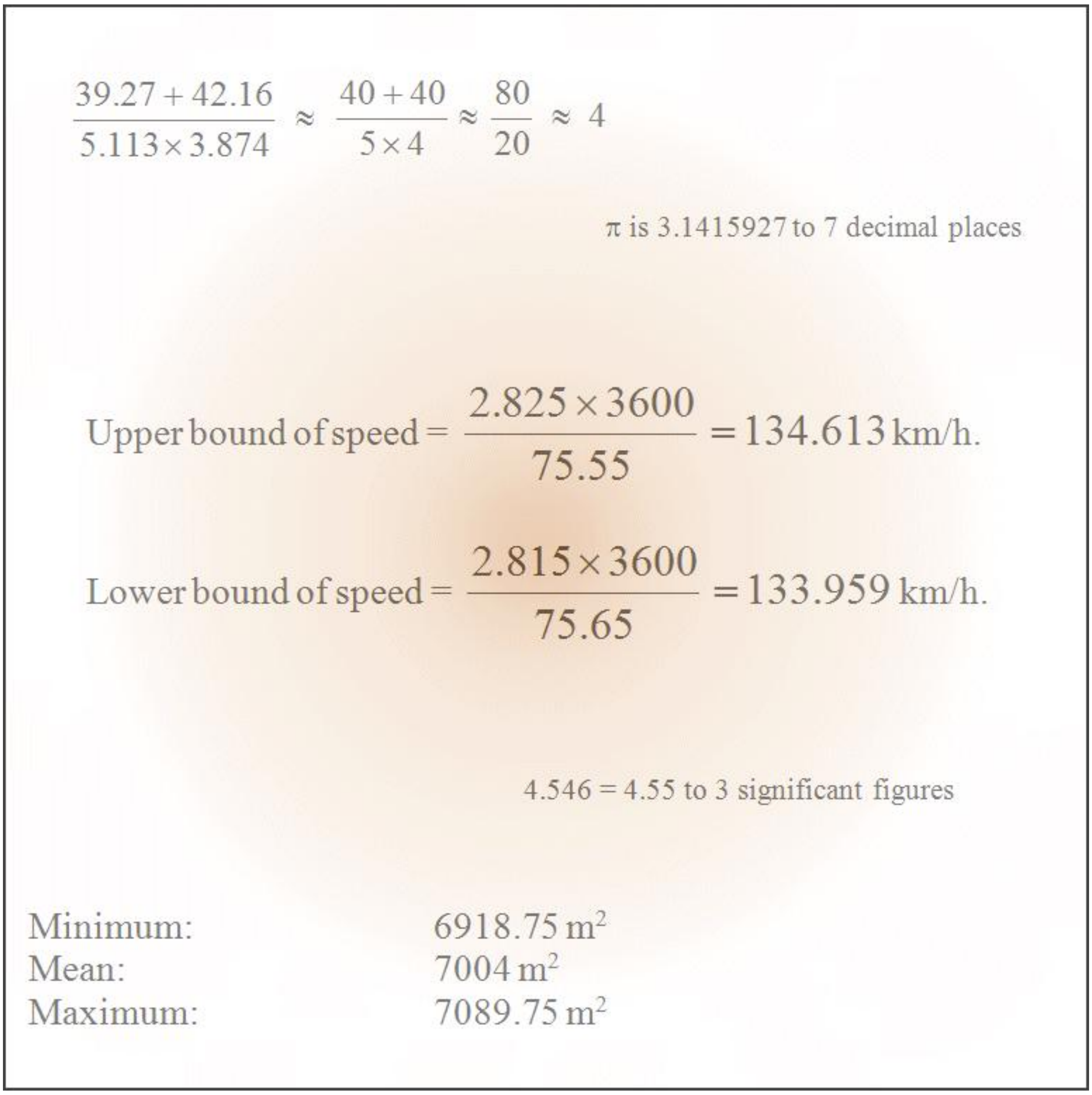
Author: Mark Kudlowski

M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

APPROXIMATION



Version: 2.1 Date: 22-10-2015

APPROXIMATION

Numbers are approximated (or rounded) for various reasons, mainly because they are measurements and cannot be exact, or because it is pointless to be ‘over-accurate’.

Thus, if we drew a right-angled triangle whose short sides were 10 cm each, then it is only possible to measure the long side to the nearest millimetre (or 0.1 cm), to give a value of 14.1cm.

Calculations would give a theoretical value would be 14.14213562.... cm, but it is impossible to measure beyond the first decimal place, and so all the extra digits are pointless, because they imply a level of accuracy that does not exist in the context of the question.

If we were asked to find the circumference of a wheel whose diameter was 60cm, we could multiply by π on the calculator to get 188.4955...cm, but it is again pointless to quote more than three or at most 4 figures, so we would give an answer of 188cm or 188.5 cm.

Rounding methods.

Numbers can be described as being rounded to the ‘nearest 10’, ‘nearest 100’, ‘ 1 decimal place’, ‘3 significant figures’ among other descriptions.

In all cases, the rounding rule is the same:

We check the final ‘desired’ digit and the first ‘discarded’ digit on either side of the ‘cut-off’ place.
If the first ‘discarded’ digit **is less than 5**, the last ‘desired’ digit is **left unchanged**.
If the first ‘discarded’ digit **is 5 or more**, the last ‘desired’ digit is **increased by 1**.

Example (1): The population of Bury Metropolitan Borough is 182,532. Round this figure to i) the nearest hundred, ii) the nearest thousand.

When we round to the nearest hundred, we are asked to ignore the tens and units digits.
The last ‘desired’ digit in 182,**5**32 is the 5, and the first ‘discarded’ one is 3.
Because 3 is less than 5, we leave the last ‘desired’ digit as 5 and say **182,500**.

When rounding to the nearest thousand, we are asked to ignore the last three digits.
The last ‘desired’ digit in 18**2**,532 is the 2, and the first ‘discarded’ one is 5.
Because 5 is greater than or equal to 5, we increase the last ‘desired’ digit by 1 to give 3, and say **183,000**.

Example (2): The value of the number π is 3.1415927 to 7 decimal places. Give its value to i) 2 decimal places ii) 4 decimal places.

When we round to 2 decimal places, the cut-off point is after the 4, as in 3.1**4**15927.
The digit after the cut-off is 1 (less than 5) so we say 3.14 and not 3.15.

When we round to 4 decimal places, the cut-off point is after the 5, as in 3.141**5**927.
The digit after the cut-off is 9 (greater than or equal to 5) so we say 3.1416 and not 3.1415.

Example (2a): There are 1.7598 Imperial pints in a litre. Express this quantity to 3 decimal places.

The cut-off point is after the 9, as in 1.75**9**8.
The digit after the cut-off is 8 (5 or greater), so we round the 9 up to 10, but in so doing, we must carry 1 forward into the second decimal place, giving a value of 1.760.

Note that we must still include the final zero in the 1.760, as the question asks for 3 decimal places.
Disregarding the zero and stating 1.76 is wrong, as this implies only 2 decimal places.

Another way of describing rounding is by the use of **significant figures.**
The first significant figure in any number is either the first digit of a number greater than or equal to 1, or the first non-zero digit of a number less than 1.

Example (3): Give the significant figures in the following numbers:
i) 265.4; ii) 0.00534; iii) the rounded number 716,000; iv) 0.03075

i) The significant figures in 265.4 are 2, 6, 5 and 4. (4 in total)
ii) The leading zeros in 0.00534 are not significant; the significant figures are 5, 3 and 4.
iii) The fact that the number 716,000 is rounded implies that the trailing zeros are not significant, and thus the significant figures are 7, 1 and 6.
iv) In the number 0.03075, the significant figures begin with the 3, and continue with 0, 7 and 5. (Embedded zeros within a string of non-zero digits are significant).

Example (4):
i) One mile is equal to 1.609344 kilometres. Give this number to 3 significant figures.
ii) One Imperial gallon is equal to 4.546 litres. Give this number to 3 significant figures, and also to 2 significant figures.
iii) The area of the United Kingdom is 243,610 km^2. Give this number to 3 significant figures.

i) The first 3 significant figures of 1.609344 are 1, 6 and (significant !) 0; the fourth one is 9.
Because the 4^{th} s.f. is a 9, (i.e greater than or equal to 5), this value is 1.61 kilometres.
ii) 4.546 = 4.55 to 3 significant figures (the 4^{th} s.f, 6, is greater than or equal to 5)
4.546 = 4.5 to 2 significant figures (the 3^{rd} s.f, 4, is less than 5)
(It is very dodgy to round in stages: 4.546 = 4.55 to 3 s.f. ; 4.55 = 4.6 to 2 s.f. which is incorrect. Don't do it !)
iii) 243,610 = 244,000 to 3 significant figures. The significant figures are 2, 4 and 4, but the answer is not 244 (a common error). The place value of the 2 is still 200,000 and not 200 !

There are other reasons in real-life situations where the approximation rules do not follow the usual pattern, usually as a result of division sums where a whole number result is required and there is a remainder.

Example (5): Electrical cable is produced in 500-metre long spools. How many 3-metre lengths of electrical flex can be cut from such a spool ?

Division gives $\frac{500}{3}$ = 166.66.... and since we are interested in a whole number result, the usual rules of rounding might suggest that it is possible to cut 167 lengths of flex, each 3 metres long.
However, $167 \times 3 = 501$, and we only have 500 metres of cable. The best we can do is to cut 166 lengths of flex from the cable, with 2 metres left over as waste.

In this case we had to round down from 166.66... to 166.
Another term for this rounding down is **truncation**.

Example (6): Paint is sold in 1-litre cans, where a can covers 11 square metres. How many cans are needed to paint a wall 48 square metres in area ?

Division gives $\frac{48}{11}$ = 4.3636.... suggesting that 4 cans of paint are needed, by the usual rounding rules.

However, $11 \times 4 = 44$, and so 4 cans of paint can only cover 44 square metres. We need to break into another can of paint to cover those remaining 4 square metres of the 48, i.e. we need 5 cans.

In this case we had to round up from 4.3636... to 5.

Use of rough estimates in calculations.

Another reason for working out approximate values for calculations is to act as a double-check on results obtained using a calculator.

Example (7): A pupil was given the sum $\frac{39.27 + 42.16}{5.113 \times 3.874}$ to work out on his calculator.

He obtained an answer of 71.2136417. Was this correct or not ?

We can estimate what the answer should have been by simplifying the numbers so that the sum could be worked out mentally – this means reducing the numbers to one, or at most two, significant figures.

39.27 and 42.16 can each be rounded to 40, 5.113 to 5 and 3.874 to 4.

This simplifies the sum to $\frac{40 + 40}{5 \times 4}$, or approximately $\frac{80}{20}$ or 4.

The pupil had forgotten to put brackets around the top and bottom lines, so the calculator treated the sum as

$39.27 + \left(\frac{42.16}{5.113} \times 3.874 \right)$ instead of $\frac{(39.27 + 42.16)}{(5.113 \times 3.874)}$.

By placing brackets round the top and bottom line expressions, we obtain the answer 4.1110.... which agrees well with the rough estimate of 4.

Example (8): Find the value of $\frac{(1.083 + 7.212) \times (0.326 + 1.176)}{0.78^2}$ using a calculator, and verify the result by using a rough estimate.

The calculator result comes out as 20.47845...

We can add 1.083 to 7.212 to obtain 8.295, simplifying to 8.
Likewise we can simplify 0.326 + 1.176 to 1.5 (1 or 2 are a little too far away from the result).
0.78 is just under 0.8, whose square is 0.64, so we can rough-guess 0.78^2 as 0.6.

The simplified sum becomes $\frac{8 \times 1.5}{0.6}$ or $\frac{12}{0.6}$, or roughly 20.
The rough approximation of 20 agrees well enough with the actual result to suggest that the correct values were entered in the calculator.

Example (9): Find the value of $\sqrt{\frac{28.9 + 74.2}{3.25 - 1.16}}$ using a calculator, and verify the result by using a rough estimate.

The true value is 7.023542......

For the rough estimate check, the numerator can be simplified to 30 + 70 or 100, and the denominator to 3 – 1 or 2.

Now $\sqrt{\frac{100}{2}} = \sqrt{50}$, or approximately 7. (√50 is close to √49 which equals 7).

Using Rounded Values in Arithmetic – Bounds.

Any practical or experimental measurements are accurate only to a limited number of figures, which can affect the accuracy of arithmetic results.

Example (10): A football pitch measures 103 metres by 68 metres to the nearest metre. Work out:

i) the bounds for perimeter of the pitch in metres;
ii) the bounds for the area of the pitch in square metres.
Give your answers to an appropriate degree of accuracy.

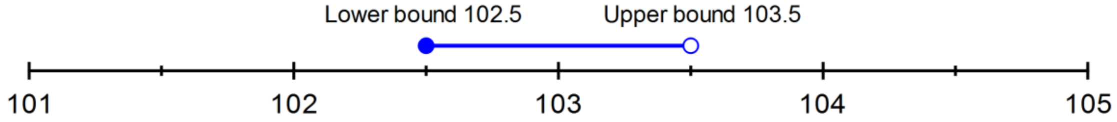
Using the given values, the perimeter of the pitch works out at 2 × (103 + 68) or 342 metres, and the area, 103 × 68 = 7004 square metres.

However the dimensions quoted are only correct to the nearest metre, and so the length of the pitch in metres can be anything between 102.5 m and 103.49.. m, or 0.5 metres, either side of 103 m.

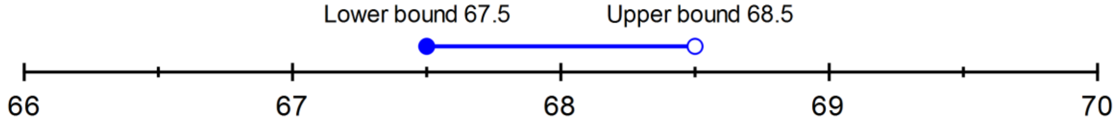
In practice the upper bound is taken as 103.5 m rather than 103.49... m, even though 103.5 m is 104 m to the nearest metre using the standard rounding rules.

We can also use inequality symbols: $102.5 \leq l < 103.5$, where l is the length of pitch in metres. :

Number line representation shown below:



Similarly the width of the pitch can take any value between 67.5 and 68.49.... m, with the upper bound rounded to 68.5 m for convenience. Using inequality symbols, we can say $67.5 \leq w < 68.5$ where w is the width of the pitch in metres.
.



i) The lower bound for the perimeter of the pitch can be found by adding together the lower bounds for the length and width and then doubling the result – giving 2 × (102.5 + 67.5) or 340 metres.
Similarly, the upper bound for the perimeter of the pitch can be found by adding together the upper bounds in the same way, giving 2 × (103.5 + 68.5) or 344 metres.

The perimeter of the pitch can be between 340 and 344 metres, so we can quote it as 340 metres to two significant figures.

ii) It can be seen that multiplying the maximum length by the maximum width would give the largest area, namely 103.5 × 68.5 = 7089.75 square metres.

Likewise, multiplying the minimum length by the minimum width would give the smallest area, namely 102.5 67.5 = 6918.75 square metres.

The computed area cannot therefore be relied on even to two significant figures, so we can only say that the area of the pitch is 7000 square metres, to 1 significant figure.

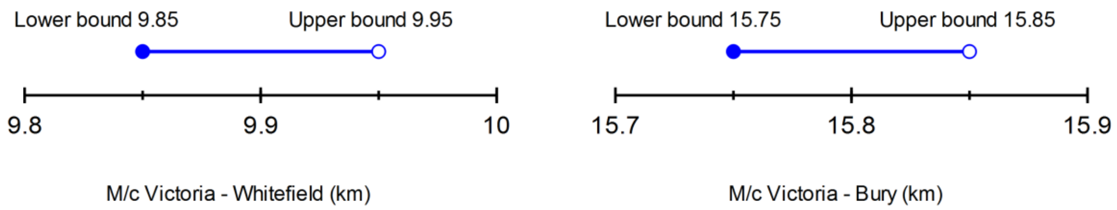
The last example involved finding the upper and lower bounds of sums and products, and it was fairly obvious that adding / multiplying upper bounds gave the upper bound for the result, and that adding / multiplying lower bounds gave the lower bound for the result

When working out the upper and lower bounds for differences and quotients, it is not as clear which combinations would give the required results.

Example (11): On the Metrolink tram system, Whitefield and Bury stations are 9.9 km and 15.8 km from Manchester Victoria respectively, as measured to the nearest 0.1 km.

What are the upper and lower bounds for the Whitefield-Bury distance, given the above information ?

The lower bound for the Manchester Victoria-Whitefield distance is 9.85 km ; the upper one, 9.95 km.
The lower bound for the Manchester Victoria-Bury distance is 15.75 km ; the upper one, 15.85 km.



The lower bound for the distance from Whitefield to Bury is obtained by subtracting the **upper** bound for the Victoria-Whitefield distance from the **lower** bound for the Victoria-Bury distance, namely 15.75 – 9.95 km or **5.8 km**.

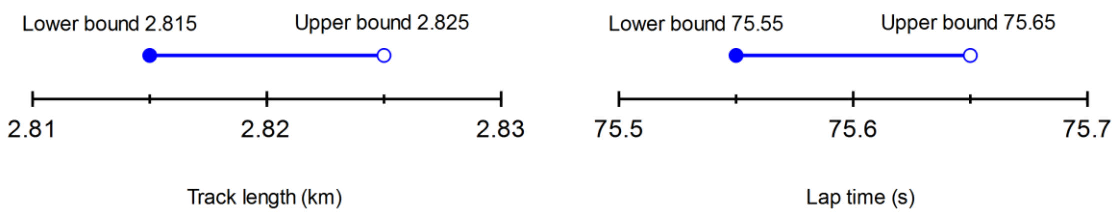
Conversely, the upper bound is obtained by subtracting the **lower** bound for the Victoria-Whitefield distance from the **upper** bound for the Victoria-Bury distance, namely 15.85 – 9.85 km or **6.0 km**.
.

Example (12): The *Top Gear* test track is 2.82 km long to 3 significant figures, and the Stig set a lap record at 1 minute 15.6 seconds, to the nearest tenth of a second. Find the upper and lower bounds for the lap record speed in km/h.

The speed in km/h based on the given data is $\frac{2.82 \times 3600}{75.6} = 134.286$ km/h

Note the scale factor of 3600, the number of seconds in an hour; also note the conversion of 1 min 15.6 sec to 75.6 sec.

The test track length l (in km) is represented by the inequality $2.815 \leq l < 2.825$, and the lap time t (in seconds) by the inequality $75.55 \leq t < 75.65$.



Here, we divide the distance by the time to give the speed.

The upper bound of the speed is taken by dividing the **upper** bound for the distance by the **lower** bound for the time.

The lower bound of the speed is obtained by dividing the **lower** bound for the distance by the **upper** bound for the time.

Upper bound of speed = $\frac{2.825 \times 3600}{75.55} = 134.613$ km/h.

Lower bound of speed = $\frac{2.815 \times 3600}{75.65} = 133.959$ km/h.

In the next two examples, the error bounds are slightly more awkward.

Example (13): A builder's lorry can carry a maximum safe load of 25 tonnes to the nearest tonne and delivers pallets of sand to a building site. A pallet of sand weighs 750 kg to the nearest 50 kg.

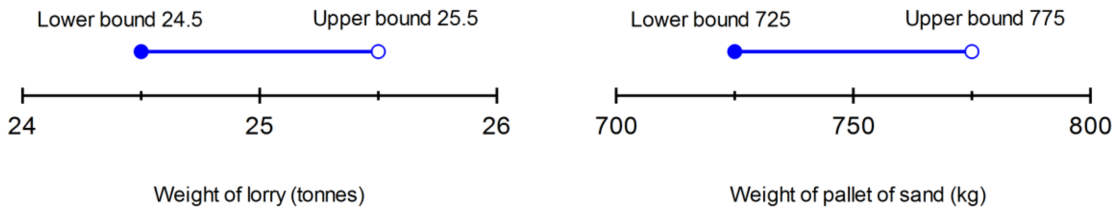
i) State the lower and upper bounds for the weight of a pallet of sand.

ii) Calculate the maximum number of pallets that can be loaded safely on to the lorry.

i) Since the weight of the pallet of sand is quoted to the nearest 50 kg, the margin of error either way is half of 50 kg, or 25 kg.

The lower weight limit is therefore 750 - 25, or 725 kg ; the upper limit is 750 + 25, or 775 kg.

Or we can state that $725 \leq w < 775$, where w is the weight of the pallet of sand in kg.



ii) We have to consider the worst-case scenario here, and that would occur when the weights of the pallets of sand have the *highest* possible value of 775 kg and the lorry's 'safe' loading capacity takes the *lowest* value of 24.5 tonnes or 24500 kg.

Maximum number of pallets allowed = $\frac{24500}{775} = 31.61$ or 31 to the nearest integer below.

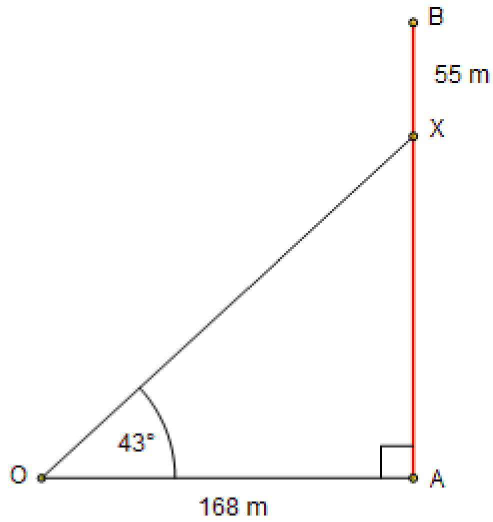
Example (14): A TV transmitter mast *AB* is anchored to the ground by a cable *OX*.

The distance from point *X* to the top of the mast is 55 m to the nearest metre.

The angle of elevation θ from the anchor at *O* to the mast at *X* is 43° to the nearest half-degree.

The ground distance *OA* from the anchor at *O* to the foot of the mast at *A* is 168 m to the nearest metre.

Find, to the nearest centimetre, the lower and upper bounds for the *total* height of the mast *AB*.



The angle of elevation , θ , from the anchor at *O* to the mast at *X* is measured to the nearest half-degree, so the margin of error is half of that, i.e. 0.25° either way. In other words, $42.75° \leq \theta < 43.25°$.

Similarly the distance *OA in metres* can vary according to the inequality $167.5 \leq OA < 168.5$

Since *OA* is adjacent to the angle *O* and *AX* is opposite to it, the height *AX* can take any value between 167.5 tan (42.75°) , or 154.84 m, and 168.5 tan (43.25°), or 158.51 m.

(The sine and tangent of an *acute* angle increase as the angle increases, but the cosine decreases.)

The height *XB* can take a value between 54.5 m and 55.5 m.

The lower bound of the mast's total height is thus (154.84 + 54.5) m or **209.34 m**.
Similarly the upper bound of the mast's total height is (158.51 + 55.5) m or **214.01 m**.

Percentage errors.

Error bounds are also often quoted in terms of percentage errors.

The values for the area of the football pitch in Example (10) were;

Minimum:	6918.75 m^2
Mean:	7004 m^2
Maximum:	7089.75 m^2

The maximum percentage error is worked out as follows:

Find the positive difference between the mean and each extreme, divide by the extreme, and multiply by 100, finally quoting the higher value.

Based on the minimum value, the percentage error is $100 \times \frac{7004 - 6918.75}{6918.75} = 1.23\%$

Based on the maximum value, the percentage error is $100 \times \frac{7089.75 - 7004}{7089.75} = 1.21\%$

The maximum percentage error in the computed area of the football pitch is thus 1.23%.

Example (14): Find the maximum percentage error in the computed value for the Stig's lap record speed given in the *Top Gear* example (12).

We have the following results:

Minimum:	133.959 km/h
Mean:	134.286 km/h
Maximum:	134.613 km/h

Taking the minimum value, the percentage error is $100 \times \frac{134.286 - 133.959}{133.959} = 0.24\%$

Taking the maximum value, the percentage error is $100 \times \frac{134.613 - 134.286}{134.613} = 0.24\%$

The maximum percentage error in the computed lap record is thus 0.24%.