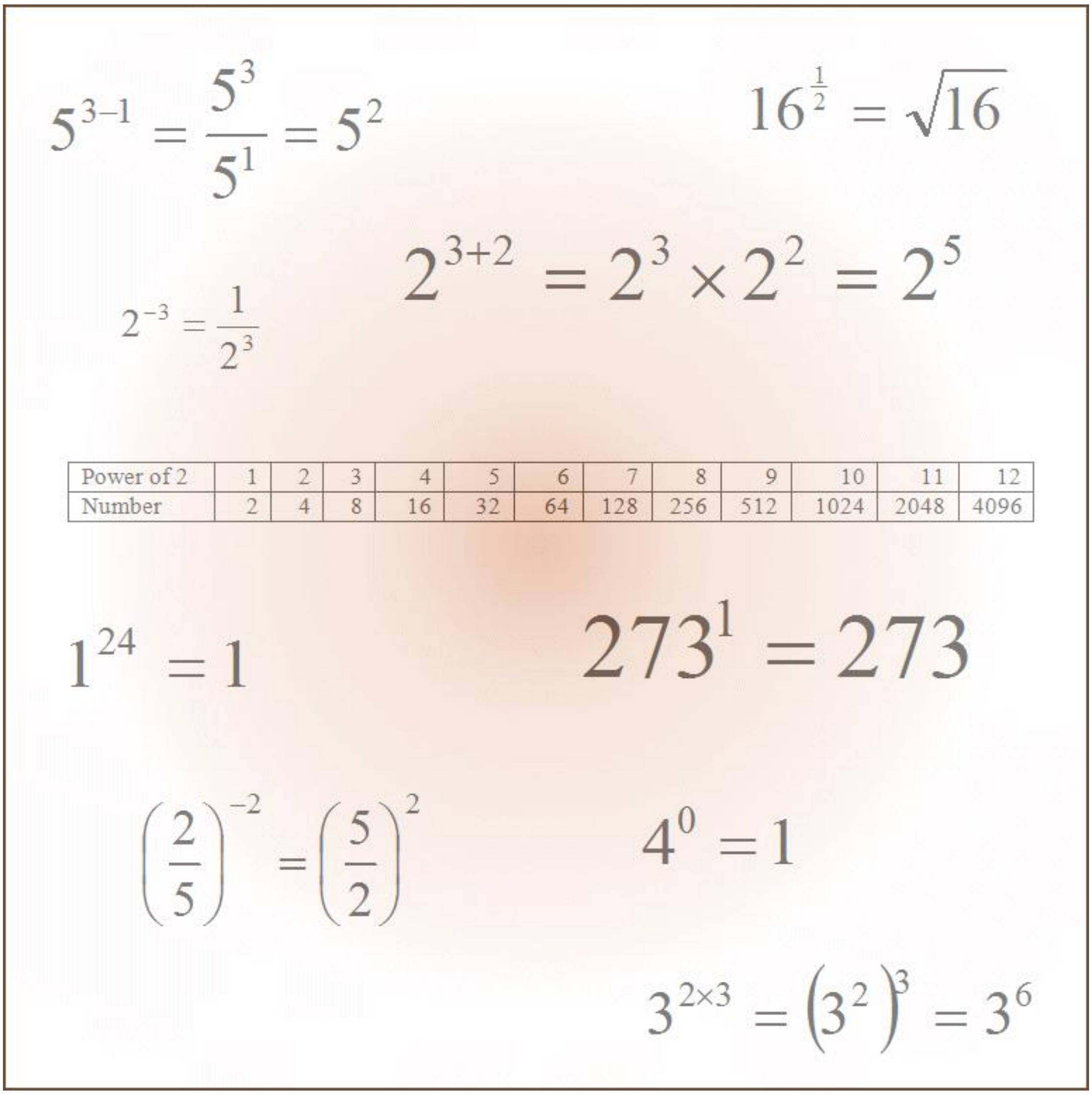
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

LAWS OF INDICES



Version: 2.3 Date: 10-11-2015

LAWS OF INDICES.

When we multiply a number by itself, we are said to square it, or raise it to the power of 2.
Thus we write $6 \times 6 = 6^2 = 36$, i.e "six squared"

When we multiply a number by itself twice, we are cubing it, or raising it to the power of 3.
Therefore we write $5 \times 5 \times 5 = 125$, or $5^3 = 125$, i.e "5 cubed".

Higher powers also exist, thus $2^4 = 2 \times 2 \times 2 \times 2 = 16$ ("2 to the fourth").

In the expression 5^3, the 5 is the **base** and the 3 is the **index** (plural: **indices**).

The basic laws of indices are as follows, applicable to all positive numbers *a*.

Multiplication and division:

<u>The Multiplication Law :</u> $a^{m+n} = a^m \times a^n$

Addition of indices corresponds to multiplication of actual numbers.

Examples (1): $2^{3+2} = 2^3 \times 2^2 = 2^5$, or $8 \times 4 = 32$.

This law also holds for fractions, as do the division law and the "powers of powers" law :

$$\left(\frac{2}{3}\right)^{1+2} = \left(\frac{2}{3}\right)^{1} \times \left(\frac{2}{3}\right)^{2} = \left(\frac{2}{3}\right)^{3}, \text{ or } \frac{2}{3} \times \frac{4}{9} = \frac{8}{27}.$$

<u>The Division Law :</u> $a^{m-n} = \dfrac{a^m}{a^n}$

Subtraction of indices corresponds to division of actual numbers.

Example (2): $5^{3-1} = \dfrac{5^3}{5^1} = 5^2$, or $\dfrac{125}{5} = 25$.

<u>Brackets ("Powers of powers"):</u> $a^{m \times n} = \left(a^m\right)^n$

When we multiply indices, we take a "power of a power".

Example (3): $3^{2\times 3} = \left(3^2\right)^3 = 3^6$, or $9^3 = 729$.

Here are some power tables:

Powers of 10:

Power of 10	1	2	3	4	5	6
Number	10	100	1000	10000	100000	1000000

Powers of 2:

Power of 2	1	2	3	4	5	6	7	8	9	10	11	12
Number	2	4	8	16	32	64	128	256	512	1024	2048	4096

Using powers of 2 is more convenient for illustration, because the numbers in question do not become unmanageably large.

Examples (4): Show how you would use the 'Powers of 2' table to work out:

i) 32×64 ; ii) $\frac{2048}{128}$; iii) 32^2

32 × 64 **add these powers** **get result**

Power of 2	1	2	3	4	**5**	**6**	7	8	9	10	**11**	12
Number	2	4	8	16	**32**	**64**	128	256	512	1024	**2048**	4096

Reading upwards, we see that $32 = 2^5$ and $64 = 2^6$. Adding the indices gives 2^{11}, or $64 \times 32 = 2048$.

$\frac{2048}{128}$ **subtract this** **from this**

Power of 2	1	2	3	**4**	5	6	**7**	**8**	9	10	**11**	12
Number	2	4	8	**16**	32	64	**128**	**256**	512	1024	**2048**	4096

get result

Reading upwards, we see that $2048 = 2^{11}$ and $128 = 2^7$. Subtracting indices gives 2^4, or $\frac{2048}{128} = 16$.

32^2 **double this** **get result**

Power of 2	1	2	3	4	**5**	6	7	8	9	**10**	11	12
Number	2	4	8	16	**32**	64	128	256	512	**1024**	2048	4096

Reading upwards, we see that $32 = 2^5$. Doubling the index gives 2^{10}, or $32^2 = 1024$.

So far, we have restricted ourselves to arithmetic involving positive whole number powers of numbers, but we can also have zero, fractional and negative powers as well.

The next section shows how to understand such expressions as 2^{-3} , 8^0 and $64^{½}$.

Looking back at the "Powers of 2" table, we can see how the entries in the table are **doubled** every time the power of 2, namely the index, is **increased** by 1. Conversely, when the index is **decreased** by 1, the corresponding entry in the table is **halved**.

Power of 2	**-4**	**-3**	**-2**	**-1**	**0**	1	2	3	4	5	6	7
Number	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	**1**	2	4	8	16	32	64	128

Negative Indices - Reciprocals:

$$a^{-m} = \frac{1}{a^m}$$

Any number raised to a negative power is the reciprocal of the same number raised to the corresponding positive power.

Examples (5) : $2^{-3} = \frac{1}{2^3}$, or $\frac{1}{8}$.

Avoid the common error: 2^{-3} is not -8.

$\left(\frac{1}{4}\right)^{-2} = 4^2$ or 16

Here we convert the expression with a negative power to an expression with a positive power by reversing the sides of the original fraction.

$\left(\frac{2}{5}\right)^{-2} = \left(\frac{5}{2}\right)^2$ or $\frac{25}{4}$

Zero index:

$$a^0 = 1$$

Any positive number raised to the zero power is equal to 1.

This can be demonstrated by the multiplication law:

$a^{m+0} = a^m \times a^0$, but as $m + 0$ is simply m, $a^m \times a^0 = a^m$.

Adding zero to a number leaves it unchanged; so does multiplying by 1.

Powers of 1:

The number 1 raised to any power is just 1.
Any positive number raised to the power 1 is equal to the number itself.

Examples (6): $1^{24} = 1$; $273^1 = 273$

This leaves us with having to find a meaning to expressions like $64^{½}$.

We will show the process of squaring 8 to obtain 64.

8^2 **double this** **get result**

Power of 2	1	2	**3**	4	5	**6**	7	8	9	10	11	12
Number	2	4	**8**	16	32	**64**	128	256	512	1024	2048	4096

To **square** a number, we **double** the index.

The inverse of squaring is to take square roots, and to do so, we **halve** the index.

$64^{½}$ **halve this**

Power of 2	1	2	**3**	4	5	**6**	7	8	9	10	11	12
Number	2	4	**8**	16	32	**64**	128	256	512	1024	2048	4096

get result

Hence $64^{\frac{1}{2}} = \sqrt{64} = 8.$

<u>Fractional Indices - Roots:</u>

This last result can be generalised:

$$a^{\frac{1}{m}} = \sqrt[m]{a}$$

Any positive number raised to the reciprocal of an index *m* is equivalent to the m^{th} root of that number.

Examples(7): $16^{\frac{1}{2}} = \sqrt{16} = 4.$

Recall the 'powers of powers' rule: $(16^{\frac{1}{2}})^2 = 16^{\frac{1}{2}\times 2} = 16^1 = 16.$

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5.$$

When we cube a number, we triple the index; when we take cube roots, we divide the index by 3.

Harder Fractional Indices.

Fractional indices of the form $a^{\frac{m}{n}}$ or $a^{\frac{-m}{n}}$ are less easy to manage, but they all use the standard laws above.

$a^{\frac{m}{n}}$ is the same as $a^{\frac{1}{n}}$ raised to the power of *m*; $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Examples (8):

$64^{\frac{2}{3}} = (64^{\frac{1}{3}})^2$ - namely 4^2 or 16.

We take the cube root of 64, namely 4, and then square 4 to obtain the final answer, 16.

$$\left(\frac{9}{16}\right)^{\frac{3}{2}} = \left(\left(\frac{9}{16}\right)^{\frac{1}{2}}\right)^3 = \sqrt{\left(\frac{9}{16}\right)}^3 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

We take the square root, followed by cubing the intermediate result.

$a^{-\frac{m}{n}}$ is the same as the reciprocal of $a^{\frac{1}{n}}$ raised to the power of *m*, as the bracket and reciprocal laws above show.

$$a^{-\frac{m}{n}} = \left(\frac{1}{a}\right)^{\frac{m}{n}} = \left(\left(\frac{1}{a}\right)^{\frac{1}{n}}\right)^m$$

Examples (9):

$$100^{-\frac{1}{2}} = \left(\frac{1}{100}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{100}} = \frac{1}{10};$$

(We take the reciprocal, followed by the square root.)

$$81^{-\frac{3}{4}} = \left(\frac{1}{81}\right)^{\frac{3}{4}} = \left(\left(\frac{1}{81}\right)^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{\frac{1}{81}}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

(Take the reciprocal, then the fourth root, and finally take the cube.)

$$\left(\frac{125}{64}\right)^{-\frac{2}{3}} = \left(\frac{64}{125}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{64}{125}}\right)^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

(Take reciprocal, then cube root, and finally square.)

More on arithmetic using indices.

The index laws can also be used to simplify algebraic expressions. .

Examples (10): Evaluate the following, giving answers in their simplest form :

i) $2a \times 3a^2$; ii) $\frac{12a^4b^5}{2a^2b}$; iii) $(4a^2)^3$; iv) $\sqrt{25a^4b^6}$; v) $(7a^3)^{-2}$

i) $2a \times 3a^2$ = $6a^3$ (Note that *numbers* are multiplied, but *indices* added !)

ii) $\frac{12a^4b^5}{3a^2b}$ = $4a^2 b^4$ (Numbers divided, but indices subtracted)

iii) $(4a^2)^3$ = $64a^6$ (The number 4 has been cubed, but power of *a* has been tripled !)

iv) $\sqrt{25a^4b^6}$ = $5a^2 b^3$ (Halve the powers of *a* and *b* , but take square root of 25)

v) $(7a^3)^{-2}$ = $7^{-2}a^{-6} = \frac{1}{49}a^{-6} = \frac{1}{49a^6}$ (Take reciprocal of square of 7)

Still other questions might have a mixture of bases:

Examples (11): Simplify i) $9^2 \times 3^3$ ii) $\frac{25^2}{5^{-3}}$ and iii) $\frac{16^4}{4^2 \times 8^3}$, leaving the answer as a power of a prime number.

The first step when dealing with this type of question is to reduce all terms in the expression *to the same base*. For example, $2^3 \times 4^2$ is definitely not 8^5 !

In i) we use $9 = 3^2$ to rewrite the expression as $\left(3^2\right)^2 \times 3^3 = 3^4 \times 3^3 = 3^7$.

In ii), we use $25 = 5^2$: $\frac{\left(5^2\right)^2}{5^{-3}} = \frac{5^4}{5^{-3}} = 5^7$.

In iii), we use $4 = 2^2$, $8 = 2^3$ and $16 = 2^4$:

$$\frac{16^4}{4^2 \times 8^3} = \frac{\left(2^4\right)^4}{\left(2^2\right)^2 \times \left(2^3\right)^3} = \frac{2^{16}}{2^4 \times 2^9} = 2^3.$$