing for bias is a separate topic, but usually such tests involve many trials. The greater the number of trials, the more accurate the test. See examples 4a – 4b.

Example (2): Three dice were thrown 600 times each and the results recorded.

						
Dice 1	104	98	101	105	95	97
Dice 2	94	104	0	205	101	96
Dice 3	71	95	102	97	103	132

i) One of the dice is mis-spotted. Which one is it ?
ii) Describe the properties of the other two dice.

Because each of the six numbers is equally likely on a dice throw, its probability is $\frac{1}{6}$ and therefore each number would be expected to turn up $\frac{1}{6}$ of 600 times, or 100 times.

i) Dice 2’s results show normal results for four of the numbers, but a result of 3 does not occur even once, whilst 4 occurs twice as often as it should. The dice is mis-spotted, with the 4 on two faces and the 3 absent.

ii) The results for Dice 1 are close to the theoretical (they only vary by a few each way), and so this dice can be passed as fair.

Dice 3’s results show that 1 occurs less often, and 6 occurs more often, than they should do. A discrepancy of about 5 in 100 is acceptable, but not one of about 30. This dice is loaded so as to favour a score of 6.

Relative Frequency.

The dice example in part (3) brings us to the idea of **relative frequency**. This is used to estimate the long-term probability of an event if the dice, coin or spinner is suspected of bias.

If 'heads' were to come up 65 times out of 100 coin tosses, then the relative frequency of 'heads' would be $\frac{65}{100}$ or 0.65, which is some way above the theoretical outcome of 0.5.

Example (3): A dice suspected of bias was thrown 600 times and the results recorded.
Complete the relative frequency table, and from it estimate the expected number of sixes tossed after 2000 throws.

						
Frequency	66	117	101	97	84	135
Relative frequency	0.11	0.195				

The dice was thrown 600 times, so each relative frequency is the actual frequency divided by 600.

The completed table therefore looks like this:

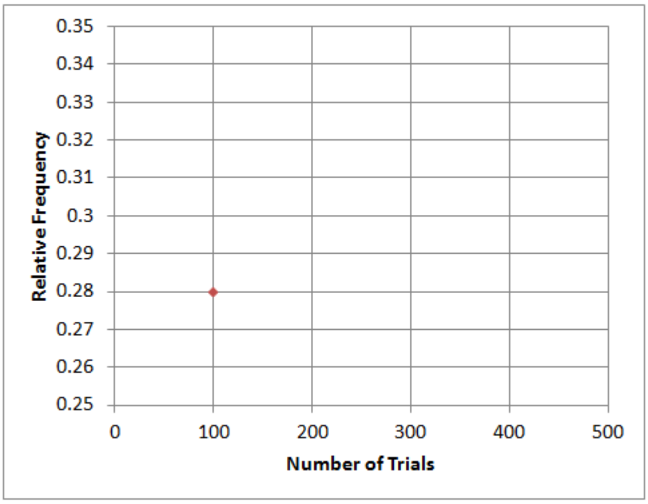
						
Frequency	66	117	101	97	84	135
Relative frequency	0.11	0.195	0.168	0.162	0.14	0.225

There seems to be a strong bias for throwing a 6 and against throwing a 1, and a slightly milder bias for throwing a 2 as opposed to a 5.

The likely number of sixes to be thrown after 2000 throws is
(relative frequency of a 6) × (number of throws), here 0.225 × 2000 or 450.

Example (4a): Jade has been testing a four-sided spinner supplied with a board game. She noticed that a 3 had turned up 28 times after 100 spins. She reckons that the spinner is biased, because a fair one should have only turned up 25 times. Is her reasoning correct here ?

The outcome here is not especially significant, as a relative frequency of $\frac{28}{100}$ or 0.28 is close enough to the 'fair' value of one quarter or 0.25 to fall within experimental error. She would need to take more trials to confirm or reject the bias.



Example (4b): Jade continues her experiment counting the number of '3's after various trials. She obtains 57 '3' s after 200 spins, 99 after 300 spins, 128 after 400 spins and finally 162 after 500 spins.

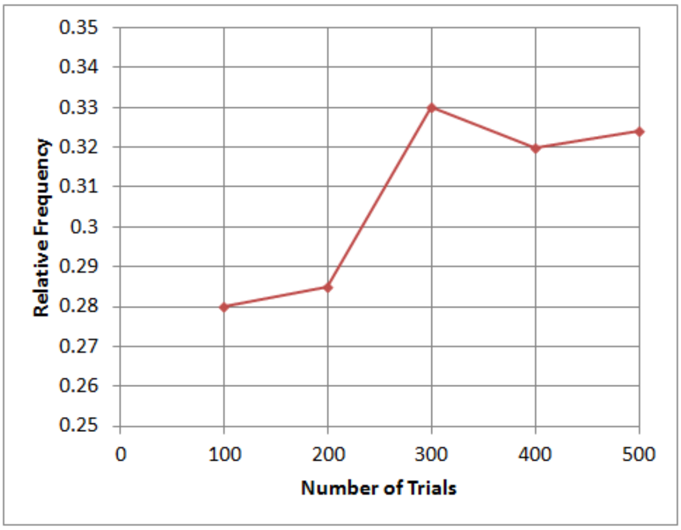
Produce a relative frequency table and use the results to complete the graph on the right.

Is there now a stronger suspicion of the spinner being biased ?

Completing the relative frequency table and graph gives the following:

Number of spins	100	200	300	400	500
Frequency of '3'	28	57	99	128	162
Relative freq. of '3'	0.28	0.285	0.33	0.32	0.324

The relative frequency of spinning a 3 has not 'evened' to a value closer to 0.25, or one quarter, as would be expected of a fair spinner. It appears to have settled to a level of between 0.32 and 0.33, or almost one third, which is a substantial bias.



Independent events.

Two events are said to be independent if the result of one has no bearing on the other.
Thus, if a player was to toss a coin and throw a dice, the result of the coin toss will have no effect on the result of the dice throw.

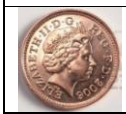
Another example of a series of independent events is a sequence of tosses of an identical fair coin – remember that the coin has no ‘memory’ of past events.

It is therefore wrong to think on the lines of “We’ve had tails twenty times, the next toss MUST be a head”. The fact that the last twenty tosses resulted in ‘tails’ makes no difference to the probability a head turning up – it will still be exactly one half.

The possible outcomes can be shown in a **possibility space diagram**.

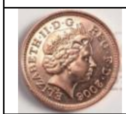
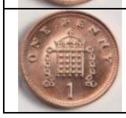
Example (5): A player tosses a fair coin and throws a fair dice.

Draw a possibility space diagram and use it to work out the probabilities of the following events:
i) a head and a number less than 5; ii) a tail and an even number; iii) a head and a 3, or a tail and a 4.

						
	Head, 1	Head, 2	Head, 3	Head, 4	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	Tail, 4	Tail, 5	Tail, 6

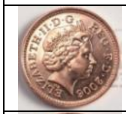
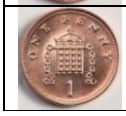
The possibility space diagram shows all the possible combinations of the coin toss and dice throw. Since there are two possible results of the coin toss and six of the dice throw, there are 2 $\times$ 6 or 12 possible combined results. Also, because both the coin and the dice are fair, each outcome has a probability of $\frac{1}{12}$. (Remember, all probabilities in the space must add to 1).

i) A head and a number less than 5.

						
	**Head, 1**	**Head, 2**	**Head, 3**	**Head, 4**	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	Tail, 4	Tail, 5	Tail, 6

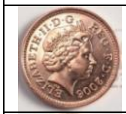
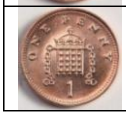
Four of the twelve combinations satisfy the given condition, so its probability is $\frac{4}{12}$ or $\frac{1}{3}$.

ii) A tail and an even number.

						
	Head, 1	Head, 2	Head, 3	Head, 4	Head, 5	Head, 6
	Tail, 1	**Tail, 2**	Tail, 3	**Tail, 4**	Tail, 5	**Tail, 6**

Three combinations satisfy the criteria, so the probability of the event is $\frac{3}{12}$ or $\frac{1}{4}$.

iii) A head and a 3, or a tail and a 4.

						
	Head, 1	Head, 2	**Head, 3**	Head, 4	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	**Tail, 4**	Tail, 5	Tail, 6

These two mutually exclusive combinations can have their probabilities added. Each individual event has a probability of $\frac{1}{12}$, so the probability of either event is $\frac{2}{12}$ or $\frac{1}{6}$.

Example (6): Two fair dice are tossed and the sum of their spots recorded.

Draw the possibility space diagram and hence work out the probabilities of the following events:

i) a sum of 7
ii) a ‘double’ (two equal numbers)
iii) a sum of 6 or 8
iv) a double or a sum of 7
v) a double or a sum of 8

The possibility space diagram looks like this:

Sum of throws						
	**2**	**3**	**4**	**5**	**6**	**7**
	**3**	**4**	**5**	**6**	**7**	**8**
	**4**	**5**	**6**	**7**	**8**	**9**
	**5**	**6**	**7**	**8**	**9**	**10**
	**6**	**7**	**8**	**9**	**10**	**11**
	**7**	**8**	**9**	**10**	**11**	**12**

There are now 6 × 6 or 36 different results of the dice throws, but only 11 possible sums of the spots, namely 2 to 12.

Moreover, not all the sums are equally likely; a sum of 2 can only be obtained in one way (1, then 1) but a sum of 4 can be obtained in three ways :

1 on first dice, 3 on second
2 on first dice, 2 on second
3 on first dice, 1 on second

i) Sum of 7

Sum of throws	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	**7**
⚁	3	4	5	6	**7**	8
⚂	4	5	6	**7**	8	9
⚃	5	6	**7**	8	9	10
⚄	6	**7**	8	9	10	11
⚅	**7**	8	9	10	11	12

There are 6 possible ways of making a sum of 7 with 2 dice, and so the probability of this happening is $\frac{6}{36}$ or $\frac{1}{6}$.

ii) A Double

Sum of throws	⚀	⚁	⚂	⚃	⚄	⚅
⚀	**2**	3	4	5	6	7
⚁	3	**4**	5	6	7	8
⚂	4	5	**6**	7	8	9
⚃	5	6	7	**8**	9	10
⚄	6	7	8	9	**10**	11
⚅	7	8	9	10	11	**12**

There are 6 possible ways of making a Double, so the probability of this happening is also $\frac{6}{36}$ or $\frac{1}{6}$.

iii) Sum of 6 or 8

Sum of throws						
	2	3	4	5	**6**	7
	3	4	5	**6**	7	**8**
	4	5	**6**	7	**8**	9
	5	**6**	7	**8**	9	10
	**6**	7	**8**	9	10	11
	7	**8**	9	10	11	12

There are 5 possible ways of making a sum of 6, and 5 ways of making 8. There are therefore 10 ways of making either score.

The probability is therefore $\frac{10}{36}$ or $\frac{5}{18}$.

iv) Double or sum of 7

Sum of throws						
	**2**	3	4	5	6	**7**
	3	**4**	5	6	**7**	8
	4	5	**6**	**7**	8	9
	5	6	**7**	**8**	9	10
	6	**7**	8	9	**10**	11
	**7**	8	9	10	11	**12**

There are 6 possible ways of making a Double, and 6 ways of making a 7. There are therefore 12 ways of making either. The probability is therefore $\frac{12}{36}$ or $\frac{1}{3}$.

v) Double or sum of 8

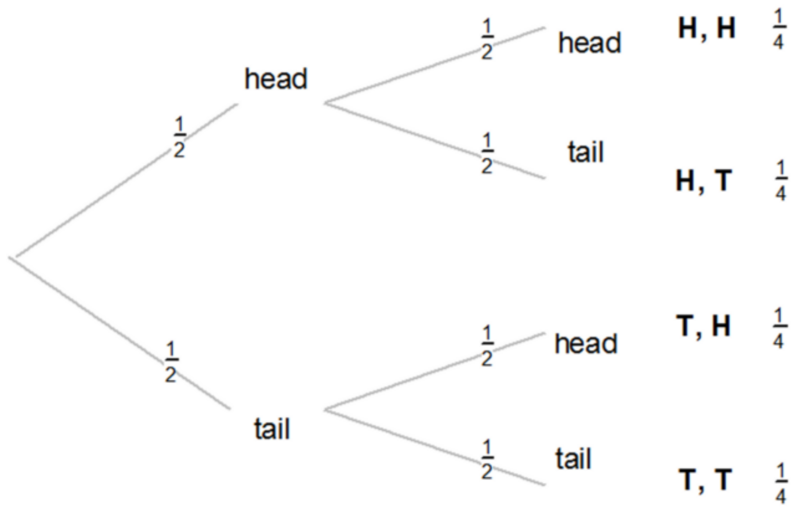
Sum of throws						
	**2**	3	4	5	6	7
	3	**4**	5	6	7	**8**
	4	5	**6**	7	**8**	9
	5	6	7	**8**	9	10
	6	7	**8**	9	**10**	11
	7	**8**	9	10	11	**12**

There are 6 ways of making a Double, and 5 ways of making an 8. This might lead you to think that there are 11 possible ways of making either, but counting the highlighted squares only gives 10, since you can score both a Double and a score of 8 in one throw, namely by throwing two 4's.

The probability is therefore $\frac{10}{36}$ or $\frac{5}{18}$.

Probability Tree Diagrams.

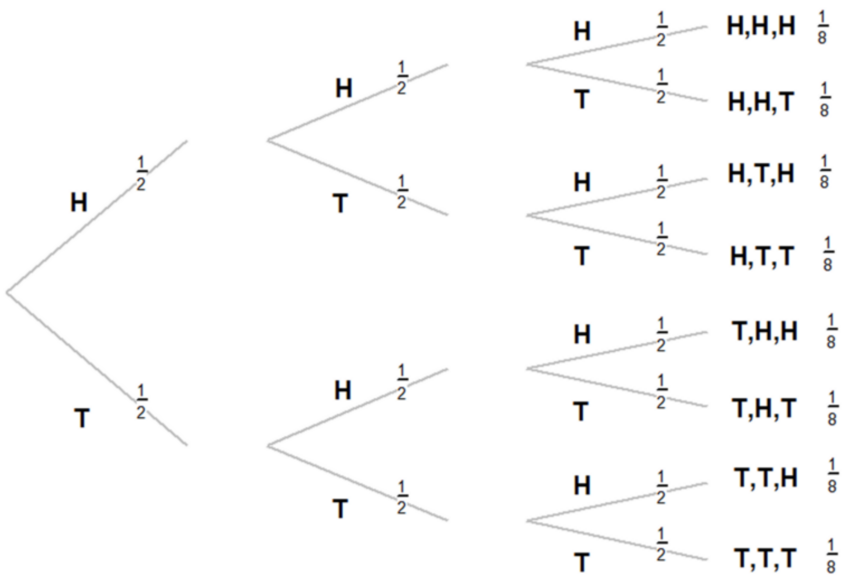
It is often convenient to use a tree diagram to work out probability problems, such as the one below showing the possible outcomes of tossing a fair coin twice.



This tree shows how the multiplication rule is used to calculate combined probabilities.
Note how there are two ways of obtaining a head and a tail, giving a total probability of $\frac{1}{2}$ for that event.

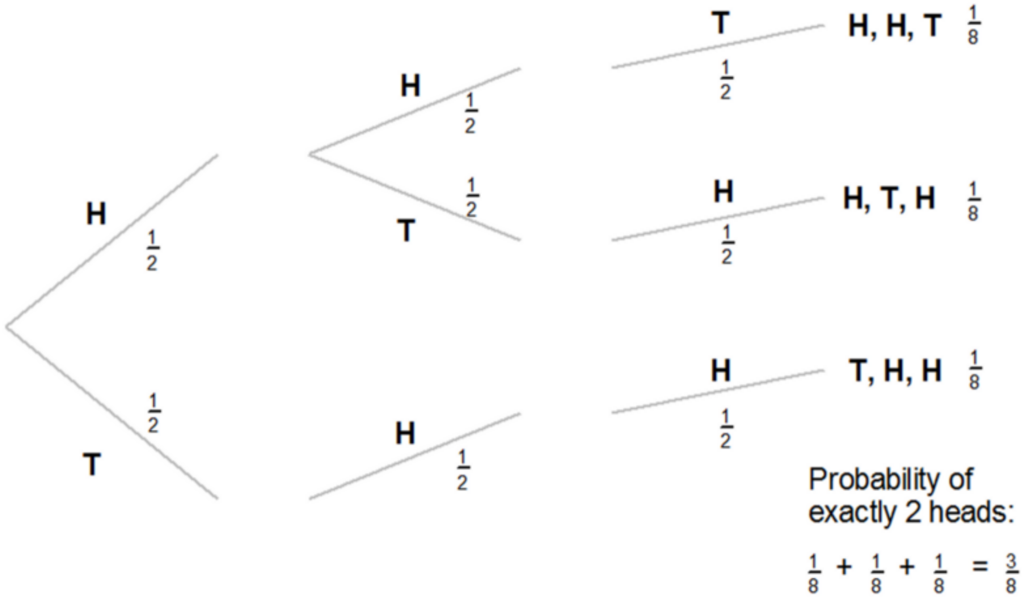
Example(7): Draw a probability tree diagram to show all the possible outcomes of tossing a coin three times.

This time, we have 8 distinct possibilities, each with equal probabilities of $\frac{1}{8}$.



It is not always necessary to draw a whole tree when solving probability problems, as the next example will show.

Example (8): Using a tree diagram, find the probability of tossing *exactly* two heads in three tosses of a fair coin.



Comparing the diagram with that for Example (8), we can see how the redundant branches of the tree have been 'pruned out', with only the required outcomes displayed on the right.

For example, if a 'tail' has been thrown on the first go, then both the following throws must be heads to satisfy the condition.

Thus it can be seen that the probability of exactly two heads in three tosses is $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$.

Example (9): A marble is drawn from a bag containing 2 blue and 3 red marbles, its colour noted, and **the marble replaced** in the bag.

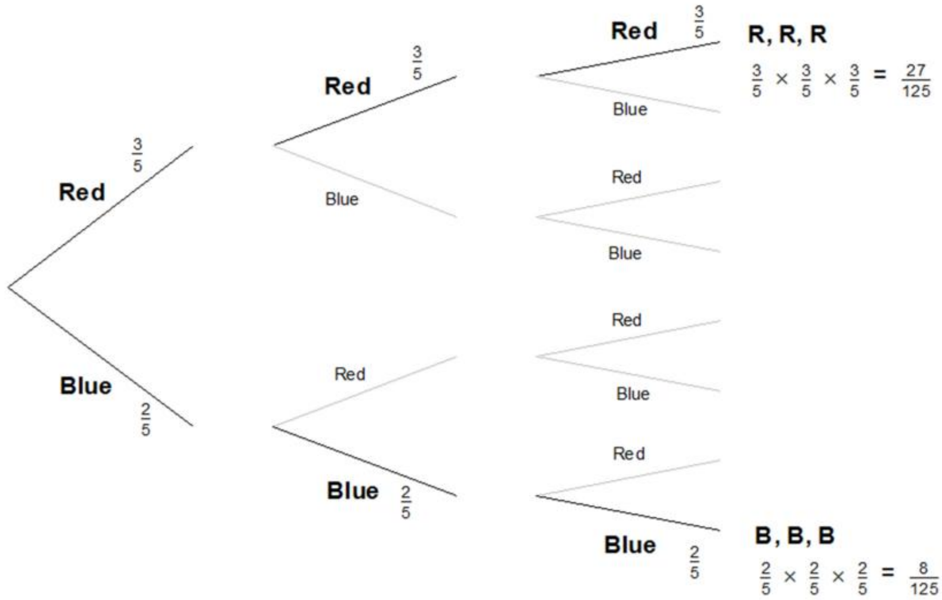
Find the probability that at least one blue and one red marble will be selected after 3 such draws.

The condition of "at least one blue and one red " can be interpreted to mean "do not include the case of three reds or three blues".

To refresh, there are eight possible outcomes: BBB, BBR, BRB, RBB, BRR, RBR, RRB and RRR.
It will be easier to find out the probabilities of three reds and three blues respectively, adding them, and subtracting from 1, a total of *two* outcomes, rather than trying to calculate the probabilities of *six* outcomes and adding them. The tree diagram illustrates this method.

Since there are 5 marbles in the bag in total, the probability of drawing a red each time is $\frac{3}{5}$ and that of drawing a blue is $\frac{2}{5}$. Because the marbles are drawn *with* replacement, the probabilities of a red and a blue do not differ between the first draw and the second.

(Redundant branches of the tree shown greyed out with smaller text).



Probability of "all reds" or "all blues" : $\frac{27}{125} + \frac{8}{125} = \frac{7}{25}$

Hence prob. of "at least one of each colour" : $1 - \frac{7}{25} = \frac{18}{25}$

Conditional probabilities.

So far, all the examples of tree diagrams referred to compound independent events, where the result of the first event had no bearing on the second. This does not apply to the next example !

Example (10):
There are 12 chocolates in a box, where 5 are milk chocolates and the remainder dark chocolates.
Carol chooses a chocolate at random, eats it, and then chooses a second one.
What is the probability that her second chocolate is a milk chocolate ?

The probability of choosing a milk chocolate first time is $\frac{5}{12}$, but if Carol were to choose and eat one of them, there would be only 4 milk chocolates left out of a box of 11. Hence the probability of her choosing a milk chocolate second time would be $\frac{4}{11}$.

On the other hand, had Carol chosen a dark chocolate first time, there would still be 5 milk chocolates left in the box of 11 remaining chocolates, and the probability of her choosing a milk chocolate second time would be $\frac{5}{11}$.

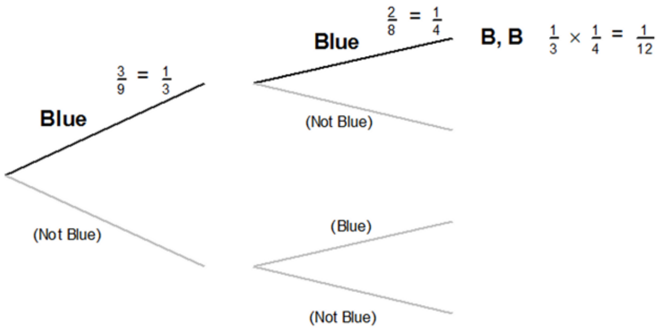
The example above demonstrates conditional probability, and this should be used whenever the question asks for **selection without replacement**.

Example (11): A marble is drawn from a bag containing 4 red, 3 blue and 2 green marbles, and then **not replaced** in the bag.

Find the following probabilities after two such draws, using tree diagrams:
i) both blue ii) exactly one red; iii) at least one green

Unlike the previous example, the marbles are *not* put back in the bag, and this alters the way in which the probabilities are calculated.

i) Both blue

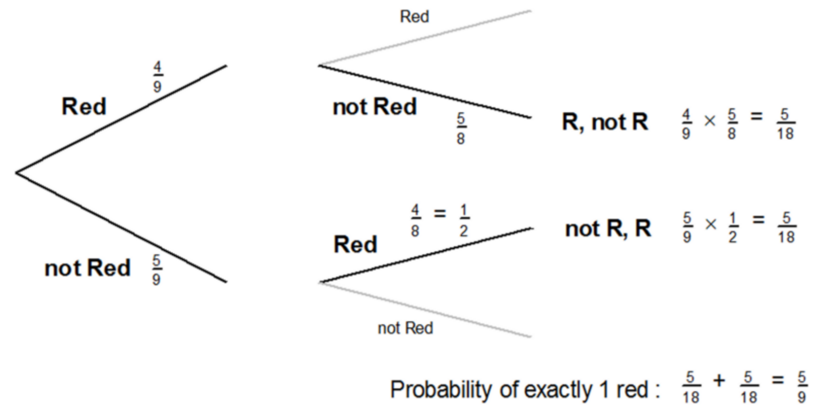


We are only interested in finding the probability of a blue, so we can combine the 'red' and 'green' draws into a 'not blue' category, which in any case is redundant.

Before the first draw, the probability of drawing a blue is $\frac{1}{3}$.
If the first marble drawn is blue, then there will be only 2 blue marbles left out of a total of 8 in the bag for the second draw, because there is no replacement.

Therefore, given that the first marble drawn is blue, the probability that the second one will be blue will be $\frac{1}{4}$, and hence by the product rule the probability that both will be blue is $\frac{1}{12}$.

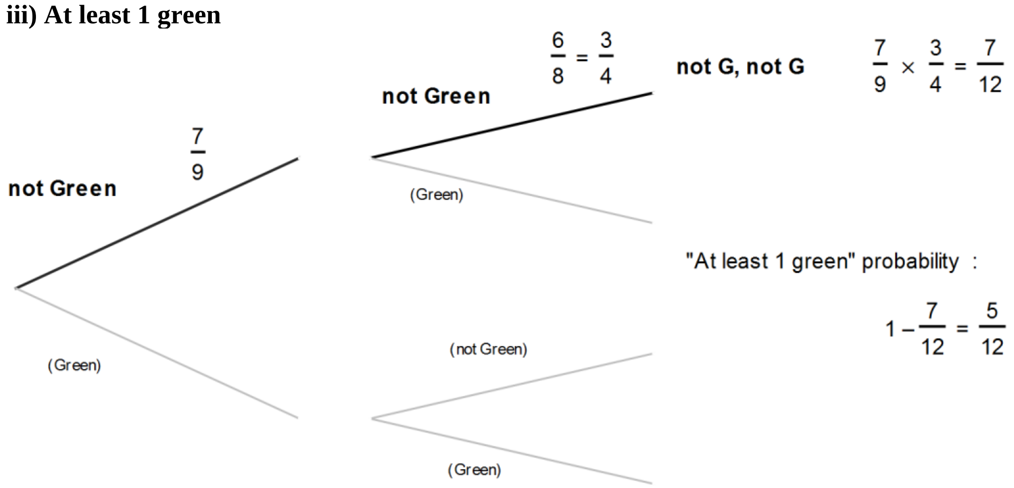
ii) Exactly one red marble



The only possible pair of draws satisfying the condition appears above.
We are not interested if the 'non-red' marble is blue or green, so we have combined the probabilities of 'blue' and 'green' into a 'not red' category.

If a red is drawn on the first draw, then 3 reds, and hence 5 'non-reds' will remain out of 8 in the bag, hence the probability of $\frac{5}{8}$ for a 'non-red' on the second draw.
If a 'non-red' is drawn on the first draw, then 4 reds will remain out of 8 in the bag, hence the probability of $\frac{1}{2}$ for a red on the second draw.
The probabilities of the two valid draws are then summed to give the overall probability of $\frac{5}{9}$.

iii) At least 1 green



The condition of "at least one green " can be interpreted to mean "do not include the cases where there are no greens at all".

Again, we can lump the red and blue events into one category, 'not green'.

It will be easiest to find out the probability of 'not green' followed by another 'not green'..

The probability of 'not green' first time is that of green, namely $\frac{2}{9}$, subtracted from 1, hence the $\frac{7}{9}$.
If a 'not green' is drawn on the second draw, then 6 'not greens' will remain out of 8 in the bag, hence the probability of $\frac{3}{4}$ for a 'not green' on the second draw.
The probability of two 'not greens' works out as $\frac{7}{12}$, and so the probability of the opposite event, namely at least one green, works out as 1-$\frac{7}{12}$ or $\frac{5}{12}$.

Showing probabilities on Venn diagrams.

An alternative way of expressing probabilities of compound independent events is by using Venn diagrams, although this is limited to events with just two possible outcomes, such as heads / tails in a coin throw, or win / lose in a game where a draw is impossible.

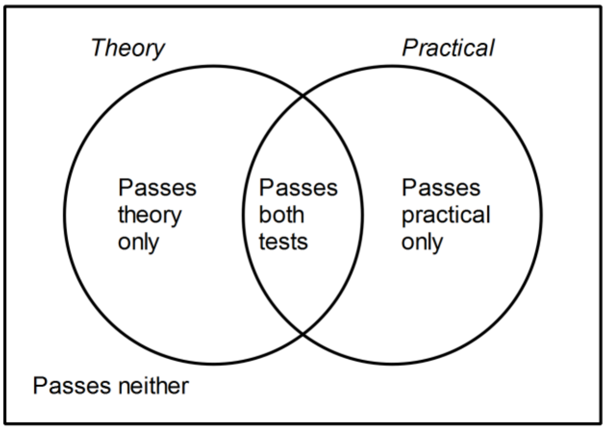
Example (12): Keith is taking his driving test, and has a 70% chance of passing his theory test, and an 80% chance of passing his practical test.

Show all of the possible outcomes, and their percentage probabilities, in a Venn diagram.

The events are shown as circles, where the inside of each circle represents a pass.
Since Keith can pass both tests, one test or none at all, there are four possible outcomes:

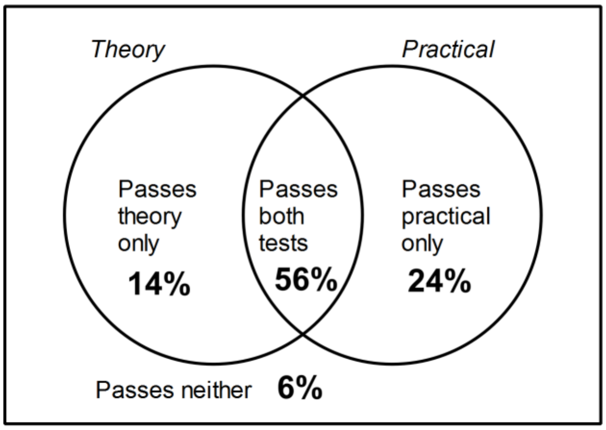
The region where the two circles overlap represents the case where Keith passes both tests.

The region in the box outside both circles represents the case of his failing both tests.



By the multiplication law, the probability of passing both the theory and practical tests is $0.7 \times 0.8 = 0.56$ (convert percentages to decimals !) or 56%

Since there are only two outcomes in each test, the probability of failing the theory test is $1 – 0.7$ or 0.3, and the probability of failing the practical test is 1- 0.8 or 0.2
Hence the probability of failing both tests is $0.3 \times 0.2 = 0.06$ or 6%.



This leave the cases where Keith passes only one test out of the two. These correspond to the 'outer' regions in each circle.

He has a 70% probability of passing the theory test, but we must subtract the case where he passes both, and so the probability of Keith passing the theory test alone is 70% - 56% = 14%.

He has a 80% probability of passing the theory test, so again we must subtract 56%. The probability of Keith passing the practical test alone is 80% - 56% = 24%.

As a final check, all the probabilities add up to 56% + 6% + 14% + 24% = 100%., as they should !

Frequency Trees.

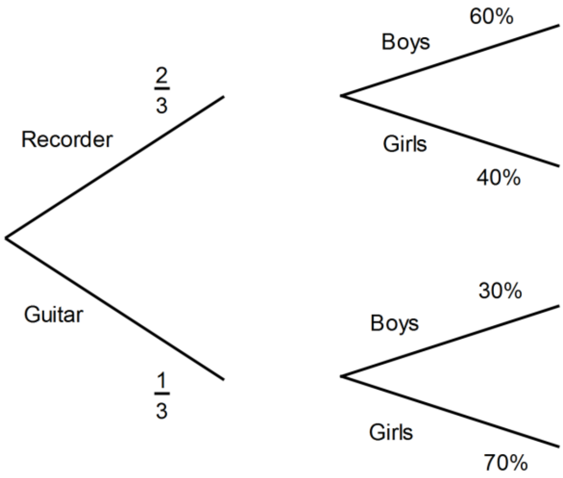
These diagrams are not unlike probability trees, but with extra numerical data added

Example (13): The pupils of Year 8 have a choice of learning to play either the recorder or the guitar should they wish to play a musical instrument.

In total, 60 pupils took up the challenge, and two-thirds chose to learn to play the recorder.
Of the pupils who chose the recorder, 60% of them were boys.
Of the pupils who chose the guitar, 70% were girls.

i) Complete the frequency tree.
ii) A girl pupil is chosen at random. What is the probability that she is learning to play the guitar ?

i) We can begin by setting up a probability tree as on the right.

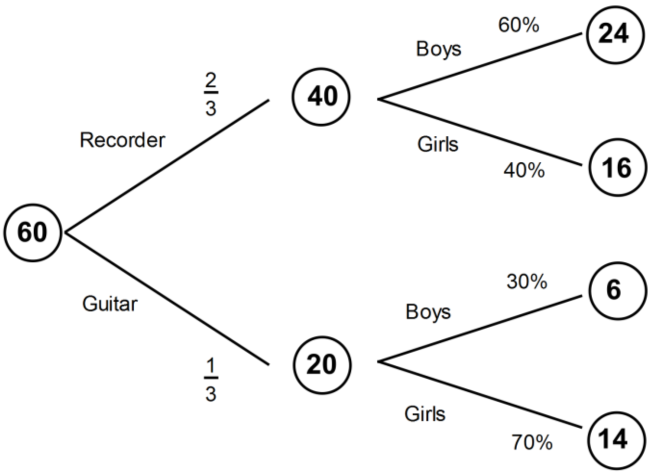


As we have 60 pupils in total, we can place the number 60 to the left of the leftmost branch.

Since two-thirds of 60 is 40, it follows that 40 pupils are learning the recorder and the remaining 20 are learning the guitar.

We therefore place 40 at the end of the “Recorder” branch and 20 at the end of the “Guitar” branch.

Now 60% of 40 is 24, so 24 boys are learning the recorder, as are 16 girls.



Similarly, 70% of 20 is 14, so 14 girls are learning the guitar, as are 6 boys.

ii) Since 14 girls are learning the guitar, and 16 the recorder, it follows that a girl chosen at random has a probability of $\frac{14}{14+16} = \frac{14}{30} = \frac{7}{15}$ that she is learning the guitar.