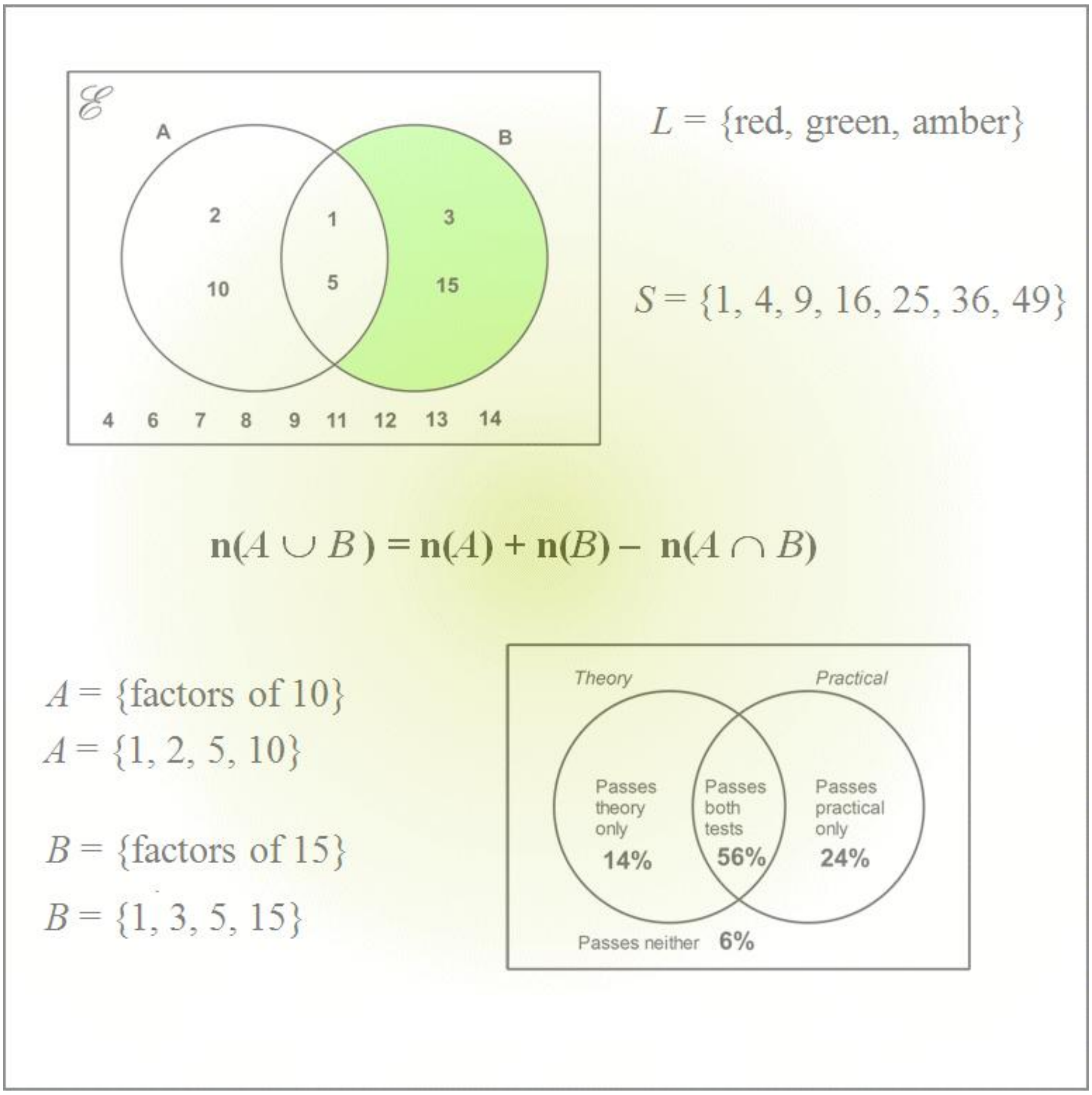
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Foundation Tier

SET THEORY AND VENN DIAGRAMS



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SET THEORY AND VENN DIAGRAMS

This is an introduction to the theory of sets, and the use of Venn diagrams.

Definitions.

A **set** is a collection of items which can be letters, numbers or any random objects. Each item in the set is called a **member** or an **element.** Sets can be finite or infinite in size, but we are only interested in finite ones for GCSE.

Sets can be defined by listing their members directly, or by a general description.

Thus we can have L = {colours of traffic lights} or L = {red, green, amber}.
Both definitions are equally valid, referring to the same set.

Another example would be S = {square numbers between 1 and 50} or S = {1, 4, 9, 16, 25, 36, 49}.

Since a set can be defined by listing its elements, we could decide on something like
D = {Frodo Baggins, Blackpool Tower, 26}. A bizarre set, but still a set !

Larger sets are best described by a definition, such as: P = {prime numbers less than 100}.

The elements of a set must be distinct. If we define set M to be the set of letters in the word MISSISSIPPI, that set would only have four members in it, i.e. M = (I, M, P, S).

Cardinality or "member count" of a set.

The **cardinality** of a (finite) set is simply the count of members in it – the cardinality of set A is given as n(A).

Example (1): Take sets D = {possible results of a standard dice throw} and P = {cards in a standard pack without Jokers}. Give their member counts.

There are six possible dice throw results, so n(D) = 6; there are 52 cards in a pack, so n(P) = 52.

The Universal set and the Null (empty) set.

The set of all elements under consideration is called the universal set, but there seems to be a lack of a defined standard. Some authorities give $\boldsymbol{U}$, others $\mathscr{E}$, others ξ and still others $\boldsymbol{\varepsilon}$**.**

We shall use $\mathscr{E}$ in this document to avoid confusion with the set union symbol $\cup$.

Examples of finite universal sets are the 26 letters of the alphabet and the positive integers up to 20.

A set containing no members is termed the **null set** or the **empty set**, and is denoted by the symbol $\varnothing$.

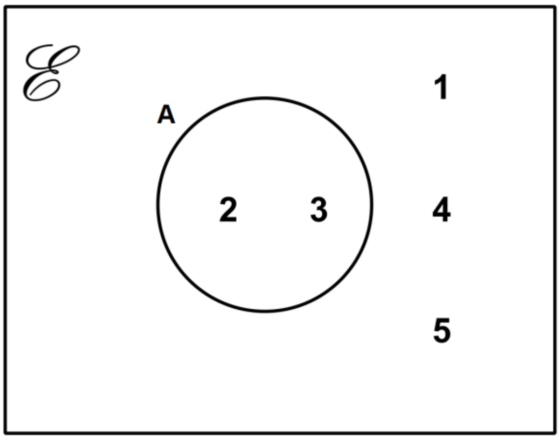
The Complement of a Set.

The complement of a set comprises the elements of the universal set that are not in the stated set.
Thus, if we had a universal set containing the numbers {1, 2, 3, 4, 5} and set A = {2, 3}, then the complement of A , written A', would contain the elements {1, 4, 5}.

The Venn Diagram.

Relationships between sets can be conveniently shown on **Venn diagrams**.

They consist of two or three overlapping circles enclosed in an outer rectangle.
We can also have the trivial case with one circle, of which an example is the universal set containing the numbers {1, 2, 3, 4, 5} and set A = {2, 3} stated earlier.



Since 2 and 3 are members of set A, they are included inside the Venn circle.

The numbers 1, 4 and 5 are part of the universal set but are not in set A, so they are outside the circle. In fact, they form the complement of A, or A'.

Intersection and Union of Sets.

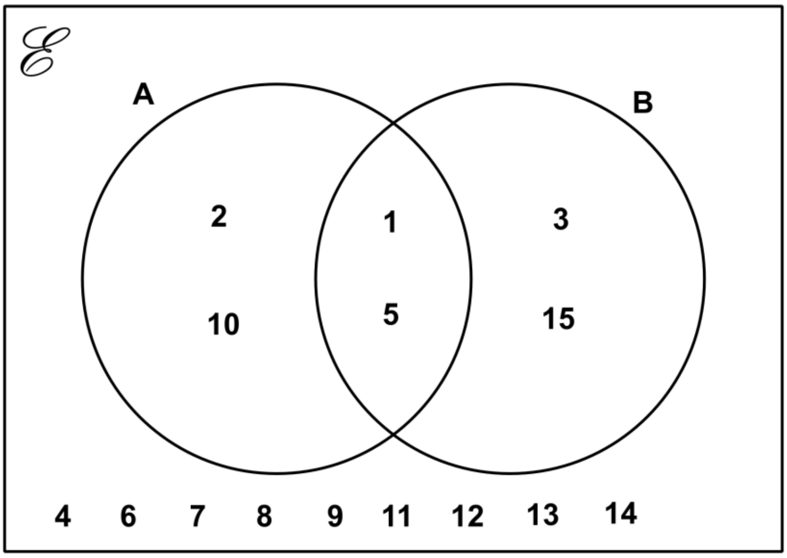
Suppose we were to take two sets as follows:

A = {factors of 10} and B = {factors of 15}, we can list their elements as
$A = \{1, 2, 5, 10\}$ and $B = \{1, 3, 5, 15\}$.

We can see that two elements, namely 1 and 5, are common to both sets.
The elements common to both sets form the **intersection** of the two sets, or $A \cap B$.
Note that the order is irrelevant; $A \cap B = B \cap A$.

Suppose we were to combine the elements of sets A and B into a larger set.
The new set would have the elements {1, 2, 3, 5, 10, 15}, defined as the **union** of the two sets, or $A \cup B$. Again, the order does not matter: $A \cup B = B \cup A$.

The Venn diagram below illustrates the situation. (We shall take the universal set to be all the positive whole numbers between 1 and 15.)



The intersection $A \cap B$ is the area of overlap of the two circles, containing the elements 1 and 5.

The union of the two sets, $A \cup B$, is the area enclosed by both circles combined, containing the elements 1, 2, 3, 5, 10 and 15.

The other nine members of the universal set are placed in the outside rectangle.

Versions of the Venn diagram can also be found under probability (see "Probability") and when finding the H.C.F and L.C.M. of two numbers (see "Factors, Primes, H.C.F., L.C.M.")

Example (2): Let set M = {letters in MONDAY} and set T = {letters in TUESDAY}.
(The universal set is that of the 26 letters of the alphabet).

i) Find a) the member count of M' ; b) $M \cap T$; c) $M \cup T$.
What do you notice about the member counts of M, T, $M \cap T$ and $M \cup T$?

ii) Illustrate the two sets M and T , and their elements, on a Venn diagram.
(Do not include the rest of the universal set.)

i) a) Since M has six members, M' consists of all the other entries in the universal set, i.e. the other twenty letters of the alphabet. Hence n(M') = 20.

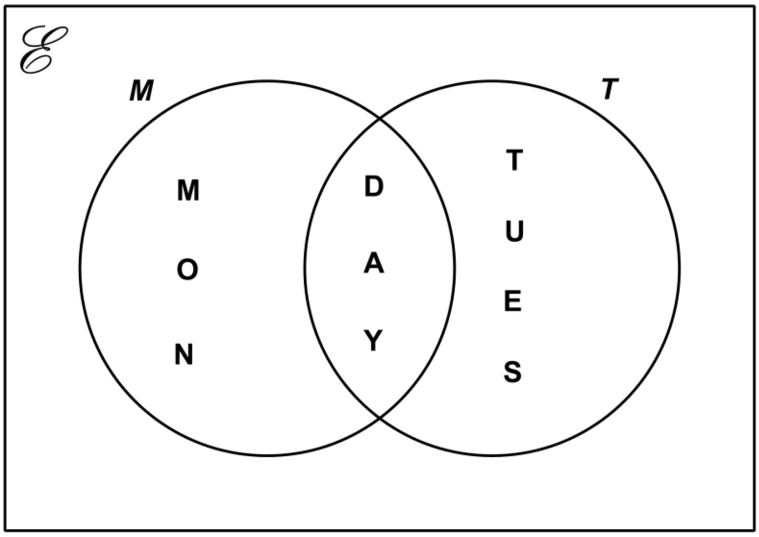
b) $M \cap T$ contains the letters common to both MONDAY and TUESDAY, i.e. {A, D, Y}.
c) $M \cup T$ contains the letters of both words combined, i.e. {A, D, E, M, N, O, S, T, U, Y}.

Note that n(M) = 6 and n(T) = 7, whilst n($M \cap T$) = 3. When we listed the 10 elements of $M \cup T$ we had to exclude one occurrence of $M \cap T$ to avoid duplication.

Therefore n($M \cup T$) = n(M) + n(T) – n($M \cap T$).

So to find the member count of the union of two sets, we add the individual member counts and subtract the member count of their intersection.

ii) The Venn diagram for the two sets is shown below.

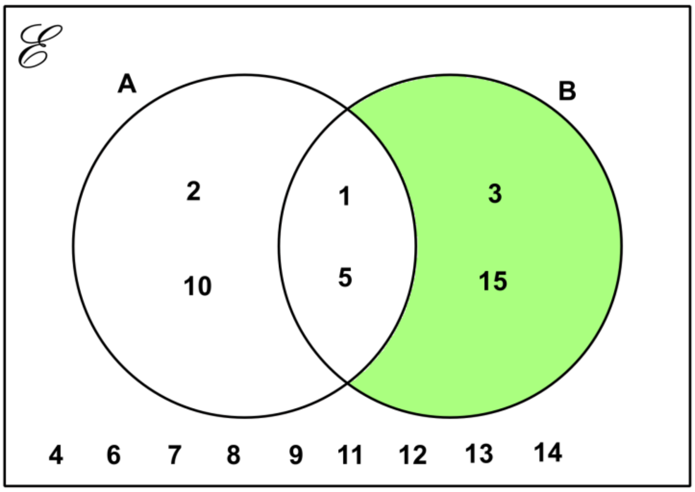


For any two sets A and B, n($A \cup B$) = n(A) + n(B) – n($A \cap B$).

Complements of Intersections and Unions.

Example (3): Take the following sets : A = {factors of 10} and B = {factors of 15}.

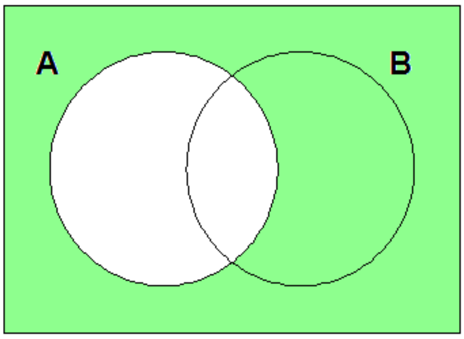
Describe the shaded area using set notation.



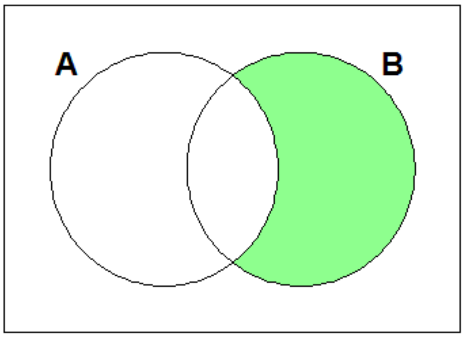
The numbers 3 and 15 are members of set B but not of set A. How would we describe this region of the Venn diagram in set notation ?

It would not be enough to simply say that the region corresponds to A', because that would end up including everything else that is in the universal set.

To exclude those "outside" elements, we must perform the intersection of B with the complement of A, namely $A' \cap B$.



A' (the complement of A)



$A' \cap B$ (in B, but not in A)

Example (4): A language class has 27 pupils, of which 18 are learning French, 12 are learning Spanish, and 4 are not learning either French or Spanish.

i) How many pupils are learning both languages ?
ii) How many pupils are learning French, but not Spanish ?
iii) A pupil is randomly chosen out of the Spanish class. What is the probability that this pupil is not also learning French ?

iv) Display the member counts of all the regions on a Venn diagram.

i) Let the universal set $\mathscr{E}$ be the class of 27 pupils.
Let F and S be the sets of pupils learning French and Spanish, respectively.

We also know that n(F) = 18 and n(S) = 12.

Also, because 4 of the 27 pupils are not learning either French or Spanish, it means that the remaining 23 are learning at least one of those languages.

Hence n($F \cup S$) = 23, i.e. there are 23 pupils in the union of the two sets.

The set of pupils learning both languages corresponds to the intersection of those sets.
We need to find n($F \cap S$), using the law of intersection and union of sets,
i.e. n($F \cup S$) = n(F) + n(S) – n($F \cap S$).

We rearrange the formula as n($F \cap S$) = n(F) + n(S) - n($F \cup S$)

(Add the member counts of both sets and subtract that of their union.)

Substituting n(F) = 18, n(S) = 12 and n($F \cup S$) = 23, we have n($F \cap S$) = 7.

Hence 7 out of the 27 pupils are learning both French and Spanish.

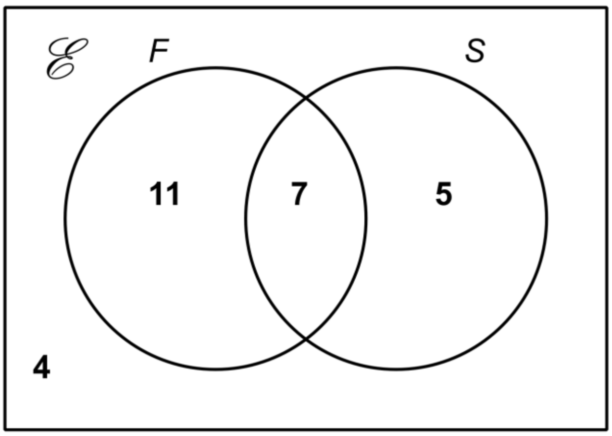
ii) Since 18 pupils are learning French, and 7 are learning both French and Spanish, it follows that 11 are learning French, but not Spanish.

iii) There are 12 pupils in the Spanish class, and 7 of them are also learning French, so there are 5 remaining who are not learning French.

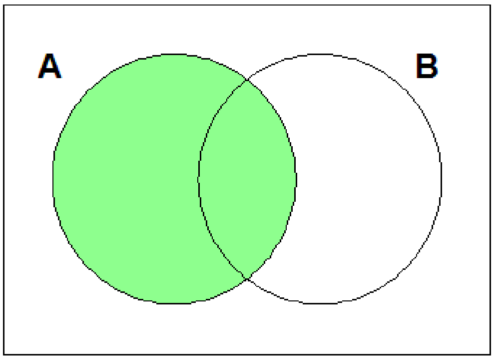
The probability that a randomly-chosen student out of the Spanish class is not learning French is therefore $\frac{5}{12}$.

iv) The Venn diagram is shown on the right.

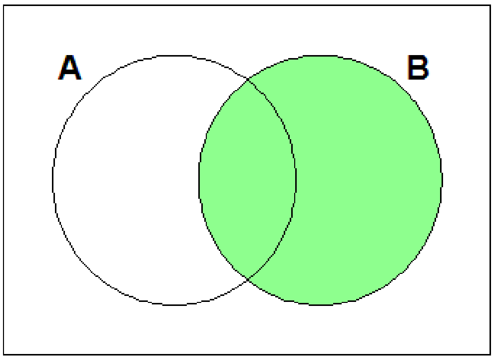
Note that the entries in each region are **counts** of elements, and not actual elements !



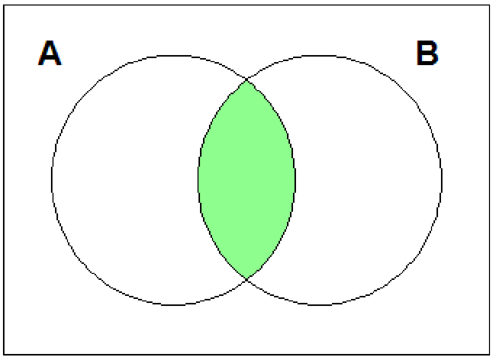
Summary of Venn Diagrams with 2 circles.



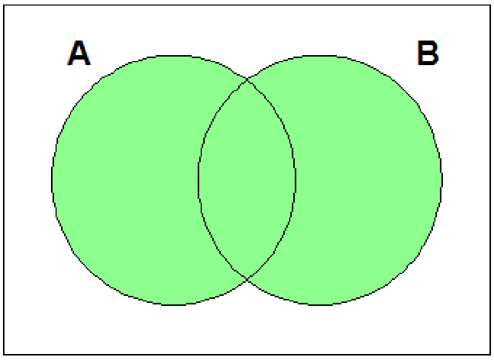
Shaded : Set A



Shaded : Set B



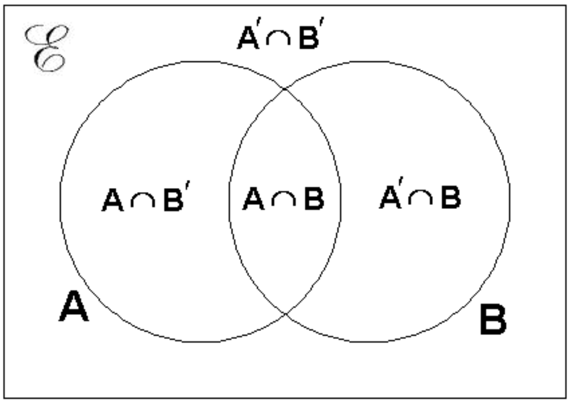
Intersection of sets: $A \cap B$



Union of sets: $A \cup B$

The sets are shown as a pair of overlapping circles; their intersection is the area common to both, while their union is the total area enclosed by both.

The diagram on the right is the 'key' for the intersection of two sets *A* and *B.*



Venn diagrams in number problems.

Example (5).

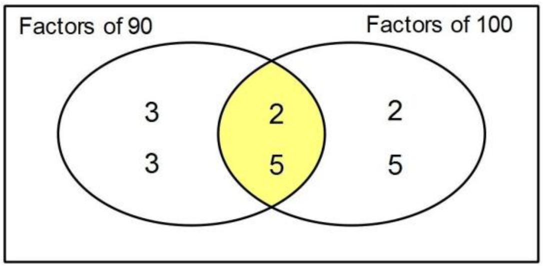
i) Given that $90 = 2 \times 3 \times 3 \times 5$ and $100 = 2 \times 2 \times 5 \times 5$, use a Venn diagram to place the factors and hence find the H.C.F. of 90 and 100
ii) Use the Venn diagram from i) to also find the L.C.M. of 90 and 100.

i) We can see that one of the 2's and the 5 are common factors of 90 and 100.

$$90 = 2 \times 3 \times 3 \times 5$$
$$100 = 2 \times 2 \times 5 \times 5$$

The Venn diagram will include two sets of numbers, namely the factors of 90 and the factors of 100.

Because 2 and 5 are common factors of both, we place them in the intersection of the two sets.
The two occurrences of 3 are factors of 90 but not of 100, so we place them in the non-intersecting 'Factors of 90' region.
Similarly, there is an occurrence of 2 and an occurrence of 5 which must be placed in the non-intersecting region of the 'Factors of 100' region.

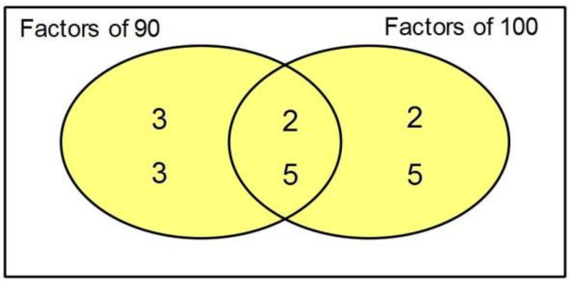


The numbers in the intersection of the two sets, 2 and 5, give us the H.C.F. of 90 and 100 when we multiply them together.

The H.C.F. of 90 and 100 is therefore 2×5 or 10.

(Note that this Venn diagram breaks the rules that elements of a set must be distinct, but we can ignore it in this context)

ii) Just as the H.C.F. of 90 and 100 was found by multiplying together the numbers in the **intersection** of the two sets of factors, their L.C.M. can be found by multiplying together all the numbers in the **union** of the two sets of factors.



The L.C.M. of 90 and 100 is therefore $2 \times 5 \times 3 \times 3 \times 2 \times 5$ or 900.

Venn diagrams in probability problems.

(See Example (4) for another probability tie-in.)

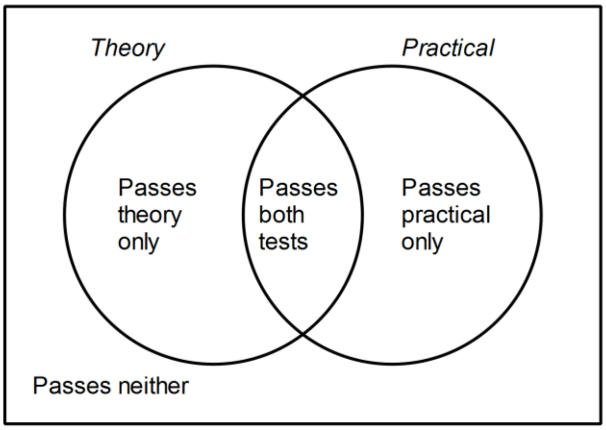
Example (6): Keith is taking his driving test, and has a 70% chance of passing his theory test, and an 80% chance of passing his practical test.

Show all of the possible outcomes, and their percentage probabilities, in a Venn diagram.

The events are shown as circles, where the inside of each circle represents a pass.
Since Keith can pass both tests, one test or none at all, there are four possible outcomes:

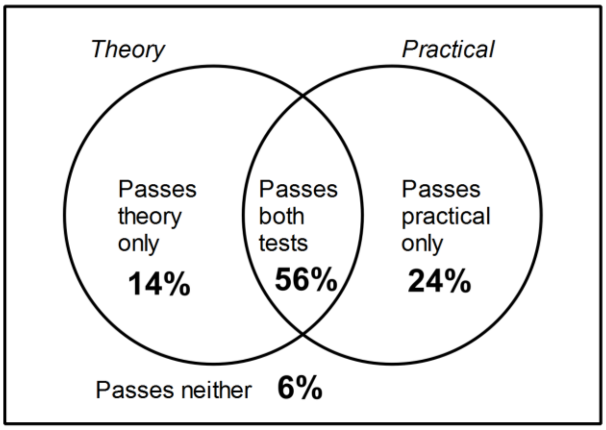
The region where the two circles overlap represents the case where Keith passes both tests.
In set notation, this is {Theory} ∩ {Practical}

The region in the box outside both circles represents the case of his failing both tests.
The set notation is ({Theory} ∪ {Practical})'.



By the multiplication law, the probability of passing both the theory and practical tests is $0.7 \times 0.8 = 0.56$ (convert percentages to decimals !) or 56%.

Since there are only two outcomes in each test, the probability of failing the theory test is $1 - 0.7$ or 0.3, and the probability of failing the practical test is 1- 0.8 or 0.2
Hence the probability of failing both tests is $0.3 \times 0.2 = 0.06$ or 6%.



This leave the cases where Keith passes only one test out of the two. These correspond to the 'outer' regions in each circle.

He has a 70% probability of passing the theory test, but we must subtract the case where he passes both, and so the probability of Keith passing the theory test alone is 70% - 56% = 14%.

He has a 80% probability of passing the practical test, so again we must subtract 56%. The probability of Keith passing the practical test alone is 80% - 56% = 24%.

As a final check, all the probabilities add up to 56% + 6% + 14% + 24% = 100%., as they should !