M.K. HOME TUITION

Mathematics Revision Guides:

Level: GCSE Higher Tier

NON-CALCULATOR ARITHMETIC

NON-CALCULATOR ARITHMETIC

Revision of the BIDMAS rule.

The BIDMAS rule is used to determine the order in which a complex calculation is carried out.
It is a set of precedence rules for the different operations in arithmetic, where:

B –Brackets have the highest priority

I - Indices (i.e. powers such as squares and cubes) are next, followed by

DM – (Division and Multiplication) having **equal** priority, and lastly

AS – (Addition and Subtraction) again having **equal** priority.

Note: Some authorities call the rule BODMAS, where O = "Others" / "Of".

If all the operations in the calculation have equal priority, then the processing is carried out from left to right.

Note as well that the four basic operations, namely addition, subtraction, multiplication and division, are in fact **binary** operations because they connect **two** numbers or quantities.

Examples(1a): Work out i) $2 \times 3 + 4 \times 5$ ii) $3 \times 5^2 - 4$ iii) $12 - 5 + 3$

i) Multiplication has priority over addition and is performed first.
Therefore; $2 \times 3 + 4 \times 5$ is reckoned as $2 \times 3 + 4 \times 5 = 6 + 20 = $ **26**.

ii) There are three operations here, a squaring ('indices'), a multiplication and a subtraction.
The index operation (here squaring) has the highest priority, so we start with

$3 \times 5^2 - 4 = 3 \times 5^2 - 4 = 3 \times 25 - 4$.

Next, multiplication has priority over subtraction, so we work $3 \times 25 - 4 = 75 - 4 = $ **71** .

iii) There are two operations here, a subtraction and an addition. Both have equal priority, so the sum is simply worked out from left to right as $12 - 5 + 3 = 12 - 5 + 3 = 7 + 3 = $ **10**.

We do *not* reckon $12 - 5 + 3 = 12 - 5 + 3 = 12 - 8 = $ **4** by wrongly thinking that addition has a higher priority than subtraction. This is one of several BIDMAS 'traps'.

As another example, the sum $2 \times 3 + 4 \times 5$ would have its processing order and result altered if we were to put brackets around it :

$(2 \times 3 + 4) \times 5 = 10 \times 5 = 50$.

The expression inside the brackets is worked out first, using BIDMAS where the multiplying is carried out before adding to give 10, and then the 10 is multiplied by 5 to give 50.

Other BIDMAS 'traps'.

Uses of the minus sign.

Taking an expression such as $12 - 8$, we can see that the minus sign represents ordinary subtraction; in other words, we subtract 8 from 12. Subtraction is a **binary** operation since the minus sign acts on **two** quantities, i.e. the numbers 8 and 12.

We have to consider the other use of the minus sign, as in the expression $-4.$ Here the purpose of the sign is not to subtract one number from another, but to change the sign of the number 4 from positive to negative. This operation takes only **one** number and not two, and is termed a **unary** operation.

An expression like -4^2 can cause problems when written this way, because there are two operations here, namely a unary minus and a power. In fact, different calculators and computers could give differing results when such an expression is entered !

We could square the 4 first and then change the sign to get -16, or we could change the sign and then square -4 to get 16. We need brackets to distinguish between the two.

Thus $-(4^2) = -16$ and $(-4)^2 = 16$.

Expressions in non-linear form.

Example (1b) : i) Find the value of $\dfrac{15+9}{3+5}$; ii) Re-express $\dfrac{15+9}{3+5}$ in one-line BIDMAS form.

i) This expression is in algebraic fraction form , so we would work out the numerator and denominator separately, and divide last. Hence $\dfrac{15+9}{3+5} = \dfrac{24}{8} = 3$.

ii) Putting the expression into one-line BIDMAS form needs a little more care.

The operation here clearly includes division, but with a fraction bar used instead of a division sign. We cannot simply convert the expression to $15 + 9 \div 3 + 5$ since division has priority over addition in BIDMAS, and the sum would be worked out as

$15 +$ <mark>$9 \div 3$</mark> $+ 5 = 15 + 3 + 5 = 23$ which is incorrect.

To prevent this misinterpretation, we must put brackets around the expressions on both the top and bottom lines in order to perform the division last, so that the sum works out as $(15 + 9) \div (3 + 5)$ and hence $24 \div 8 = 3$, which was our expected answer.

In general, if an expression is written in fractional form, it is best to work out the numerator and denominator separately using the normal BIDMAS rules, and then perform the fractional division last.

(Modern calculators now have the option of inputting expressions such as the last one in fractional form, and thus preventing such errors in interpretation).

In general, if we want to force the calculation to override the usual BIDMAS rules, we must place brackets in the right place in the expression.

Another example of algebraic shorthand in an arithmetic expression is the omission of the implied multiplication sign before a bracketed expression, such as 2(1 + 3), the short form of $2 \times (1+3)$.

Example (1c) : i) Find the value of $\dfrac{4 \times 6}{2(1+3)}$; ii) Re-express $\dfrac{4 \times 6}{2(1+3)}$ in linear BIDMAS form.

i) Notice that the denominator of the algebraic expression has a missing implied multiplication sign, so

it works out as $\dfrac{4 \times 6}{2(1+3)} = \dfrac{24}{2(4)} = \dfrac{24}{2 \times 4} = \dfrac{24}{8} = 3$.

ii) In order to put the expression into one-line BIDMAS form, we need to put the multiplication sign into the denominator, to give $2(1 + 3) = 2 \times (1 + 3)$.

Secondly, we need to force the final division to be performed last, so we put brackets around both the top and bottom of the fraction :

$(4 \times 6) \div (2 \times (1 + 3))$. .

Check; $(4 \times 6) \div (2 \times (1 + 3)) = (4 \times 6) \div (2 \times (1 + 3)) = (4 \times 6) \div (2 \times 4)$

(we have worked out the addition in the innermost brackets first)

Next, we deal with the outer brackets, followed by the last division:
$(4 \times 6) \div (2 \times 4) = 24 \div 8 = 3$.

It is all too easy to omit additional brackets when placing a more complicated expression into BIDMAS form.

In this example we could have begun with $(4 \times 6) \div 2(1 + 3) = (4 \times 6) \div 2 \times (1 + 3)$.

We have made the error of forgetting to put brackets around the denominator $2 \times (1 + 3)$, and that has altered the meaning of the whole expression.

If we were to apply BIDMAS now, we would have the incorrect result

$(4 \times 6) \div 2 \times (1 + 3) = 24 \div 2 \times 4 = 48$.

Note the difference in processing here; the brackets are worked out first, but then we are left with the expression $24 \div 2 \times 4 = 48$ which is worked left-to-right since multiplication and division have equal priority.

Always put brackets around the *whole of the* numerator and denominator when putting fractional expressions into linear BIDMAS form.

Example (2a): The *Countdown* game show features a game where contestants are given a set of six numbers and are then asked to combine them to give a larger number, using only brackets and the four basic arithmetic operations.
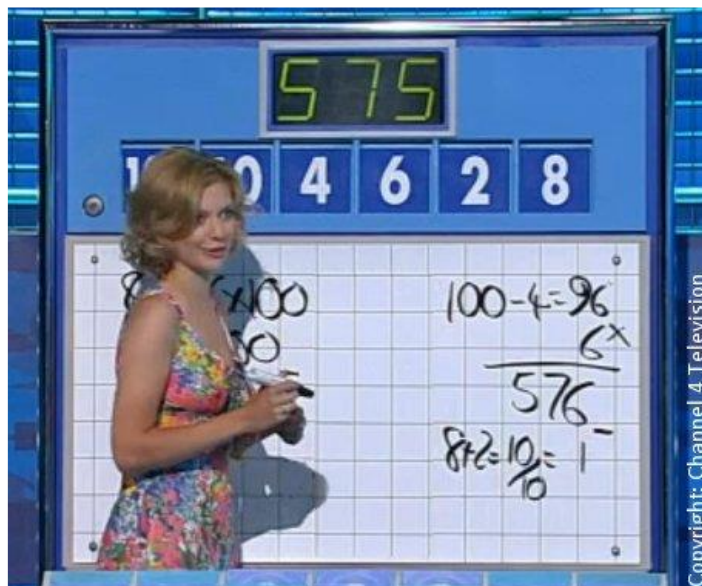
Here, the given numbers are 100, 10, 4, 6, 2 and 8 and the target number is 575.

The first part of the calculation goes

100 – 4 = 96; 96 × 6 = 576

The second part goes

8 + 2 = 10 ; 10 ÷ 10 = 1

Finally, the result of the second part is subtracted from that of the first, giving 576 – 1 = 575.

i) Use your knowledge of BIDMAS to write an expression for the first part of the calculation, putting brackets in where necessary.

ii) Repeat part i) for the second part of the calculation.

iii) Are brackets necessary for completing the final part of the calculation ? If not, why not ?

i) It would do no good to say 100 – 4 × 6 , as BIDMAS rules would force the multiplication to be carried out before the subtraction, giving 100 – 24, or 76.

We therefore need to insert brackets around the subtraction sum to override the BIDMAS rule.
This part of the sum is therefore (100 – 4) × 6 , namely 96 × 6 or 576.

ii) Again, it would be wrong to say 8 + 2 ÷ 10, as BIDMAS rules would force division before addition, giving 8 + 0.2, or 8.2.

We insert brackets around the addition sum to override BIDMAS;
(8 + 2) ÷ 10 , which gives 10 ÷ 10 or 1.

iii) By writing the whole calculation in one go, we have the result:

(100 – 4) × 6 - (8 + 2) ÷ 10 = 575.

We could have used extra brackets around each 'sub-total' and obtained

((100 – 4) × 6) – ((8 + 2) ÷ 10) = 575.

The extra brackets are not needed here, because the final operation, namely subtraction, has a lower priority then either the brackets, the multiplication or the division.

However, adding extra matched brackets never does any damage when done correctly !

Example (2b): Here, the given numbers are 25, 75, 50, 2, 7 and 10 and the target number is 441.

The calculation goes $10 \times 50 = 500$; $500 - 75 = 425$; $425 + 25 - 7 - 2 = 441$.

Express this working as a single sum, using BIDMAS and with no more brackets than necessary.

This time we can express the entire sum as

$10 \times 50 - 75 + 25 - 7 - 2 = 441$.

The multiplication is performed first, and then the additions and subtractions are performed left-to-right as they have equal precedence. Hence no brackets are needed !
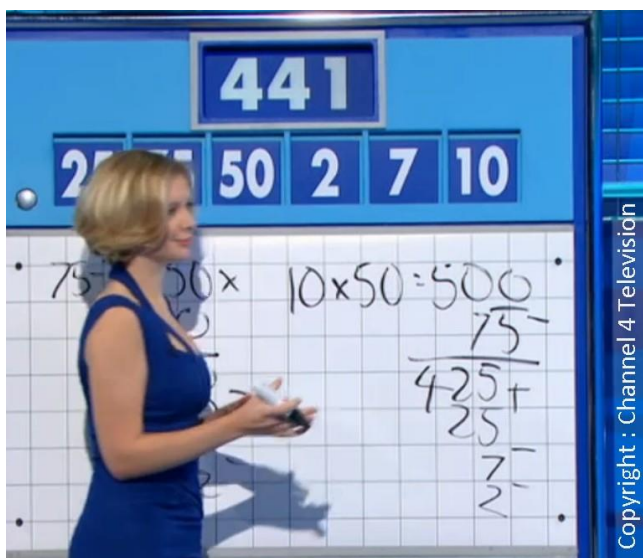
Two other solutions are:

i) $75 - 25 = 50$; $50 \times 10 = 500$; $500 - 50 - 7 - 2 = 441$.

ii) $75 - 10 - 2 = 63$; $63 \times 7 = 441$.

Express each as a single sum in BIDMAS form, and explain what would happen if the brackets were omitted.

i) $(75 - 25) \times 10 - 50 - 7 - 2 = 441$.

Here the brackets are needed to force the sum $75 - 25$ to be performed before the multiplication by 10, since $75 - 25 \times 10 - 50 - 7 - 2 = -234$.

The multiplication $25 \times 10 = 250$ would be performed first, and then the rest of the sum worked out as $75 - 250 - 50 - 7 - 2 = -234$.

ii) $(75 - 10 - 2) \times 7 = 441$.

Here the brackets are needed to force the sum $75 - 10 - 2$ to be performed before the multiplication by 7, since $75 - 10 - 2 \times 7 = 51$.

The multiplication $2 \times 7 = 14$ would be performed first, and then worked out as $75 - 10 - 14 = 51$.

Note on the correct use of the equals sign.

Take the statement $100 - 4 = 96$ from Example (1).

This statement is an example of an **equation**, where two quantities are separated by an equals sign.

The equals sign states that the expression on the left, i.e. the result of subtracting 4 from 100, is equal in value to the expression on the right, namely the number 96.

Similarly the equation $96 \times 6 = 576$ states that the values of the expressions on each side of the equality sign are equal.

The two stages of the calculation can then be combined :

$100 - 4 = 96; \quad 96 \times 6 = 576$

Notice how the stages are combined using a semicolon to separate them.

Never try combining such stages by equality signs as below :

$100 - 4 = 96 = 96 \times 6 = 576$ **Incorrect !**

According to the second equals sign, we have $96 = 96 \times 6$ which is clearly nonsense.

Equally bad is an attempt to think through the expression along the lines of "100 minus 4 is 96, then times 6 is 576", and then "condense" it to

$100 - 4 = 96 \times 6 = 576$ **Incorrect !**

There will be more about the correct use of the equality sign in the section on algebra, since this is where there are the most opportunities for its misuse.

Mental Methods.

To obtain good speed in mental arithmetic methods, it is necessary to know the tables up to 12×12 confidently. Rapid halving and doubling is particularly useful, as will be seen later.

Some multiplications and divisions are particularly simple, especially by 10, 100 and 1000.

Multiplying by 10, 100, 1000.....

All of those operations are related, and involve moving the decimal point (or adding zeros to the end). The following examples show how this works:

Using whole numbers:

456 × 10 =	**4560**	one zero added to end of number.
456 × 100 =	**45600**	two zeros added to end of number.
456 × 1000 =	**456000**	three zeros added to end of number.

∴ **When multiplying a whole number by 10, 100, 1000 ...., we add 1, 2, 3... zeros to the end of it.**

Using decimal numbers:

21.64 × 1 =	**21.64**	original number
21.64 × 10 =	**216.4**	decimal point moved one place right.
21.64 × 100 =	**2164**	dec. pt moved two places right and 'dropped'.
21.64 × 1000 =	**21640**	note the extra zero needed !

The need for the extra zero could have been avoided in the last case by expressing 21.64 as 21.640 :

21.640 × 1000 =	**21640**	dec. pt moved three places right and 'dropped'.

∴ **When multiplying a decimal number by 10, 100, 1000 ...., we move the decimal point 1, 2, 3... places to the right.**

Note: This method is also often illustrated by keeping the decimal point fixed and moving the *numbers* to the left instead. In fact, it demonstrates the principles of place value more clearly.

Example (3):   Find the values of a) 327×10; b) 803×100; c) 3.629×100; d) 0.0732×10 e) 9.546×1000; f) 4.95×1000.

a) 327 is a whole number, so **327 × 10 = 3270** (add 0 to the end).

b) 803 is also whole, so **803 × 100 = 80300** (add 2 zeros to the end).

c) **3.629 × 100 = 362.9** (decimal point moved two places right).

d) **0.0732 × 10 = 0.732** (decimal point moved one place right).

e) **9.546 × 1000 = 9546** (decimal point moved three places right and 'dropped').

f) If we try to move the decimal point three places to the right, we seem to 'run out' of decimal places. We can avoid this problem by writing 4.95 as 4.950, so **4.950 × 1000 = 4950**.

Dividing by 10, 100, 1000.....

The methods of division by 10, 100, 100 ..... are closely related to those of multiplication.
It is best, however, to treat whole numbers as decimals with an 'invisible' point at the end unless they
have enough zeros at their ends for easy cancellation.

Thus, $3760 \div 10 = 376$ (one zero cancelled each side) and $417000 \div 100 = 4170$ (two zeros cancelled
each side).

Using decimal numbers:

$$51.37 \div 1 = \quad 51.37 \quad \text{original number}$$

$$51.37 \div 10 = \quad 5.137 \quad \text{decimal point moved one place left.}$$

$$51.37 \div 100 = \quad 0.5137 \quad \text{decimal point moved two places left.}$$

$$51.37 \div 1000 = \quad 0.05137 \quad \text{note the extra zero after the decimal point !}$$

The need for the extra zero could have been avoided in the last case by expressing 51.37 as 051.37 :

$$051.37 \div 1000 = 0.05137 \text{ decimal point moved three places left.}$$

∴ **When dividing a decimal number by 10, 100, 1000 ...., we move the decimal point 1, 2, 3...
places to the left.**

Note: Again, this method is also often illustrated by keeping the decimal point fixed and moving the
numbers to the right instead.

Example (4): Find the values of a) $287 \div 10$; b) $756000 \div 100$; c) $518.6 \div 100$; d) $4.53 \div 10$;
e) $67.3 \div 1000$.

a) 287 is a whole number, so **287. $\div$ 10 = 28.7** (add 'dummy' dec. pt to the end and move it 1 place left)

b) As 756000 has three zeros on the end and 100 has two, we can merely cancel out two zeros to get
756000 $\div$ 100 = 7560.

c) **518.6 $\div$ 100 = 5.186** (decimal point moved two places left).

d) **4.53 $\div$ 10 = 0.453** (decimal point moved one place left).

e) If we try to move the decimal point three places to the left, we seem to 'run out' of decimal places.
We can avoid this problem by writing 67.3 as 067.3, so **067.3 $\div$ 1000 = 0.0673**.

Using the above methods, we can perform quick multiplications and divisions by 20, 30 and so forth.
Also we can multiply by 5 quickly by multiplying by 10 and halving.

Other quick tricks; to multiply by 15, we can multiply by 30 and halve, since $15 = 30 \div 2$.
To multiply by 25, we can multiply by 100 and halve twice, i.e. quarter, since $25 = 100 \div 4$.
Try finding some more out by yourself.

Examples (5):

Work the following out quickly;
a) 17×20;   b) 3.1×40;   c) 23×300;   d) $1560 \div 30$
e) 284×5;   f) 83×25;   g) 2.8×15;   h) 37×500

In a) we quickly find that $17 \times 2 = 34$, then we add a 0 at the end to get 340, $\therefore$ **$17 \times 20 = 340.$**

For b) it is easier to do $3.1 \times 10 = 31$ first, and then multiplying by 4 to get 124.
$\therefore$ **$3.1 \times 40 = 124$**.

In part c) we find $23 \times 3 = 69$; add two zeros to the end to get 6900, $\therefore$ **$23 \times 300 = 6900.$**

For d), both 1560 and 30 have a common factor of 10, so the answer is the same as that of $156 \div 3$,
namely 52, $\therefore$ **$1560 \div 30 = 52$**.

In e), we can halve 284 to obtain 142, then add a 0 to the end to obtain 1420 $\therefore$ **$284 \times 5 = 1420$**.
(Had we been asked to multiply an odd number by 5, it would have been better to add the 0 at the end
and then halve the result.)

For part f), put two zeros at the end of 83 to obtain 8300, then halve once for 4150, and again for 2075.
$\therefore$ **$83 \times 25 = 2075$**.

In g), we firstly multiply by 10 to get 28, then multiply by 3 to get 84, and finally halve to get 42.
$\therefore$ **$2.8 \times 15 = 42.$**

Finally, in h) we see that 500 is half of 1000. We thus put 3 zeros after 37 and halve to get 18500.
$\therefore$ **$37 \times 500 = 18500.$**
 (If multiplying an even number by 500, it's better to halve before putting the 3 zeros at the end.)

There are also some tricks available in addition and subtraction. Look at the sums below:

296 + 427 = 723
300 + 423 = 723

We have simplified the addition sum by *adding* 4 to the left-hand number, 296, whilst *subtracting* the
same number, 4, from the right-hand number, 427. This makes the second sum easier to work out
without losing any accuracy.

Examples (5a) : Work out the following quickly : i) $336 + 495$; ii) £5.99 + £7.99 + 65p.

i) We add 5 to 495 and subtract 5 from 336 to form the sum $331 + 500 = 831$.
ii) We add 1p to both £5.99 and £7.99, then subtract 2p from 65p (remember, we've added 1p twice !)
to change the sum into £6.00 + £8.00 + 63p = £14.63.

Look also at the following:

854 – 292 = 562
862 – 300 = 562

We have simplified the subtraction sum by adding 8 to both numbers on each side of the minus sign. Again, this makes the second sum easier to work out without losing any accuracy.

Notice that, with subtraction, we need to perform the same changes on each side, unlike with addition.

Another "trick" is as follows:

1002 – 648 = 354
999 – 645 = 354

The second sum does not look like a simplification at first, but it makes the sum easier to perform because there is no "borrowing" of digits. All of the digits of 645 can be subtracted from those of 999 without any need to "borrow". Thus 9 – 5 = 4 (units), 9 – 4 = 5 (tens) and 9 – 6 = 3 (hundreds).

Examples (5b) : Work out the following quickly : i) 978 - 396; ii) £10.00 - £7.39

i) We add 4 to both 978 and 396 to form the sum 982 - 400 = 582.

ii) We subtract 1p from both £10.00 and £7.39, to change the sum into £9.99 - £7.38 = £2.61.
We reckon 9-8 = 1 (pence), 9-3 = 6 (10p's), 9-7 = 2 (£s) as we work from right to left.

Mechanical Methods.

Non-calculator exam questions will also usually include mechanical arithmetic, using standard written methods. We recall them here:

Addition.

Example (6): Add together 395 and 841.

Working is shown below:
5 + 1 = 6, so we write down 6.
Then 9 + 4 = 13, so we write down 3 and 'carry' the 1 on to the next column.
Finally 3 + 8 = 11, plus the 1 carried over in the last step = 12.
As there are no more columns to add, we write down 12 and the sum is complete.

```
                    1               1
   3 9 5         3 9 5           3 9 5
 + 8 4 1       + 8 4 1         + 8 4 1
 ───────       ───────         ───────
       6           3 6         1 2 3 6
```

∴ **395 + 841 = 1236.**

We can add together as many numbers as we like in columns.

Example (7): Find the value of 795 + 361 + 482.

5 + 1 + 2 = 8, so write down 8.
9 + 6 + 8 = 23, so write down 3 and carry 2.
7 + 3 + 4 = 14, plus the 2 carried over = 16,
and finally write down 16 to finish the sum.

```
     2
     7 9 5
 +   3 6 1
 +   4 8 2
 ─────────
   1 6 3 8
```

∴ **795 + 361 + 482 = 1638.**

When adding together decimals, we must ensure that decimal points are lined up correctly.

Example (8): What is 27.56 + 4.3 + 116 ? Hence find £27.56 + £4.30 + £116.

In each case, the working is the same:
6 + 0 = 6 , so write down 6.
5 + 3 = 8 , so write down 8.
7 + 4 + 6 = 17, so write down 7 and carry 1.
2 + 1 = 3 , plus 1 carried over = 4, so write down 4.
1 + 0 = 1, so write down 1.

```
                                       £       p
     1                             1
   2 7 . 5 6                     2 7 . 5 6
       4 . 3                         4 . 3 0
 1 1 6 .                       1 1 6 . 0 0
 ───────────                   ───────────
 1 4 7 . 8 6                   1 4 7 . 8 6
```

∴ **27.56 + 4.3 + 116 = 147.86,**
and **£27.56 + £4.30 + £116 = £147.86.**

In the decimal sum, there is no need to write in extra trailing zeros to 'bulk up' the columns, but when writing money, the zeros should be included, as we write £4.30 and not £4.3.

Subtraction.

Example (9) : Subtract 762 from 1538.

The working is shown below:

Subtracting 2 from 8 gives us 6, so we write down 6.

We cannot subtract 6 from 3, so we must 'borrow' 1 unit from the column to the left (changing it from 5 to 4) so that we can subtract 6 from 13, which gives 7, so we write down 7.

Next, we must try and subtract 7 from 4 (*not* 5), but again we cannot do that, so we borrow 1 from the next column to the left (thus leaving it as 0) to subtract 7 from 14, which gives us 7 to write down.

As there are no more columns
to subtract from, the sum is
complete.
∴ **1538 – 762 = 776.**

$$
\begin{array}{r}
1\ 5\ 3\ 8 \\
-\ \ 7\ 6\ 2 \\
\hline
6
\end{array}
\qquad
\begin{array}{r}
1\ {}^4\!5\ {}^1\!3\ 8 \\
-\ \ \ 7\ 6\ 2 \\
\hline
7\ 6
\end{array}
\qquad
\begin{array}{r}
{}^0\!1\ {}^{14}\!5\ {}^1\!3\ 8 \\
-\ \ \ \ 7\ 6\ 2 \\
\hline
7\ 7\ 6
\end{array}
$$

Example (10): Find 4007 – 1358.

We cannot subtract 8 from 7, so we must 'borrow' 1 unit from the column to the left.

Unfortunately, this column has a 0 in it, as does the next one to the left, so we cannot borrow until we come across a non-zero digit, here 4.

To borrow 1, we must take 1 from 400 to obtain 399, altering the digits on the top line from 4, 0, 0 to 3, 9, 9 and then take 8 from 17, writing down 9.

After this messy bit, the rest is
easy.
9 - 5 = 4, so write down 4.
9 - 3 = 6, so write down 6.
3 - 1 = 2, so write down 2.
∴ **4007 – 1358 = 2649.**

$$
\begin{array}{r}
4\ 0\ 0\ 7 \\
-\ 1\ 3\ 5\ 8 \\
\hline
\end{array}
\qquad
\begin{array}{r}
{}^3\!4\ {}^9\!0\ {}^9\!0\ {}^1\!7 \\
-\ 1\ 3\ 5\ 8 \\
\hline
9
\end{array}
\qquad
\begin{array}{r}
{}^3\!4\ {}^9\!0\ {}^9\!0\ {}^1\!7 \\
-\ 1\ 3\ 5\ 8 \\
\hline
2\ 6\ 4\ 9
\end{array}
$$

When subtracting decimals or money, we must again remember to line up decimal points correctly.

Example (11): Work out the difference between £178.60 and £34.25.

We cannot subtract 5 from 0, so
we must 'borrow' 1 unit from the
column to the left, (changing it
from 6 to 5).

Then, 10 – 5 = 5 so write down 5.
5 – 2 = 3, so write down 3.
8 – 4 = 4, so write down 4.
7 – 3 = 4, so write down 4.
1 – (0) = 1, so write down 1.

∴ **£178.60 - £34.25 = £144.35.**

$$
\begin{array}{r}
£ \qquad\ p \\
1\ 7\ 8\ .\ 6\ 0 \\
-\ \ 3\ 4\ .\ 2\ 5 \\
\hline
\end{array}
\qquad
\begin{array}{r}
£ \qquad\ {}_5p_1 \\
1\ 7\ 8\ .\ 6\ 0 \\
-\ \ 3\ 4\ .\ 2\ 5 \\
\hline
.\ 3\ 5
\end{array}
\qquad
\begin{array}{r}
£ \qquad\ {}_5p_1 \\
1\ 7\ 8\ .\ 6\ 0 \\
-\ \ 3\ 4\ .\ 2\ 5 \\
\hline
1\ 4\ 4\ .\ 3\ 5
\end{array}
$$

Short Multiplication.

Example (12): Multiply 348 by 7.

Working is shown below:

$8 \times 7 = 56$, so write down 6 and carry 5.
$4 \times 7 = 28$, plus the 5 carried over = 33, so write down 3 and carry 3.
$3 \times 7 = 21$, plus the 3 carried over = 24, so write down 24 (as it's the last column).
∴ **348 × 7 = 2436.**

```
    3 4 8          3 4 8          3 4 8          3 4 8
  x     7        x     7        x     7        x     7
  ─────────      ─────────      ─────────      ─────────
                       ⁵6           ³3 ⁵6      2 4 ³3 ⁵6
```

Example (13): Find the product of 974 and 5.

$4 \times 5 = 20$, so write down 0 and carry 2.
$7 \times 5 = 35$, plus the 2 carried over = 37, so write down 7 and carry 3.
$9 \times 5 = 45$, plus the 3 carried over = 48, so write down 48 (as it's the last column).
∴ **974 × 5 = 4870.**

```
    9 7 4          9 7 4          9 7 4          9 7 4
  x     5        x     5        x     5        x     5
  ─────────      ─────────      ─────────      ─────────
                       ²0           ³7 ²0      4 8 ³7 ²0
```

Multiplying Decimals / Money.

Unlike addition and subtraction, it is best to handle multiplication of decimals (including money) as a whole-number exercise, and then put the decimal points in afterwards.

If the question involves money, it is best to work in whole units, which might be pounds or pence.

If the question involves decimal numbers, there are two methods of placing the decimal point correctly after having performed the whole-number sum.

Method (1): counting decimal places

In this method, we count the number of digits after the decimal point in each number that is being multiplied. Add those two numbers to give the number of digits after the decimal point.

This rule breaks down, however, when multiplying a whole number by a decimal.

Method (2): using rough approximation

In this method, we rough-guess one or both of the numbers concerned to work out an estimate, and then put the decimal point in the most appropriate place.

Example (14): Using the result $974 \times 5 = 4870$ from Example 11, find the value of 9.74×0.5.

Using the 'decimal point counting' method, we see that 9.74 has **two** digits after the decimal point and 0.5 has **one**. The result will therefore have **three** digits after the point, and so the result will be 4.870. (Note that we must not ignore the trailing zero when positioning the point.)

Using the 'rough approximation' method, we can estimate 9.74×0.5 as about 10×0.5, or about 5. The nearest possible answer to 5 is 4.870, and definitely not 48.70 or 0.4870.

$\therefore$ **$9.74 \times 0.5 = 4.87(0)$.**

Solving "many sums from one".

Example (15): Given that $376 \times 294 = 110544$, find the values of :

i) 3.76×2.94 ; ii) $1105.44 \div 29.4$; iii) 0.376×0.0294; iv) $110.544 \div 3760$; v) 3760×0.294

i) Using the 'decimal point counting' method, we see that 3.76 has **two** digits after the decimal point and 2.94 has **two**. The result will therefore have **four** digits after the point, and so will be 11.0544.

Using the 'rough approximation' method, we can estimate 3.76×2.94 as about 4×3 or 12.
The nearest possible answer to 12 is 11.0544, and definitely not 110.544 or 1.10544.

∴ **3.76 × 2.94 = 11.0544.**

ii) The original sum can be rewritten as $110544 \div 294 = 376$, but the given sum of $1105.44 \div 29.4$ involves dividing by a decimal. We therefore multiply the quantities on each side of the division sign by 10 to give $11054.4 \div 294$, which is the original sum with a dividend 10 times smaller.

The quotient will also be 10 times smaller than the 376 of the original, namely 37.6.

Another method is to express the original sum as a fraction : $\dfrac{110544}{294} = 376$

Then $\dfrac{1105.44}{29.4} = \dfrac{11054.4}{294}$ (multiply top and bottom by 10 to make bottom line a whole number)

Hence $\dfrac{11054.4}{294} = 37.6$

This result has the same digits as the original, but is 10 times smaller because top line is also 10 times smaller.

iii) Using the 'decimal point counting' method, we see that 0.376 has **three** digits after the decimal point and 0.0294 has **four**. The result will therefore have **seven** digits after the point, and so will be 0.0110544.

iv) The original sum can be rewritten as $110544 \div 376 = 294$, but the given sum of $110.544 \div 3760$ has a divisor 10 times larger. We therefore divide the quantities on each side of the division sign by 10 to give $11.0544 \div 376$, which is the original sum with a dividend 10,000 times smaller.

The quotient will also be 10,000 times smaller than the 294 of the original, namely 0.0294.

Or in fraction form: $\dfrac{110544}{376} = 294$, and so $\dfrac{110.544}{3760} = \dfrac{11.0544}{376}$

(dividing top and bottom by 10 to make bottom line correspond with original)

Hence $\dfrac{11.0544}{376} = 0.0294$

v) We cannot use decimal point counting here along the lines of "0 + 3 = 3, so there'll be 3 digits after the decimal point to give 110.544."

We therefore make an approximation: 3760×0.294 is roughly 4000×0.3, or about 1200.
The only possible answer of the right order of size is 1105.44, so **3760 × 0.294 = 1105.44**.

Long Multiplication.

Long multiplication is an extension of methods used in the shorter form.
Basically, it is a case of carrying out more than one short multiplication and adding the results.

Example (16): Multiply 132 by 24.

This sum can be broken down into two short multiplication sums, namely 132 × 4 and 132 ×20.

The first step is to multiply 132 by 4 using the standard short multiplication method.

Next, we prepare to multiply 132 by 20, but we must remember that 20 is two *tens,* so we prevent any possible errors by placing a zero in the units place.

In Step 3, we perform 132 × 2 , but the column results (beginning with 2 × 2 = 4) are offset one place left because we are multiplying by 20 and not 2. (That's why placing a zero in the units place helps !)

In Step 4, we add the sub-totals together to obtain the final result, **132 × 24 = 3168.**

```
    1 3 2                           1 3 2
  x   2 4                         x   2 4
  _____                       _____
    5 ¹2 8   = 132 x 4              5 ¹2 8   = 132 x 4
                                        0    = 132 x 20

  _____                       _____
```

Step 1 - multiply 132 by 4   Step 2 - prepare to multiply 132 by 20

```
    1 3 2                           1 3 2
  x   2 4                         x   2 4
  _____                       _____
    5 ¹2 8   = 132 x 4              5 ¹2 8   = 132 x 4
  2 6 4 0    = 132 x 20          2 6 4 0    = 132 x 20
                                1
  _____                     _____
                                3 1 6 8    = 132 x 24
```

Step 3 - continue to multiply 132 by 20   Step 4 - add the two sub-totals

Example (17): Find the value of 418×107.

The working is very similar to the previous example, i.e. we are performing 418×7, followed by 418×100.

The second multiplication is by 100, so we insert *two* zeros on the right-hand side of the 418×100 row, and thus the 8×1 result and all the others have been offset *two* places left.
The final result is **$418 \times 107 = 44726.$**


```
        4  1  8                           4  1  8
  x     1  0  7                   x       1  0  7
  ─────────────────               ─────────────────
     2  9 ¹2 ⁵6   = 418 x 7          2  9 ¹2 ⁵6   = 418 x 7
                                    4  1  8  0  0  = 418 x 100
  ─────────────────               ─────────────────
```

Step 1 : work out 418 x 7   Step 2 : work out 418 x 100


```
        4  1  8
  x     1  0  7
  ─────────────────
     2  9 ¹2 ⁵6   = 418 x 7
  4  1  8  0  0  = 418 x 100
     1
  ─────────────────
  4  4  7  2  6  = 418 x 107
```

Step 3: add the subtotals

The grid method for long multiplication.

An alternative method for long multiplication is shown below.
Here we are multiplying 48 by 37, and we are treating it as $(40 + 8) \times (30 + 7)$.

The results of each partial calculation are shown in the cells of the grid as follows:
upper left, $40 \times 30 = 1200$; right, $8 \times 30 = 240$;
lower left, $40 \times 7 = 280$; right, $8 \times 7 = 56$.

We then add the contents of all the cells together to obtain the total, 1776.

The grid method is useful as it lends itself well to visualising multiplication in algebra.

X	40	8
30	1200	240
7	280	56

```
1200
 240
 280
  56
```

$48 \times 37 = \overline{1776}$

Example (18): Work out 128×53 using the grid method.

X	100	20	8
50	5000	1000	400
3	300	60	24

```
5000
1000
 400
 300
  60
  24
```

$128 \times 53 = \overline{6784}$

The completed solution is shown below:

Because the number of calculations increases rapidly the more digits there are in the two numbers to be multiplied, it is best to use the 'traditional' method for any sums involving 3-digit numbers and above.

The "Gelosia" , "Chinese" or "Napier's Bones" method.

This uses multiplication tables in a rather unusual way, as per the diagram below

	0	1	2	3	4	5	6	7	8	9
0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0
1	0/0	0/1	0/2	0/3	0/4	0/5	0/6	0/7	0/8	0/9
2	0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8
3	0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7
4	0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

The original device was made from strips of bone, hence the name, and the grid resembles the multiplication table to 9×9 with a diagonal line separating the tens and units.

This diagram can be copied and pasted onto strips of card for classroom activity.

Example (19): Multiply 132 by 24 using the Gelosia method.

We have selected only the cells needed to carry out the calculation.

Notice how we work along diagonal lines starting at the bottom right triangle, corresponding to the units digit in the result, 8, and then work left along the diagonals.

The digits in the next diagonal to the left are added together, i.e. 4 + 2 = 6. This is the tens digit in the result.

The next diagonal is for the hundreds : 6 + 1 + 4 = 11. We write down 1, but must also carry 1 into the thousands.

The last diagonal is for the thousands: 0 + 2 + 0 = 2, plus the 1 carried over in the last step, so we write down 3.

Hence **132 × 24 = 3168.**
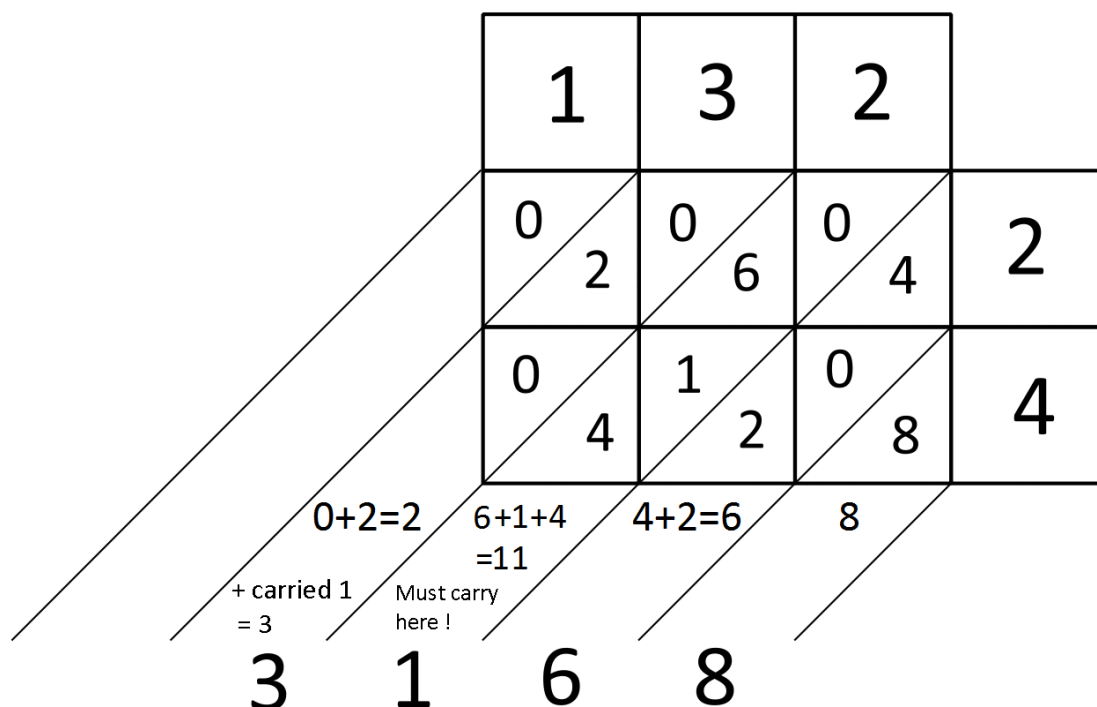
Multiplication using Fractional Parts.

This is a method of multiplication where a complicated multiplier is divided up into easier fractional
parts, which are then added (or sometimes subtracted) . The best fractional parts to work with are
halves, quarters, fifths and tenths, as they make for easy division.
Many non-calculator percentage problems are often solved in that manner.

Example (20): (Percentages) Find 15% of £246.

We split the 15% into 10% + 5%.

100% of £246 is obviously £246. Now 10% is one tenth of 100%, so 10% of £246 is one tenth of
£246, or £24.60. Finally, 5% is half of 10%, so 5% of £246 is half of £24.60, or £12.30.

Adding those last two sub-totals gives us £24.60 + £12.30 = £36.90.

100% of £246 =	£246.00	
10% of £246 =	£24.60	(Divide £246 by 10, as 10% = one tenth)
5% of £246 =	£12.30	(5 is half of 10, so divide 10% by 2)

15% of £246 =   £36.90   (Add last two sub-totals)

Example (21): Find the cost of 21calculators at £5.99 each.

This time, we can choose two different methods.

We can split the number of calculators sold, namely 21, as $(10 \times 2) + 1$:

10 calculators @ £5.99 =	£59.90	
20 calculators @ £5.99 =	£119.80	(Double last result)
1 calculator @ £5.99 =	£5.99	

21 calculators @ £5.99 =   £125.79   (Add the two sub-totals above)

Alternatively, we can treat the £5.99 as being £6 – 1p :

$21 \times$ £1 =	£21.00	
$21 \times$ £6 =	£126.00	(Multiply last result by 6)
$21 \times$ 1p =	£0.21	(To be subtracted)

$21 \times$ £5.99 =   £125.79   (Subtract $21 \times$ 1p from $21 \times$ £6)

Short Division.

Example (22): Divide 992 by 8.

Working shown below:

$$\begin{array}{r} 1 \\ 8\,\overline{)\,9\,{}^{1}9\,2} \end{array} \qquad \begin{array}{r} 1\ 2 \\ 8\,\overline{)\,9\,{}^{1}9\,{}^{3}2} \end{array} \qquad \begin{array}{r} 1\ 2\ 4 \\ 8\,\overline{)\,9\,{}^{1}9\,{}^{3}2} \end{array}$$

First, we see that 8 goes into 9 once with a remainder of 1, so we place 1 above the first 9 and the remainder 1 to the left of the second 9, making 19.
Then, we see that 8 goes into 19 twice (8 × 2 = 16) with 3 left over, so we place 2 above the second 9 and 3 to the left of the 2, making 32.

Finally, 8 goes into 32 exactly 4 times, so we put 4 above the 2.
There are no more numbers to divide into, so the sum ends here, i.e. **992 ÷ 8 = 124.**

Example (23): Find the quotient and remainder of 2165 ÷ 7.

We see that 7 does not go into 2, so we leave the space above the 2 blank.
We then try if 7 goes into 21, which it does, exactly 3 times, so we put 3 above the 1, without and remainder to carry over.
The next digit is 6, but 7 does not go into 6, so we must put a zero above the 6.
Finally 7 goes 9 times into 65 (9 × 7 = 63) with 2 left over.
We have reached the end of the sum, so the quotient of **2165 ÷ 7** is **309** and the remainder **2**.

$$\begin{array}{r} 3 \\ 7\,\overline{)\,2\ 1\ 6\ 5} \end{array} \qquad \begin{array}{r} 3\ 0 \\ 7\,\overline{)\,2\ 1\ 6\ 5} \end{array} \qquad \begin{array}{r} 3\ 0\ 9\ \ r\ 2 \\ 7\,\overline{)\,2\ 1\ 6\ 5} \end{array}$$

Short Division of Decimals.

When dividing a decimal by a whole number, then the decimal points must be lined up correctly, as the next example shows:

Example (24): Divide 12.6 by 8.

We must reckon: "8 does not go into 1, so leave a blank space above the 1".
Then "8 goes into 12 once, with 4 left over, so put 1 above the 2, and carry 4 on".
Then, "8 goes into 46 five times , with 6 left over."

We have run out of numbers here, but as this is a decimal sum, we *can* continue inserting extra zeros to carry on with the sum. (Decimal division does not "do remainders").

We therefore add a 0 to the number we're dividing by, and put the leftover 6 in front of it
"8 goes into 60 seven times, with 4 left over, so put 7 above the 0, and carry 4 in front of *another* 0."
"8 goes into 40 exactly 5 times, so put 5 above the second 0, and end the sum".

∴ **12.6 ÷ 8 = 1.575.**

$$\begin{array}{r}1\,.\\[-2pt]\hline 8\,\big|\;1\;2\,.\,{}^{4}6\end{array}\qquad\begin{array}{r}1\,.\,5\\[-2pt]\hline 8\,\big|\;1\;2\,.\,{}^{4}6\end{array}$$

$$\begin{array}{r}1\,.\,5\;7\\[-2pt]\hline 8\,\big|\;1\;2\,.\,{}^{4}6\,{}^{6}0\end{array}\qquad\begin{array}{r}1\,.\,5\;7\;5\\[-2pt]\hline 8\,\big|\;1\;2\,.\,{}^{4}6\,{}^{6}0\,{}^{4}0\end{array}$$

If dividing by a decimal, then it is best to change the divisor (the number we're dividing *by*) into a whole number before carrying on with the sum.

Thus, if the last example were given as 0.126 ÷ 0.08 , we would have had to have multiplied both top and bottom by 100 to turn it into 12.6 ÷ 8.

Example (25): Divide 216.5 by 7, giving the result correct to 2 decimal places.

This is very similar to Example (19), but for the use of the decimal point, and the need to continue the division without the idea of a remainder.

$$\begin{array}{r}3\,.\\[-2pt]\hline 7\,\big|\;2\;1\;6.5\end{array}\qquad\begin{array}{r}3\;0.\\[-2pt]\hline 7\,\big|\;2\;1\;6.5\end{array}\qquad\begin{array}{r}3\;0.9\\[-2pt]\hline 7\,\big|\;2\;1\;6.5\end{array}$$

$$\begin{array}{r}3\;0.9\;2\\[-2pt]\hline 7\,\big|\;2\;1\;6.5\,{}^{2}0\end{array}\qquad\begin{array}{r}3\;0.9\;2\;8\\[-2pt]\hline 7\,\big|\;2\;1\;6.5\,{}^{2}0\,{}^{6}0\end{array}$$

At the fourth step, instead of giving a remainder of 2, we must add a zero and divide 7 into 20, giving 2 with a remainder of 6. Adding another zero gives a result of 8 and remainder 4, but we can stop the division here as the question only asked for two decimal places. The value 30.928 is 30.93 to 2 decimal places.

∴ **216.5 ÷ 7 = 30.93 to 2 decimal places.**

Long Division – introduction to the process.

When we performed the short divisions in earlier examples, we used shortened methods of working.
The long method of working is shown in the following examples:

When we divided 992 by 8, we first said "8 goes into 9 once with a remainder of 1, so we place 1
above the first 9 and the remainder 1 to the left of the second 9, making 19".

```
           1                        1 2                     1 2 4
        ┌────────              ┌────────              ┌────────
      8 │ 9 9 2              8 │ 9 9 2              8 │ 9 9 2
Subtract 1×8  8  ↓                  8  ↓                  8
Remainder 1   1  9                  1  9                  1  9
            ─────                   ─────                  ─────
          Bring down    Subtract 2×8  1 6  ▼           1 6
          next digit, 9 Remainder 3   3  2                3 2
                                                Subtract 4×8  3 2
                                    Bring down              ─────
                                    next digit, 2              0
```

**What we are actually doing is subtracting (1 × 8) from 9 to get 1 (which we put above the first 9),
and then bringing down the next digit of the dividend (the number we're dividing *into*), i.e. the
second 9.**
See the first diagram.

Then, we said "8 goes into 19 twice (8 × 2 = 16) with 3 left over, so we place 2 above the second 9 and
3 to the left of the 2, making 32."

**We are actually subtracting (2 × 8) from 19 to get 2 (which we put above the second 9), and then
bringing down the next digit, 2, of the dividend . See second diagram.**

Lastly we said, "Finally, 8 goes into 32 exactly 4 times, so we put 4 above the 2.
There are no more numbers to divide into, so the sum ends here".

**We are subtracting (4 × 8) from 32 to get 4 (which we put above the 2), and because there is no
remainder, the sum ends. See third diagram.**

The working for 2165 ÷ 7 is shown below. There are a few differences in the working compared to the
last sum, but the principle is the same. (We have to bring *two* digits down in the final stage of the
working, there is a zero in the quotient, and there is also a remainder).

```
         3                      3 0                    3 0 9   r 2
     ┌────────              ┌────────              ┌────────
   7 │ 2 1 6 5            7 │ 2 1 6 5            7 │ 2 1 6 5
     2 1                    2 1  ↓                  2 1  ↓  ↓   5 brought down
     ─────                  ─────                   ─────       7 can go into 65
3 × 7 = 21, so               0 6                    0 6 5
subtract it            6 brought down,                6 3
Notice no              but 7 can't go into it        ─────
remainder !                                            2      but there's a
                                                              remainder of 2.
```

Long Division examples.

Example (26): Find the value of 4075 ÷ 25.

It is best to leave some space on the paper for side working when doing long division sums.
The first nine multiples of 25 are 25, 50, 75, 100, 125, 150, 175, 200, and 225 – fortunately 25 is a fairly easy number to deal with.

Since 25 does not go into 4, we leave the space above the 4 blank. We can divide 25 into 40, so we put the answer, 1, above the 0 and subtract to find the remainder, 15. (First diagram)

Next, we bring down the next digit, 7, in the dividend (the number we're dividing *by*), and proceed to divide 25 into 157. The largest multiple of 25 below 157 is 25 × 6 or 150, so we prepare to subtract 150 from 157 to get 7. (Second diagram).

Then we bring down the next digit, 5, and proceed to divide 75 by 25. Now 75 is exactly 25 × 3, so the final subtraction leaves no remainder.

∴ **4075 ÷ 25 = 163.**

```
        1                      1 6                    1 6 3
25 | 4 0 7 5           25 | 4 0 7 5           25 | 4 0 7 5
     2 5                    2 5                    2 5
     ___                    _____                  _____
     1 5                    1 5 7                  1 5 7
                           1 5 0                  1 5 0
                           _____                  _____
                               7                      7 5
                                                     7 5
                                                     ___
                                                       0
```

Example (27): Evaluate 30.66 ÷ 2.1

We must first adjust this sum into one with a whole number divisor. This we do by multiplying both the divisor and dividend by 10, turning the sum into 306.6 ÷ 21.

We must line up the decimal points in the dividend and the quotient, but we leave them out in the 'subtraction and bringing down' process.

The first division we can perform is 21 into 30, which goes once with 9 left over, so we put the 1 in the quotient above the 0 in the dividend (First diagram).

We then bring down 6 from the dividend and proceed to divide 21 into 96. Now 21× 5 or 105 is too much, so the required digit in the quotient is 4, and the total to be subtracted is 21× 4 or 84.
This leaves a remainder of 12. (Second diagram).

Lastly we bring down the second 6 from the dividend and proceed to divide 126 by 21.
We could guess 5, but that would give 21× 5 or 105, and subtracting that from 126 would leave 21.
This is unacceptable, since the remainder *must be less than the divisor.*

We must therefore try 6, and this time 21 × 6 = 126, which leaves a remainder of 0, thus finishing the sum. Hence **30.66 ÷ 2.1 = 14.6.**

```
        1   .                    1 4.                     1 4.6
     ┌─────────              ┌─────────               ┌─────────
  21 │ 3 0 6.6            21 │ 3 0 6.6             21 │ 3 0 6.6
      2 1                      2 1                      2 1
     ─────                    ─────                    ─────
        9                        9 6                      9 6
                                 8 4                      8 4
                                ─────                    ─────
                                 1 2                      1 2 6
                                                          1 2 6
                                                         ─────
                                                             0
```

Example (28): Find the quotient and remainder of 3523 ÷ 17.

As 17 cannot divide into 3, we try to divide 17 into 35. Since a quick check gives $17 \times 2 = 34$, we can place 2 above the 5, and subtract 34 from 35 to give 1. (First diagram)

Bringing down 2 gives us 12, but 17 cannot go into 12 , so we put 0 above the 2. (Second diagram)
We then bring down the 3 and try dividing 17 into 123. We might try 6, but $17 \times 6 = 102$ which would leave us with a remainder of 21, which is *greater than* the divisor of 17.
We must try 7, and 17×7 gives 119, which leaves a remainder of 4 on subtracting. (Third diagram).

```
        2                    2  0                 2  0  7  r  4
17 | 3  5  2  3       17 | 3  5  2  3      17 | 3  5  2  3
     3  4                  3  4                 3  4
     -------               -------              -------
        1                     1  2                 1  2  3
                                                   1  1  9
                                                   -------
                                                         4
```

∴ **The quotient of 3523 ÷ 17 is 207 and the remainder 4.**

Another method of long division is the 'chunking' method which involves repeated subtractions of 'easy' multiples of the divisor. This is most easily illustrated by a worked example.

Example (29): Find the value of 4075 ÷ 25 using the 'chunking' method.

```
25    4  0  7  5
      2  5  0  0      100       We know that 25 × 100 = 2500, and it's
      ----------                less than 4075, so we take it away

      1  5  7  5
      1  0  0  0       40       25 × 40 = 1000, so we can 'chunk' that
      ----------                out from the remaining 1575

         5  7  5
         5  0  0       20       25 × 20 = 500, so 'chunk' that
         -------                out from 575

            7  5
            7  5        3       25 × 3 = 75, and no
            ----                remainder

               0
```

Quotient =
100 + 40 + 20 + 3 = 163   Add the partial
                                 quotients to get 163

Starting with 4075, we first subtracted out 100×25 or 2500, leaving 1575. Next, we 'chunked' out 40×25 or 1000 to leave 575. Repeating the process, we subtracted 20, then 3 'chunks' of 25 and added them together to obtain the quotient of 163.

"Chunking" is perhaps most useful when dividing by a large number.

Example (30): Collectors' football cards cost 79p per packet. How many packets can be bought with a £10 note ?

Here the sum to be worked out is $1000 \div 79$. (Remember £10 = 10 × 100p = 1000p.)

```
79    1 0 0 0
      7 9 0        10
     ─────────
      2 1 0
        7 9         1
     ─────────
    1 3 1
      7 9           1
     ─────────
      5 2

Quotient = 10 + 1 + 1 = 12;   Remainder = 52
```

It is therefore possible to buy 12 packs of football cards with £10, and also have 52p left over in change.

Product rule for counting.

Example (31): A restaurant has issued a set menu for customers at lunchtimes.
How many possible combinations are there of three-course meals ? List them all.

The customers have two choices of starter, three choices of main course and two of dessert.
We can find the total of such combinations by multiplying the numbers of choices together.
Here, the total is of $2 \times 3 \times 2$ or 12 possible combinations.

This rule also features in examples on probability, when combined events are shown on a sample space.

Numero's Ristorante
3 - COURSE SET MENU

STARTERS

Tomato soup
or
Garlic Mushrooms

MAINS

Pizza Margherita
or
Lasagne Verdi
or
Spaghetti Carbonara

DESSERTS

Ice Cream
or
Fruit Salad

The list of all 12 combinations is shown below.

Starter	Main	Dessert
Soup	Pizza	Ice cream
Soup	Pizza	Fruit salad
Soup	Lasagne	Ice cream
Soup	Lasagne	Fruit salad
Soup	Spaghetti	Ice cream
Soup	Spaghetti	Fruit salad
Mushrooms	Pizza	Ice cream
Mushrooms	Pizza	Fruit salad
Mushrooms	Lasagne	Ice cream
Mushrooms	Lasagne	Fruit salad
Mushrooms	Spaghetti	Ice cream
Mushrooms	Spaghetti	Fruit salad

Example (32) : Chris has bought a new combination lock for his bicycle,
where the combination can be any four digits from 0000 to 9999.

He has forgotten the exact combination of numbers, but can remember the
following facts:

- None of the digits is 0 or 1.
- The first digit is 4.
- The third digit is either 8 or 9.
- The fourth digit is even.

i) Show that Chris needs to test 64 different combinations of digits, given the above data.

There is only one choice of first digit, namely 4.
There are eight possible choices of second digit, i,e all the digits from 2 to 9.
The third digit can only be 8 or 9, so that makes two choices.
The fourth digit must be even, so it can take the value of 2,4,6 or 8 (remember, 0 is absent).

The total number of 4-digit combinations for Chris to test is therefore $1 \times 8 \times 2 \times 4$, or 64.

ii) Chris then remembers that all the four digits of the combination are distinct, with no repeated digits.
Show that he can reduce the number of combinations for checking to 35 at most, by pairing the third
and fourth digits.

The first digit is 4 as before, but there are now seven, not eight, choices of second digit, as the 4 cannot
be repeated. Neither can the fourth digit be a 4 for the same reason. Finally, if the third digit is an 8,
the fourth digit cannot take that value. The only 3rd/4th digit pairings allowed are thus 82, 86, 92, 96
and 98 – five in total.

The total number of 4-digit combinations for Chris to test now is $1 \times 7 \times 5$, or 35.

iii) Complete the table below and hence calculate how many combinations Chris needs to test. Do not
forget to reject any more cases where there are repeated digits in the combination.

	Digits 3 and 4				
Digits 1 and 2	**82**	**86**	**92**	**96**	**98**
42	~~4282~~	4286	~~4292~~	4296	4298
43	4382	4386	4392	4396	4398
45	4582	4586			
46	4682				
47	4782				
48					
49					

.
After completing the table and rejecting all remaining combinations with repeated digits, the final
number of combinations for Chris to test (highlighted in the table) is 25.

	Digits 3 and 4				
Digits 1 and 2	**82**	**86**	**92**	**96**	**98**
42	~~4282~~	4286	~~4292~~	4296	4298
43	4382	4386	4392	4396	4398
45	4582	4586	4592	4596	4598
46	4682	~~4686~~	4692	~~4696~~	4698
47	4782	4786	4792	4796	4798
48	~~4882~~	~~4886~~	4892	4896	~~4898~~
49	4982	4986	~~4992~~	~~4996~~	~~4998~~