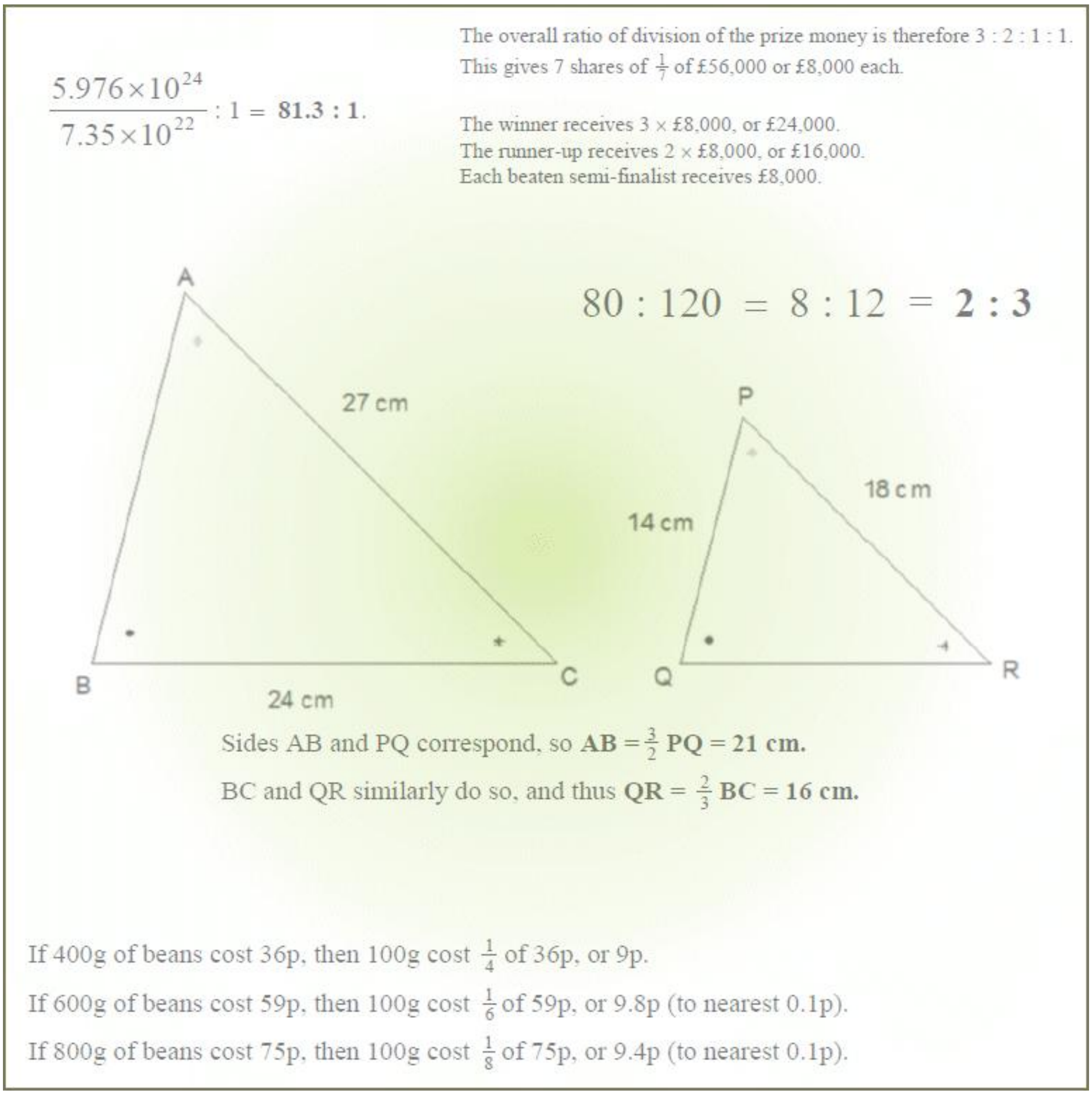
# M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

# RATIO AND PROPORTION

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# RATIO AND PROPORTION.

A ratio is a way of comparing two different quantities by size.

Thus, if a college had an intake of 80 Arts students and 120 Science students in a certain year, then we can say that the ratio of Arts to Science students is 80 : 120.

Similarly, if the label on a drinks bottle said "Dilute to taste – 1 part concentrate to 5 parts water", we say that the ratio of concentrate to water is 1 : 5.

Ratios can be simplified in various ways. One way is to cancel out common factors on each side in the same manner as ordinary fractions are cancelled.

**Example (1):** A college had an intake of 80 Arts students and 120 Science students in a certain year. Express the ratio of Arts to Science students in its simplest form.

Since 80 and 120 both have a common factor of 40, we can cancel 40 out on each side.
80 : 120 :: 8 : 12, and **8 : 12 :: 2 : 3.**

Notice how the signs of proportion are used.
The statement **8 : 12 :: 2 : 3** is equivalent to saying "8 is to 12, as 2 is to 3."

Also, the product of the outer terms, 8 and 3, is equal to the product of the inner terms. 12 and 2.

Alternatively, we could have used fractions to express the same: $\frac{80}{120} = \frac{8}{12} = \frac{2}{3}$.
(This document favours the fraction form.)

With more complicated ratios, it is often convenient to divide the larger quantity by the smaller one, so that the smaller quantity becomes equal to 1. This is known as **unitary form**.

**Example (2):** An outbreak of a viral disease resulted in 64 fatalities out of 2,520 recorded cases. Express the ratio of the fatalities to the whole in the form 1 : $x$ rounding $x$ to the nearest whole number.

We must solve 64 : 2520 :: 1 : $x$ , or in fraction form, $\frac{64}{2520} = \frac{1}{x}$.

To find $x$ , we divide 2520 by 64, giving 39.375, or 39 to the nearest whole.
Hence the ratio of fatalities to the whole is **1 : 39**.

A ratio can also be taken as being a scale factor, obtained by dividing the second number by the first. Thus a ratio stated as 5 : 2 is equivalent to a scale factor of $\frac{5}{2}$ or 2.5.

**Example (3):**
i) Increase 260 in the ratio 5 : 2.
ii) What number gives a result of 385 when increased in the ratio 5 : 2 ?

In i) we multiply 260 by a scale factor of $\frac{5}{2}$ or 2.5, to obtain a result of 650.
We can also say, as a result, that 5 : 2 and 650 : 260 are the same ratio.

In part ii) we are given the result of 385 **after** the increase in the ratio 5 : 2
We must therefore **divide** 385 by the scale factor of 2.5 to obtain 154.
Hence 5 : 2 and 385: 154 are the same ratio.

**Example (4):** The catering requirements at a children's charity event include 500ml of diluted fruit squash for 480 children. The recommended dilution for fruit squash concentrate to water is 1 : 5.
How many litres of concentrate will be required to cater for the party ?

The required amount of squash after dilution is $0.5 \times 480 = 240$ litres, and given the squash : water ratio of 1 : 5, we might think incorrectly that $\frac{240}{5}$ litres or 48 litres of squash would be needed.

The dilution is 1 part of squash to 5 parts of *water*, which gives us 6 parts of *diluted* squash in total.

The ratio of squash concentrate to the *diluted squash* is therefore 1: 6, or a scale factor of 6.
We are given the amount of squash *after* dilution, so the correct amount of squash concentrate needed is $\frac{240}{6}$ litres or **40 litres**.

We can also say that the ratio of the squash concentrate to the *whole* is 1 : 6.

Some ratio problems might involve some trickier algebra – the next two examples use simultaneous equations.

**Example (5):** A car dealer sells both electric and petrol cars, and the current stock in his showroom is in the ratio of 1 electric car to 9 petrol cars.

He then sells 6 petrol cars and buys 8 electric cars, with the result that the ratio of electric to petrol cars is now 2 : 5.

i) How many cars of each type did he have for sale at the end of the transactions ? (Do not use trial and improvement.)

Let $p$ be the number of petrol cars for sale at the beginning, and $e$ the number of electric cars.

The ratio $e : p$ is 1 : 9, so $\frac{e}{p} = \frac{1}{9}$, or $p = 9e$.

After the dealer sells 6 petrol cars and buys 8 electric cars, he has $e + 8$ electric and $p - 6$ petrol cars remaining in the garage in the ratio of 2 electric to 5 petrol.

The ratio $e + 8 : p - 6$ is 2 : 5, so $\frac{e+8}{p-6} = \frac{2}{5}$, or $2(p - 6) = 5(e + 8)$.

This equation simplifies to $2p - 12 = 5e + 40$ , and $2p - 5e = 52$.

We now have two simultaneous equations:

$p = 9e$
$2p - 5e = 52$

Substituting $9e$ for $p$ in the second equation gives $18e - 5e = 52$, or $13e = 52$, and finally $e = 4$.

$\therefore$ The dealer had 4 electric cars and 36 petrol cars for sale before his transactions..

After the transactions, he has 4 + 8, or **12**, electric cars and 36 – 6, or **30**, petrol cars.

**Example (6):** An electrical warehouse has both standard and HD television sets in stock.

The warehouse then dispatches 90 standard sets to its shops, with the result that the ratio of standard sets to HD sets remaining in stock becomes 1 : 2.

This is followed by another dispatch of 70 high-definition sets, which in turn changes the ratio of standard sets to HD sets in stock to 3 : 1.

Work out, using algebra, the number of TV sets by type originally in the warehouse before the first dispatch.

Let $s$ be the number of standard sets in the warehouse at the beginning, and $h$ the number of HD sets.

After the first dispatch, there are ($s$ – 90) standard sets remaining in stock and $h$ HD sets, with the ratio ($s$ – 90) : $h$ = 1 : 2.

Therefore $\frac{s-90}{h} = \frac{1}{2}$, or $h = 2(s - 90)$.

This can be rearranged as $2s - 180 = h$ and thus to $2s - h = 180$.

After the second dispatch, we have ($s$ – 90) standard sets and ($h$ – 70) HD sets left in stock, with a standard to HD ratio of 3 : 1

The ratio ($s$ – 90) : ($h$ – 70) is 3 : 1, so $\frac{s-90}{h-70} = \frac{3}{1}$, or $s - 90 = 3(h - 70)$.

We rearrange this to $3h - 210 = s - 90$ and finally to $3h - s = 120$.

We now have two simultaneous equations:

$2s - h = 180$ $A$
$3h - s = 120$ $B$

$2s - h = 180$ $A$
$6h - 2s = 240$ $2B$
$5h = 420$ $A + 2B$

$h = 84$, so there were 84 HD sets in the warehouse to begin with.
Substituting for $h$ in equation $A$ gives $2s - 84 = 180$, or $2s = 264$, and finally $s = 132$.
Hence there were 132 standard sets in the warehouse at the start.

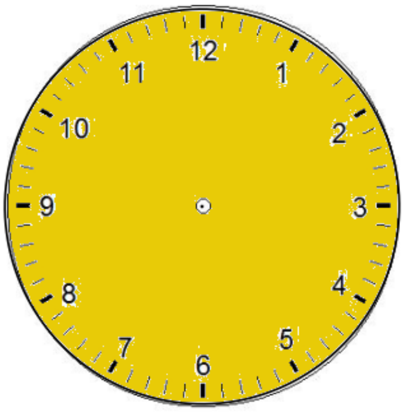
$\therefore$ There were **132 standard TV sets** and **84 HD TV sets** in the warehouse before the first dispatch.

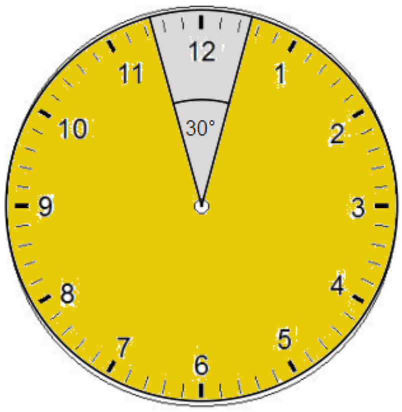
**Example (7):** A watchmaker has produced two versions of a certain style of watch face, each of which is 30 mm in diameter. The standard version has an all-bronze face, but the more expensive special edition has a 30° inset sector in silver plate, with the rest of the watch face in bronze.

The costs of bronze and silver plate are related in the ratio 2 : 5.

Show, using full working, that the ratio between the costs of the watch faces is 8 : 9.



Standard - bronze face




Special Edition - Silver-plated inset

The surprising fact about this question is that there is no need to use the circle area formula at all !

Both watch faces have the same diameter, and thus the same area, which simplifies matters when calculating their relative costs, although the proportions of metals used in each are different.

We visualise each watch face as being divided into 12 sectors of 30° each, since 30° is $\frac{30}{360} = \frac{1}{12}$ of a circle. Moreover, these sectors have the same areas on both watches, so they use the same amount of metal on each one.

The special edition has a 30° sector, i.e. $\frac{30}{360} = \frac{1}{12}$ of a circle, in silver plate, with the remaining $\frac{11}{12}$ of a circle in bronze.

The metals in the special edition watch face are in the ratio 1 part silver plate to 11 parts bronze, for a total of 12 parts of metal.

The metal content of the regular watch face can also be described as 12 parts bronze out of 12.

Since the costs of bronze and silver plate are related in the ratio 2 : 5, we can say that bronze costs 2 units per 30° sector and silver plate costs 5 units per 30° sector.

The cost breakdown of the special edition watch face is
1 share silver plate @ 5 units = 5 units, plus 11 shares bronze @ 2 units = 22 units.
∴ The cost of the special edition watch face is 5 + 22 units, or 27 units.

The regular version watch face is made entirely of bronze, so we can reckon its cost as
12 shares bronze @ 2 units = 24 units in total.

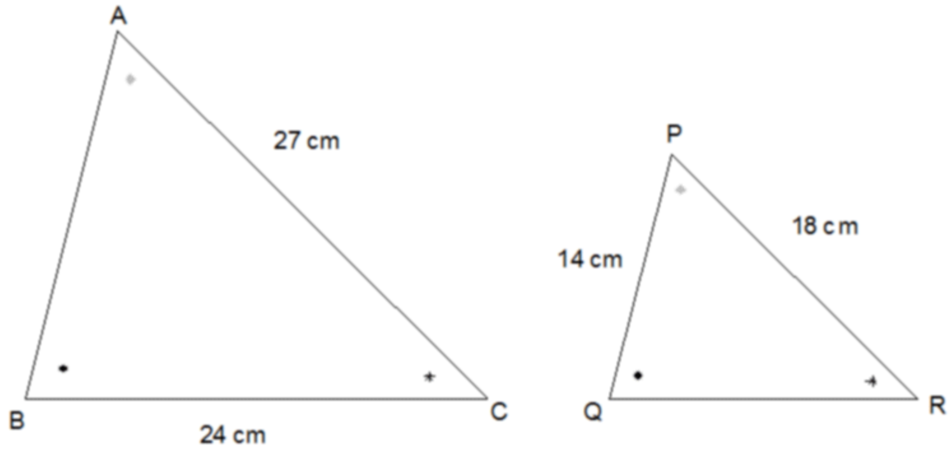
(Note that this is only true here because the areas of the watch faces are the same).

∴ The costs of the watch faces (regular : special) are in the ratio **24 : 27** , simplifying to **8 : 9.**

**Ratios and Similarity.**

Ratio can also be used to find measurements of similar figures, i.e. figures of the same **shape** but not of the same size.

**Example (8):** The triangles shown in the diagram are similar.
Find the lengths of the missing sides *AB* and *QR*.




We are given the lengths of one pair of corresponding sides from each triangle, namely *AC* = 27cm and *PR* = 18 cm.

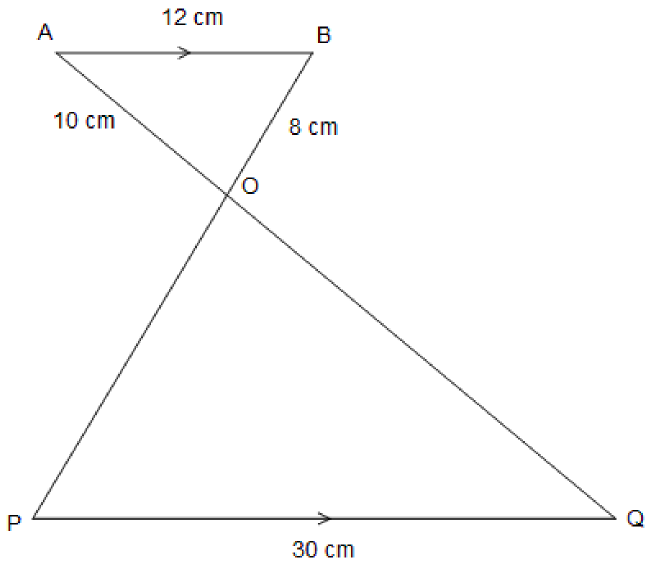
The ratio *AC* : *PR* = 27 : 18 or 3 : 2 in its simplest form.

The sides of *ABC* are therefore $\frac{3}{2}$ or $1\frac{1}{2}$ times the length of the corresponding sides of *PQR*, so we multiply by $\frac{3}{2}$
.
Conversely, the sides of *PQR* are $\frac{2}{3}$ as long as those of *ABC*, so we multiply by $\frac{2}{3}$ (or divide by $\frac{3}{2}$ ).

Sides *AB* and *PQ* correspond, so $\boldsymbol{AB = \frac{3}{2} PQ}$ **= 21 cm.**

*BC* and *QR* similarly do so, and thus $\boldsymbol{QR = \frac{2}{3} BC}$ **= 16 cm.**

**Example (9):** Find the lengths *OP* and *OQ* in the figure shown, given that lines *AB* and *PQ* are parallel.

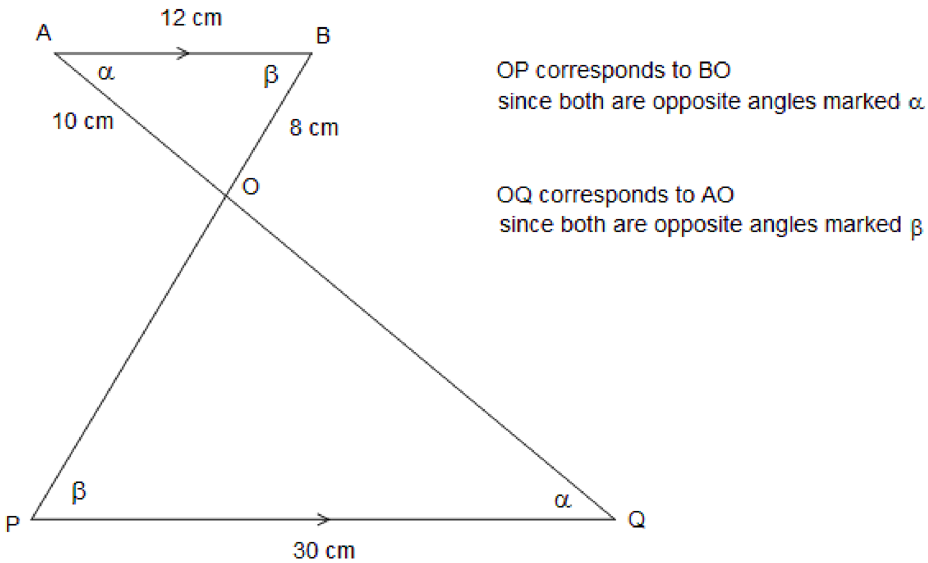



We need to know which sides of triangle *OAB* correspond to those of triangle *OPQ*.

Sides *AB* and *PQ* form a corresponding pair, where the ratio *AB* : *PQ* = 12 : 30 or 2 : 5 in its simplest form.

The sides of *OPQ* are therefore $\frac{5}{2}$ or $2\frac{1}{2}$ times the length of the corresponding sides of *OAB*.

To determine which remaining sides correspond, we use the properties of angles in parallel lines. We can see that angles *ABO* and *OPQ* form a pair of alternate angles (marked α in the diagram), as do the pair *BAO* and *OQP* (marked β).




Sides *OP* and *BO* correspond because they are both opposite the angles marked α; similarly sides *OQ* and *AO* correspond because they are both opposite the angles marked β.

The sides of *OPQ* are $\frac{5}{2}$ times the length of the corresponding sides of *OAB*, so

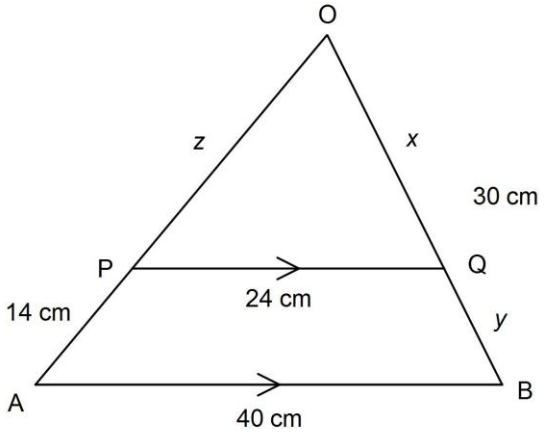
***OP*** **=** $\frac{5}{2}$ ***BO*** **= 20 cm** and ***OQ*** **=** $\frac{5}{2}$ ***AO*** **= 25 cm.**

**Example (10):** In the diagram, $OB$ = 30 cm, $AB$ = 40 cm, $PQ$ = 24 cm and $PA$ = 14 cm.

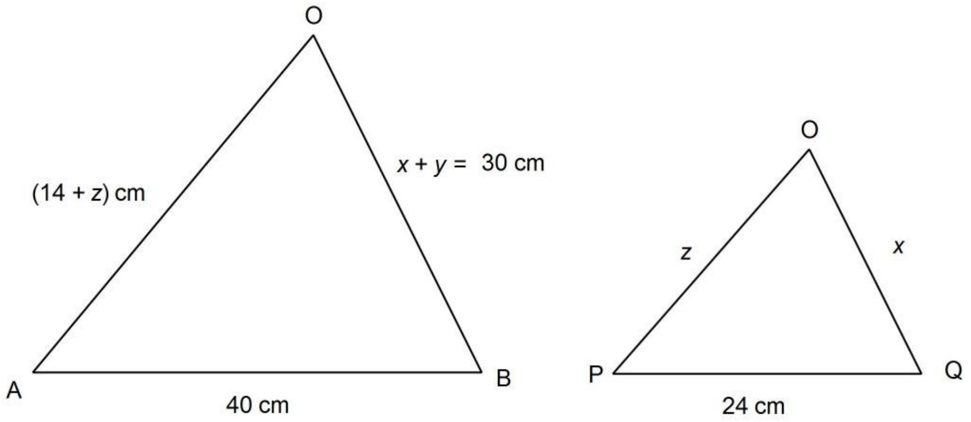
$PQ$ is also parallel to $AB$.

Find the lengths $OQ$ (labelled $x$), $QB$ (labelled $y$) and $OP$ (labelled $z$).

The working is similar to that in the last two examples, but there is a little more algebra.




Since $PQ$ and $AB$ are parallel**,** the triangles $AOB$ and $POQ$ are similar, and are shown separately in the diagram below. .




The sides $AB$ and $PQ$ correspond, so the ratio of the lengths is 40 : 24, or 5:3 in its simplest form .

Therefore $\frac{AB}{PQ} = \frac{OA}{OP} = \frac{OB}{OQ} = \frac{5}{3}$ , and conversely , $\frac{PQ}{AB} = \frac{OP}{OA} = \frac{OQ}{OB} = \frac{3}{5}$.

Given $OB = x + y = 30$ and $OQ = x$ , we can say $\frac{OQ}{OB} = \frac{x}{x+y} = \frac{x}{30}$ and since $\frac{OQ}{OB} = \frac{3}{5}$ as well, we have $\frac{x}{30} = \frac{3}{5}$ and finally $x = \frac{90}{5} = 18$. Therefore, **$x$ = 18 cm** and **$y$ = (30-18) cm = 12 cm**.

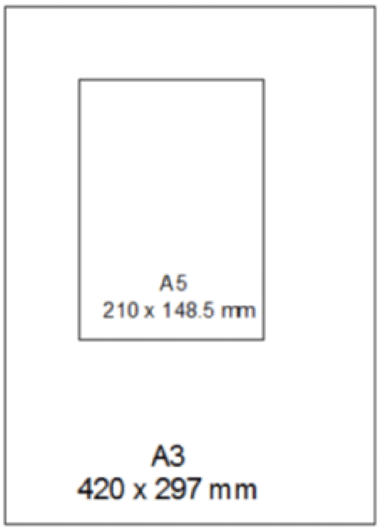
The third length, $z$ , can be found using one of two methods:

**Method 1:** Since $OA = 14 + z$, and $OP = z$ we can form the equation $\frac{OA}{OP} = \frac{14+z}{z} = \frac{5}{3}$ and rearrange

to $5z = 3(14 + z)$, then $5z = 42 + 3z$ , and finally $2z = 42$ and $z = 21$, and so $OP$ = 21 cm.

**Method 2:** We can now establish another ratio, $x$: $y$ , or $OQ$ : $QB$ = 18 :12 or 3: 2 in its simplest form .
The ratio $OP$: $PA$ is also 3:2 , and because $PA$ = 14cm, $OP$ is $\frac{3}{2}$ times that, or 21 cm.
Hence $z$ = 21 cm.

**Example (11):** A sheet of A3 printing paper has dimensions 420mm × 297mm, whereas a sheet of A5 paper measures 210mm × 148.5mm.




Are they similar, and if so, what is the ratio of the lengths of their sides?

The long sides are in the ratio 420 : 210.
This can be simplified to 2 : 1.

The short sides are in the ratio 297 : 148.5.
This can also be simplified to 2 : 1.

$\therefore$ The A3 and A5 sheets of paper are similar, with the ratio 2 : 1 between A3 and A5.

**Example (12):** This model of a Spitfire fighter plane has a wingspan of 468 mm and a length of 380 mm.


i) Given that this is a 1:24 scale model, what are the actual wingspan and length of a Spitfire in metres, to the nearest centimetre ?

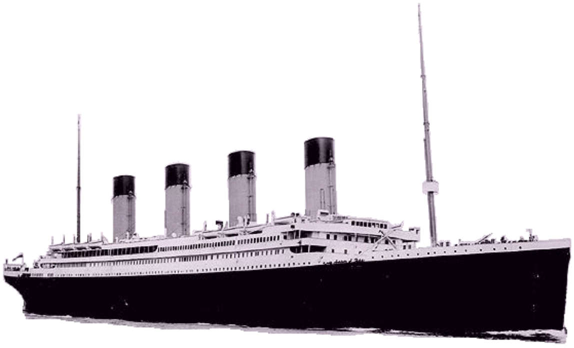
ii) If the height of an actual Spitfire is 3.48 m, calculate the height of the model in millimetres.

i) The scale factor of actual : model is 24 : 1, so all **linear** dimensions of the actual Spitfire will be 24 times those of the model.

The wingspan of an actual Spitfire is therefore ($24 \times 468$) mm, or **11.23 m**, and the length is ($24 \times 380$) mm or **9.12m.**

ii) This time we have the height of the actual Spitfire, so we divide 3.48 m, or 3480 mm, by 24 to obtain the height of the model, namely $\frac{1}{24} \times 3480$ mm, or **145 mm**.

**Example (13):** The film studio mock-up of the *Titanic* liner was built to seven-eighths the scale of the actual ship.



i) If the actual length of the *Titanic* was 270 metres, what was the length of the studio mock-up ?

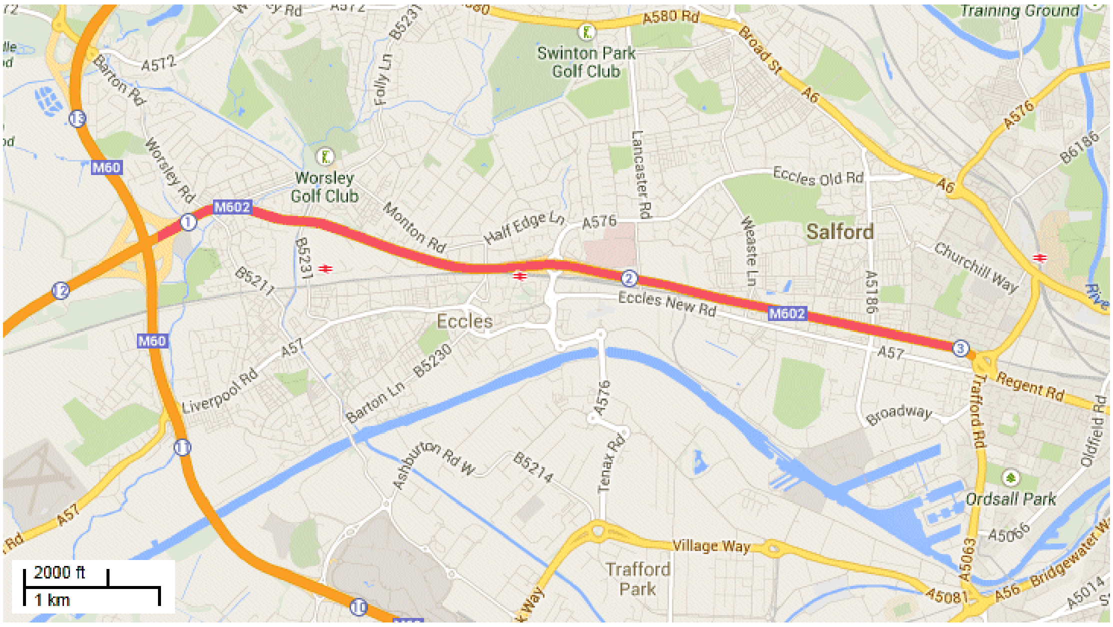
ii) If the funnels of the studio mock-up were 16.8 metres tall, how tall were the funnels of the actual *Titanic* ?

i) The length of the studio mock-up is $\frac{7}{8} \times 270$ , or **236 metres** long to the nearest metre.

ii) This time we must divide the funnel height of the mock-up by $\frac{7}{8}$ , or multiply it by $\frac{8}{7}$ , to obtain the funnel height of the real ship, i.e. $\frac{8}{7} \times 16.8$ metres, or **19.2 metres**.

**Example (14) :** The M602 motorway appears 10 cm long on a map, but is 6 kilometres long in reality. What is the scale of the map ? Give the result in the form 1 : *n* where *n* is a whole number.




The ratio of the map to reality is 10 cm : 6 km, or 0.1 m : 6000 m.
(We have converted lengths to metres for consistency.)

This can in turn be redefined as 1 m : 60,000 m, and finally as 1 : 60,000.

The map scale is therefore **1 : 60,000**.

**Example (15) :** Another map is issued to a scale of 1 : 25,000.

i) The true distance between Manchester Piccadilly and Oxford Road stations is 825 metres. How far apart are they on the map ?

ii) Oxford Road and Deansgate stations are 23 mm apart on the map. How far apart are they in reality ?

i) A scale of 1 : 25,000 can be redefined as 1 mm: 25,000 mm, or **1 mm : 25 m.**

The distance of 825 metres in real life is therefore represented by $\frac{825}{25}$, or 33 millimetres.
$\therefore$ Piccadilly and Oxford Road stations are **33 mm** apart on the map.

ii) As 1 mm on the map represents 25 metres in real life, 23 mm represents 23 × 25, or 575 metres.
$\therefore$ Oxford Road and Deansgate stations are **575 m** apart in reality.

**Proportionate Division.**

Here we add the proportional parts of a ratio to obtain the number of shares. Each part of the ratio then becomes the top line of a fraction whose bottom line is the number of shares.

**Example (16):** Abel, Baker and Charlie buy 8, 5 and 3 National Lottery tickets respectively each week and agree to share out their winnings in proportion to their weekly outlay.

One week, they win a jackpot of £720,000. How much does each player receive ?

Adding the contributions gives 8 + 5 + 3 = 16 shares in total.
Each share of the winnings is therefore $\frac{1}{16}$ of the total £720,000, or £45,000.
Abel therefore receives 8 shares or $\frac{8}{16}$ (or $\frac{1}{2}$ ) of the total, namely £360,000.
Baker receives 5 shares or $\frac{5}{16}$ of the total, or £225,000.
Charlie receives 3 shares or $\frac{3}{16}$ of the total, or £135,000.

**Example (17):** The top three cash prizes in a raffle are distributed in the ratio 5 : 2 : 1.
The first prize is £120 more than the second prize. What is the value of each prize ?

There are 5 + 2 + 1 or 8 shares in total, and we also know that the first prize is valued at 5 shares and the second at 2 shares. Their difference is 5 – 2, or 3 shares.
Also, as 3 shares = £120, one share is one-third of £120, or £40.

$\therefore$ The first prize is £(5 × 40), or £200; the second, £(2 × 40), or £80, and the third prize is £40.

**Example (18):** The top four finishers in a snooker tournament receive prize money divided as follows: the winner receives half as much again as the runner-up, who in turn receives twice as much as each beaten semi-finalist. If the total prize money is £56,000, how much does each prize winner receive ?

The ratio between the winner's earnings and those of the runner-up can be deduced as 3 : 2; the ratio between the runner-up and one semi-finalist can be worked out as 2 : 1.

The overall ratio of division of the prize money is therefore 3 : 2 : 1 : 1.
This gives 7 shares of $\frac{1}{7}$ of £56,000 or £8,000 each.

The winner receives 3 × £8,000, or £24,000.
The runner-up receives 2 × £8,000, or £16,000.
Each beaten semi-finalist receives £8,000.

**Example (19):** A bricklayer has prepared 160 kg of mortar by mixing 1 part of cement to 3 of sand.
His foreman then asks him to change the ratio of the mix to 2 parts of cement to 5 of sand.

Assuming that the bricklayer does not use any additional sand, how much cement does he need to add to the existing mortar mixture to produce the required cement to sand ratio of 2 : 5 ?

Since 1 part + 3 parts = 4 parts, the original mortar mixture is one-quarter, or 40 kg cement, and three-quarters, or 120 kg, sand.

After the foreman's request to change the cement : sand ratio to 2 :5, this 120 kg of sand now represents 5 parts, and so one part of the new mixture is $\frac{1}{5} \times 120$ kg, or 24 kg.
The bricklayer therefore needs 2 parts, or 2 × 24 kg, or **48 kg total cement** in the mixture.

As the original mixture had 40 kg of cement in it, the bricklayer needs to add another 8 kg of cement to change the ratio to the required 2 : 5.

**Example (20) :** The outside of a football is made by stitching together a number of panels, whose faces are regular pentagons and regular hexagons. The ratio of pentagons to hexagons for a complete football is 3 : 5.

A workshop currently has 840 hexagonal panels and 700 pentagonal panels available for assembling the footballs.

At the end of a working shift, there are twice as many pentagons left as there are hexagons.

i) Calculate how many panels of each shape remained at the end of the shift.

ii) A complete football has 32 panels in total.

a) State how many pentagons and hexagons make up the outside of a completed football.
b) Calculate the number of footballs assembled in the working shift.

i) If the ratio of pentagons to hexagons used is 3 : 5, then we can say that $3n$ pentagons and $5n$ hexagons are used up in making the footballs.

As there are 700 pentagons to begin with, there are $(700-3n)$ remaining at the end of the shift.
By similar reasoning, there are $(840-5n)$ hexagons left.

There are twice as many pentagons as hexagons remaining at the end of the shift, so we can say

$700-3n = 2(840-5n) \rightarrow 700-3n = 1680-10n \rightarrow 700 + 7n = 1680 \rightarrow 7n = 980$ and finally $\boldsymbol{n} = \mathbf{140}$.

The number of pentagons remaining at the end of the shift is = $700 - (3 \times 140)$, or **280**.
Likewise, the number of hexagons remaining is $840 - (5\times 140)$, or **140**.

ii) a) Dividing 32 in the ratio 3 : 5 gives 8 parts, with one part is $\frac{1}{8}$ of 32, or 4 panels.
Hence $3 \times 4$, or 12, of the panels on a football are pentagons, and $5 \times 4$, or 20, are hexagons.

$\therefore$ The outside of a football consists of **12 pentagons and 20 hexagons.**

ii) b) The total number of hexagonal panels used up is $5n$ , which, with $n = 140$, makes 700.

There are 20 hexagonal panels on a football, so the total number of footballs assembled in the working shift is $\frac{700}{20}$, or **35 footballs**.

(Working with pentagonal panels would give the same result.)

**Example (21):** A tea company produces its own special blend by using India, Ceylon and China teas in the ratio 1:3:6 by weight and then packing the mixture in cases containing 20 kg each.
The owner checks her warehouse and finds she has 24 kg of India tea, 42 kg of Ceylon tea and 120 kg of China tea in stock.

Does she have enough of each tea variety to fill 8 of those 20 kg cases ?
You must show all your working.

As $8 \times 20 = 160$, and the owner has (24 + 42 + 120), or 186 kg of the required teas, it seems at first that she has enough of each to make 160 kg of the blend.

Dividing the required 160 kg in the ratio 1:3:6 gives us 1 + 3 + 6 shares, or 10 shares, in total.
One share is $\frac{1}{10}$ of 160 kg, or 16 kg, which is the amount of India tea she needs. The required amount of Ceylon tea is 3 shares or $\frac{3}{10}$ of the total, i.e. 48 kg.
Similarly, she needs 6 shares of China tea, which is $\frac{6}{10}$ of the total, or 96 kg.
She has 24 kg of India tea in stock, but needs 16 kg to make the blend, so she has enough there.
She also has 120 kg of China tea but needs 96 kg, so she has more than enough.
She needs 48 kg of Ceylon tea, but has only 42 kg in stock, so she is 6 kg short there.

$\therefore$ She does not have enough of the required teas to fill 8 cases.

**Example (22):** Four children each receive a sum of money from their father, and this money is divided between them in the ratio of their ages.

Jessica, Jane and Jennifer are triplets.
Jack is two years younger than his sisters and receives £154.
The sum of all the children's ages is 50.

Calculate the sum of money that each of Jack's sisters receives.

We must begin by forming an equation to find the ages of the children.

Let each sister's age be $s$. Jack is two years younger than his sisters, so his age is $s - 2$.
As the three sisters are triplets, their combined age is $3s$, and adding Jack's age gives us the expression $3s + (s - 2)$ or $4s - 2$.

The sum of all the children's ages is 50, so we must solve the equation $4s - 2 = 50$.
Simplifying, we have $4s = 52$, leading to $s = 13$.

Hence the sisters are all 13 years old and Jack is 11. Since the children's sums of money are in proportion to their ages, Jack's sum of £154 is the equivalent of 11 shares, so one share of the money is $\frac{1}{11}$ of £154, or £14.

Hence each sister receives 13 shares of the money, or **£182.**

**Example (23):** Niall, Oliver and Peter collect stamps, and the numbers in each boy's collection are proportional to their ages.

Peter has two-fifths of the total number of stamps.
Niall has three-sevenths of the remainder.
Oliver has 60 fewer stamps than Peter.
All three boys are under 16 years old.

Work out the boys' ages, and hence the numbers of stamps in each of their collections.

If Peter has $\frac{2}{5}$ of the total number of stamps, then the other two boys have $1-\frac{2}{5}$, or $\frac{3}{5}$, of the total between them.

Niall has $\frac{3}{7}$ of that remaining $\frac{3}{5}$ of the total, so his share is $\frac{3}{7}\times\frac{3}{5}=\frac{9}{35}$ of the total.

Oliver therefore must have $1-\left(\frac{2}{5}+\frac{9}{35}\right)$ or $1-\frac{23}{35}=\frac{12}{35}$ of the total.

The numbers of stamps in the boys' collections, and hence the boys' ages, are divided in the ratio

$\frac{9}{35}:\frac{12}{35}:\frac{2}{5}$.

This awkward fractional ratio can be turned into a whole-number one by multiplying all its parts by 35 to give 9 : 12 : 14.

As all three boys are under 16 years old, their ages are: Niall, 9 ; Oliver, 12 ; Peter, 14.

We are also told that Oliver has 60 fewer stamps than Peter, and this difference is equal to two proportional parts, i.e. 14 – 12. One proportional part is therefore equivalent to 30 stamps.

Hence, the numbers of stamps in each collection are

Niall: $9 \times 30 =$ **270** stamps;
Oliver: $12 \times 30 =$ **360** stamps ;
Peter: $14 \times 30 =$ **420** stamps.

**Example (24):** Sarah has some cash saved for the Variety Club and the NSPCC, such that the ratio of money saved is 6 : 5 in favour of the Variety Club.

Sarah then decides to open a new cash fund for Marie Curie, starting from zero. She therefore transfers one third of her Variety Club money and a quarter of her NSPCC money into that new fund.

The total amount saved is a whole number of £s, more than £250 but less than £300.

Work out how much money Sarah has in each fund at the end of the transfers.

At the start, the money is divided in the ratio 6 : 5 : 0 between Variety Club, NSPCC and Marie Curie.

One third of 6 parts is 2 parts, so after Sarah transfers the money from Variety Club to Marie Curie, the ratio is changed to 4 : 5 : 2. (Variety Club loses 2 parts, Marie Curie gains 2 parts.)

One quarter of 5 parts is 1¼ parts, so after Sarah transfers the money from NSPCC to Marie Curie, the ratio becomes 4 : 3¾ : 3¼. (NSPCC loses 1¼ parts, Marie Curie gains 1¼ parts.)

Because the total amount is a whole number of pounds, the parts need to be multiplied by 4 to put the ratio into whole-number form as 16 : 15: 13.

Adding the parts gives us 16 + 15 + 13 = 44, so the total sum of money must be some whole-number multiple of £44. The only such multiple between £250 and £300 is 6 × £44, or £264.

Hence if 44 parts are £264, one part is $\frac{1}{44}$ of £264, or £6.

The amounts of money in each of Sarah's funds are thus :

Variety Club: 16 × £6 = **£96**;
NSPCC: 15 × £6 = **£90**;
Marie Curie: 13 × £6 = **£78**.

**Example (25) (Percentage tie-in) :** Muddler and Sulky are two agents who have been discussing their paranormal investigative cases over their careers.

Muddler: “Well, Sulky, I can say that my clear-up rate of 75% of the total was better than your 60%.”

Sulky: “Listen, Muddler – I had investigated 30% more cases in total than you have. When it comes to the number of cleared cases, I’d been able to clear nine more than you.”

i) Show, using ratios, that Sulky had cleared 4% more cases than Muddler.
ii) Hence calculate the number of cases that each agent had cleared in total.

i) Since Sulky had investigated 30% more cases than Muddler, the ratio of total cases is 130 : 100 in favour of Sulky.

Now 75% of 100 is obviously 75, and 60% of 130 is $\frac{130 \times 60}{100} = 78$.

Hence the ratio of **cleared** cases is 78 : 75, or 26 : 25, in Sulky’s favour.

This ratio can in turn be rewritten as 104 : 100, so Sulky had cleared 4% more cases than Muddler.

ii) We are also told that Sulky had cleared 9 cases more than Muddler, so we can go back to the simplified ratio of 26 : 25.

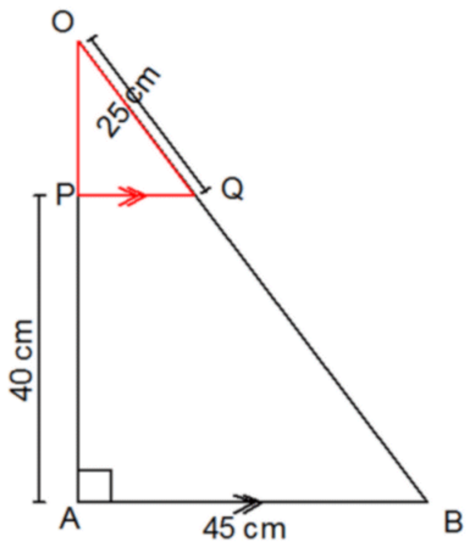
Sulky’s excess of 1 proportional part is equivalent to 9 cleared cases, so reverting to the original ratio of 26 : 25, the numbers of cleared cases are

Sulky: $26 \times 9 =$ **234** cases ;
Muddler: $25 \times 9 =$ **225** cases.

**Ratio applied to area.**

**Example (26):** Triangles *AOB* and *POQ* are right-angled and similar.
Given that point *P* divides the side *OA* in the ratio 1 : 2, find

i) the lengths of the sides *OA* and *OB* ; ii) the areas of triangles *AOB* and *POQ*.




i) Since the ratio *OP* : *PA* = 1 : 2 and *PA* = 40 cm, *OP* is half as long at 20 cm, and thus ***OA* = 60 cm**.

The ratio *OQ* : *QB* is also 1: 2. Since *OQ* = 25 cm, *QB* = 50 cm and hence ***OB* = 75 cm**.

The area of triangle *AOB* is ½ (base × height)
= ½ × 45 × 60 cm$^2$ = **1350 cm$^2$**.

Since triangles *AOB* and *POQ* are similar, we know that the ratio *OP* : *OA* = 20 : 60 or 1 : 3, as is the ratio *OQ* : *OB*. Therefore *PQ* is one-third as long as *AB*, i.e. 15 cm long.

The area of triangle *POQ* is ½ (base × height) = ½ × 15 × 20 cm$^2$ = **150 cm$^2$**.

**Example (26b):** Express the ratio of the areas of the triangles *AOB* and *POQ* in its simplest form.

The ratio of the **areas** of the triangles *AOB* : *POQ* is 1350 : 150 or **9 : 1** in the simplest form.
Triangle *AOB* has **nine** times the area of triangle *POQ*, although each side is **three** times as long.

The last example illustrates an important point.

If two similar figures have their **linear** dimensions in the ratio $a : b$ , then their **areas** will be related in the ratio $a^2 : b^2$.

In other words, the ratio between **areas** is the **square** of the ratio between **lengths**.

**Example (27):** The wing area of the Spitfire fighter plane in Example (12) is 22.48m$^2$ in real life. What is the wing area of the 1:24 scale model ?

The model's **linear** dimensions are $\frac{1}{24}$ of actual, and so the **area** dimensions are $\left(\frac{1}{24}\right)^2 = \frac{1}{576}$ of actual.

The wing area of the scale model is $\dfrac{22.48}{576}$ = 0.0390 m$^2$ or 0.039 × 100$^2$ cm$^2$ = **390 cm$^2$**.

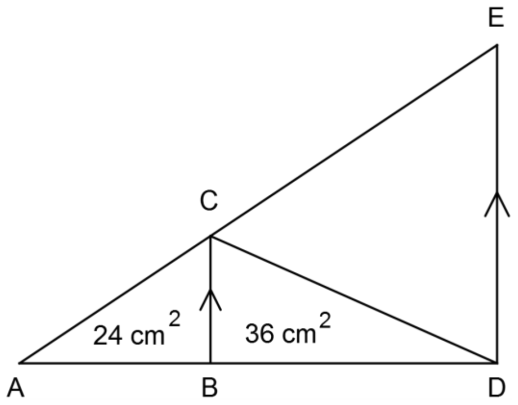
(Note that because there are 100cm in 1m, there are 100$^2$ or 10,000cm$^2$ in 1 m$^2$).

**Example (28):**

In the figure on the right, lines *BC* and *DE* are parallel.

Triangle *ABC* has an area of 24 cm$^2$ and triangle *CBD* an area of 36 cm$^2$.

Find the area of triangle *CDE*.




Because line *BC* is parallel to line *DE*, the triangles *ABC* and *ADE* are similar.
Although we have not been given any side lengths, were are given the areas of the triangles *ABC* and *CBD*, and because they both share the side *BC*, their perpendicular heights must also be equal.

The ratio between the triangles' areas is 24 : 36, or 2 : 3 in its simplest form, so the ratio between their bases *AB* and *BD* must also be 2 : 3.
$\therefore$ The ratio of the base of *ABC* to that of *ADE* is 2 : (2 + 3) or 2 : 5.

The ratio of the area of *ABC* to that of *ADE* is $2^2 : 5^2$, or 4 :25.
$\therefore$ The area of triangle *ADE* is $24 \times \frac{25}{4}$ or 150 cm$^2$.

The area of *CDE* can be found by subtracting the sum of the areas of *ABC* and *CBD* from the area of *ADE*.

Therefore the area of triangle *CDE* = 150 – (24 + 36) cm$^2$ = **90 cm$^2$**.

There is more about areas, volumes and similarity in the document "Areas,Volumes and Similarity".

**Everyday Ratio Problems.**

Ratio methods are often used to solve practical ‘everyday’ problems, such as ‘value for money’ price comparisons and altering recipes for differing numbers of people.

**Example (29a):** The following recipe has appeared in the Lancashire Cooking magazine:

**BURY HOT-POT Serves 4**

**600 g boneless lamb, diced**
**1 kg potatoes, peeled and cut into chunks**
**2 large carrots, sliced**
**1 large onion, chopped**
**4 slices of Bury black pudding**

Bob is organising a party where he needs to make enough Bury Hot-Pot to cater for 10 people. How will he need to adjust the amounts of ingredients he uses ?

The recipe quoted is for 4 people but Bob needs to cater for 10.
We therefore need to divide by 4 to find the amounts for one person, and then multiply by 10 to find the amounts for all.

Alternatively, we could deduce that 10 is 2½ times as large as 4,

600g of lamb for 4 people therefore equals $\frac{1}{4}$ of 600g or 150g for one, hence 1.5kg for 10 people.
1kg of potatoes similarly works out as 250g for one, and 2.5kg for 10.

The rest is worked out similarly:
2 large carrots for 4 works out as half a carrot each, or 10 halves = 5 carrots for all.
1 large onion for 4 works out as 2½ large onions for 10 (Bob can use 2 or 3 whole) .
Finally, 4 slices of Bury black pudding for 4 works out as 10 slices for 10.

The adjusted recipe is thus:

**BURY HOT-POT Serves 10**

**1.5 kg boneless lamb, diced**
**2.5 kg potatoes, peeled and cut into chunks**
**5 large carrots, sliced**
**2-3 large onions, chopped**
**10 slices of Bury black pudding**

Alternatively, we could have deduced that 10 is 2½ times as large as 4, and multiplied all the amounts by 2½ to get the same results.

**Example (29b):** Bob has had more people interested in his hot-pot meal than he originally thought, and he has checked his kitchen for ingredients.

He was able to find 3 kg of boneless diced lamb, three 2 kg bags of potatoes, a bag of 16 large carrots, 6 large onions and a bulk pack of 30 slices of Bury black pudding.

What is the maximum number of people he was able to feed by following the original recipe and not buying any extra items ?

Here is the original recipe :

**BURY HOT-POT** ***Serves 4***

**600 g boneless lamb, diced**
**1 kg potatoes, peeled and cut into chunks**
**2 large carrots, sliced**
**1 large onion, chopped**
**4 slices of Bury black pudding**

We need to find the "critical" ingredient in Bob's kitchen whose availability is the smallest multiple of the recipe for 4 people.

He has 3 kg of lamb, and as 3 kg is 3000g , he has $\frac{3000}{600}$ or 5 times the quantity specified in the recipe. The recipe asks for 1 kg of potatoes, but Bob has three 2 kg bags, or 6 kg in total, which is 6 times that. He needs two carrots for the recipe, but has 16, or 8 times the recipe amount. The recipe calls for one onion, but Bob has 6. Finally, he has 30 slices of black pudding, which is $\frac{30}{4}$ , or 7½, times the recipe amount.

The lamb is the "critical" ingredient here, as the 3 kg he has is 5 times the recipe amount, which is the smallest multiple among the ingredients.

The amounts in the original recipe were suggested to serve 4 people, so the contents of Bob's kitchen are enough to serve $5 \times 4$, or 20 people.

**The "Best Buy" problem.**

**Example (30):** A supermarket sells the same brand of baked beans in three different-sized cans; 400g for 36p, 600g for 59p and 800g for 75p. Which offers the best value for money ?

The method is to find a suitable 'weight unit' (100g would be a good choice here) and then find the prices for that 'unit weight'.

If 400g of beans cost 36p, then 100g cost $\frac{1}{4}$ of 36p, or 9p.

If 600g of beans cost 59p, then 100g cost $\frac{1}{6}$ of 59p, or 9.8p (to nearest 0.1p).

If 800g of beans cost 75p, then 100g cost $\frac{1}{8}$ of 75p, or 9.4p (to nearest 0.1p).

By comparing the prices per 100g, we can see that the 400g can offers the best value for money.

If the quantities are more awkward (as in a calculator question), there are two methods available.

**Example (31):** A supermarket sells the same brand of baked beans in three different-sized cans; 405g for 36p, 595g for 59p and 822g for 75p. Which offers the best value for money ?

This time, the weights are more awkward to handle, so we have two choices.

**Method (1):** Divide the price by the weight to find the **lowest price per gram (**or kg) :

Thus, the 405g can costs $\frac{36}{0.405}$ = 88.9p per kg (we have converted weight to kilograms for convenience)

Likewise the 595g can costs $\frac{59}{0.595}$ = 99.2p per kg and the 822g can $\frac{75}{0.822}$ or 91.2p per kg.
Again, the 400g size offers the best value for money.

**Here, best value for money = lowest price per g (or kg).**

**Method (2):** Divide the weight by the price to find the **highest weight per penny** (or £):

Thus, a person buying the 405g can buys $\frac{405}{36}$ = 11.3 g per penny.
Likewise, someone buying the 595g can buys $\frac{595}{59}$ = 10.1 g per penny, and someone buying the 822g can buys $\frac{822}{75}$ or 11.0 g per penny.

**Here, best value for money = highest weight per penny (or £).**

There is little to choose between the two methods, but the first one is more commonly used by supermarkets when comparing prices.

The next example of "shopping economics" is not a ratio problem as such, but features quite commonly in exams, so we have included it here.

**Example (32):** Barbara needs to buy 6 bottles of lemonade for a family party, and she sees the following offers for the same-size bottle of the same brand at three supermarkets:

Aldi: 55p per bottle
Tesco: 79p per bottle, but on a 'buy 3 for the price of 2' offer
Lidl: 72p per bottle, but on a 'buy one, get second half price' offer

Work out how much money Barbara would have to spend at each of the supermarkets, and hence find out which supermarket gives the best value for money.

If Barbara were to shop at Aldi, she would be paying 6 × 55p or **£3.30**.

At Tesco, she would be able to buy 3 bottles at 2 × 79p or £1.58 using the '3 for 2' offer
Since 6 is twice 3, she would be paying 2 × £1.58 or **£3.16**.

At Lidl, Barbara would be buying one bottle at 72p and the second one at half that or 36p, so she would be paying £1.08 for 2 bottles. Since 6 is three times 2, she would be paying 3 × £1.08 or **£3.24**.

$\therefore$ Tesco gives the best value for money of the three supermarkets.

**Direct and Inverse Proportion.**

This is an introduction to the ideas of direct and inverse proportion without too much of the algebra. The document on “Direct and Inverse Proportion” studies more of the relationships in greater detail.

**Example (33):** Laura is training on an exercise bike set to a constant speed. She cycles the equivalent of 3 km in 5 minutes. How long would it take her to cycle i) 12 kilometres, ii) 8 kilometres ?
Also, iii) how far will she cycle in 25 minutes ?

i) We see that 12 km is 4 times longer than 3 km, and if Laura’s speed remains constant, then the time taken will also be 4 times greater.

Thus, if she cycles 3 km in 5 minutes, the time taken for her to cycle 4 × 3 km, or 12 km, will be 4 × 5 minutes, or **20 minutes.**

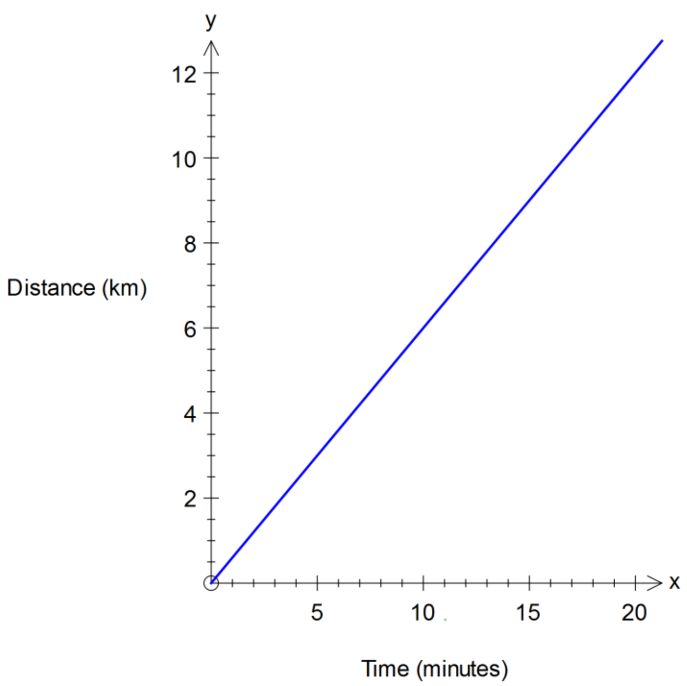
ii) This time the ratio between 3 km and 8 km is fractional, where we would have to multiply by $\frac{8}{3}$.

Laura will cycle the 8 km in $\frac{8}{3}$ × 5 minutes = $13\frac{1}{3}$ minutes = **13 minutes and 20 seconds**.

iii) Because 25 minutes is 5 times as long as 5 minutes, Laura will cycle 5 × 3 km, or 15 km, in 25 minutes.

This example was one of **direct** proportion; when one quantity (in this case, distance) was **increased** by a certain ratio, the other quantity (the time) was **increased by the same ratio**.

All relationships leading to direct proportion lead to straight-line graphs passing through the origin, with a general equation of $y = kx$ where $k$ is any positive number.



**Example (34):** An non-stop express passenger train travelling at 120 miles per hour takes 25 minutes to cover the distance from Rugby to Stafford. How long would it take i) a goods train travelling non-stop at 60 mph, and ii) a slower passenger train limited to 75 mph, to cover the same distance ?

iii) A slower goods train takes 1 hour and 15 minutes to cover the same distance. What is its speed ?

i) A speed of 60 mph is half as fast as 120 mph, and therefore the time taken to cover the same distance will be twice as long.

Therefore if the express train takes 25 minutes to cover the Rugby-Stafford distance, the goods train will take twice 25 minutes, or **50 minutes**, to do the same. So, as speed is halved, time is doubled.

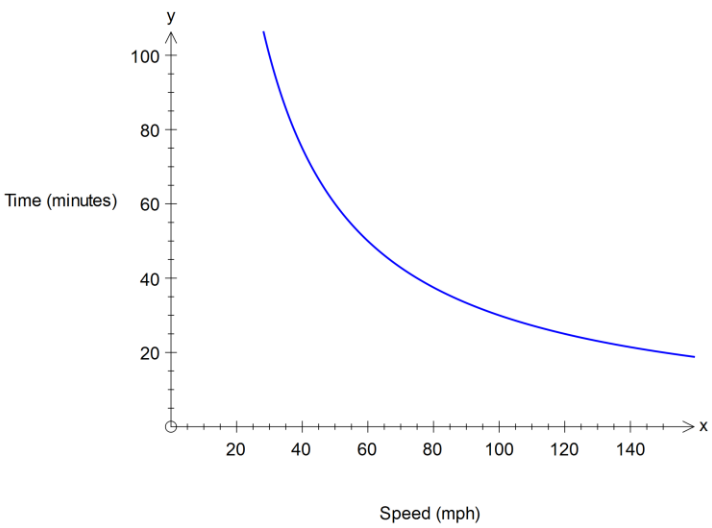
ii) The ratio between the speeds of the slow passenger train and the express train is 75 : 120, simplifying to 5 : 8. Because the slower train is travelling at $\frac{5}{8}$ of the speed of the express train, the time taken to cover the same distance will be $\frac{8}{5}$ times longer.

The express train takes 25 minutes, so the slow passenger train will take $\frac{8}{5} \times 25$ minutes = **40 minutes**.

iii) Since 1 hour and 15 minutes are equal to 75 minutes, the slow goods train takes three times as long to cover the same distance the express train travelling at 120 mph does in 25 minutes. As the time is tripled, the speed is divided by 3, and so the slow goods train is travelling at **40 mph.**

Unlike the previous example, we are dealing with **inverse** proportion; when one quantity (in this case, speed) was **decreased** by a certain ratio, the other quantity (the time) was **increased by the same ratio**.

All relationships leading to inverse proportion lead to a curved graph related to that of $y = \frac{1}{x}$, with a general equation of $y = \frac{k}{x}$ where $k$ is any positive number.




The next example is a little more tricky, as it combines both direct and inverse proportion in the same question. Such problems are best tackled in stages.

**Example (35):** Given that 12 checkout staff at a supermarket can serve 48 customers in 20 minutes,
i) how long would it take 15 checkout staff to serve 84 customers ?
ii) how many checkouts need to be open to serve 240 customers in one hour ?

If we increase the number of checkout staff, we increase the number of customers being served in the same time interval. Hence the checkout staffing level and the number of customers are in direct proportion when time is unchanged.

**More checkouts = more customers (in same time)**

Also, an increase in the number of checkout staff would lead to a decrease in the time taken to serve the same number of customers. This time, the checkout staffing level and the time taken to serve the customers are in inverse proportion when the number of customers is unchanged.

**More checkouts = less time (for same number of customers)**

Finally, if the number of checkout staff remains constant, an increase in the number of customers would lead to an increase in the time taken to serve them. In other words, the number of customers and the time taken to serve them are in direct proportion when the number of checkout staff is unchanged.

**More customers = more time (for same number of checkouts)**

i) We can see that the number of checkout staff has increased from 12 to 15, or in the ratio 15 : 12, simplifying to 5 : 4. Therefore 15 checkout staff can serve $\frac{5}{4} \times 48$, or 60 customers, in 20 minutes.
We now have the correct number of checkout staff, but we still need to increase the number of customers to 84, or in the ratio 84 : 60, simplifying to 7 : 5.
Since customer numbers and the time are in direct proportion, the time taken is $\frac{7}{5} \times 20$, **or 28 minutes**.

| | 12 checkouts; | 48 customers; | 20 minutes | Original problem |
|---|---|---|---|---|
| → | **15 checkouts;** | **60 customers;** | 20 minutes | Customers / checkouts in direct proportion, so increase both in ratio 5 : 4 (i.e. multiply by $\frac{5}{4}$ ) |
| → | 15 checkouts; | **84 customers;** | **28 minutes** | Customers / time in direct proportion, so increase both in ratio 7 : 5 (i.e. multiply by $\frac{7}{5}$ ) |

ii) This time, the number of customers has increased from 48 to 240, a ratio of 5 : 1, and so the time taken will increase by the same ratio, from 20 minutes to $5 \times 20$ or 100 minutes.

The question however asks us to work out the checkouts per hour, and so we have to decrease the time from 100 to 60 minutes, or multiply it by $\frac{60}{100}$ or $\frac{3}{5}$.

Because the time taken and the number of checkouts are inversely proportional, we have to **multiply** the number of checkouts by $\frac{5}{3}$, giving us $\frac{5}{3} \times 12$, **or 20 checkouts**.

| | 12 checkouts; | 48 customers; | 20 minutes | Original problem |
|---|---|---|---|---|
| → | 12 checkouts; | **240 customers;** | **100 minutes** | Customers / time in **direct** proportion, so increase both in ratio 5 : 1 |
| → | **20 checkouts;** | 240 customers; | **60 minutes** | Checkouts / time in **inverse** proportion, so **decrease** time in ratio 5 : 3 (i.e. multiply by $\frac{3}{5}$ )and **increase** checkouts in ratio 5 : 3 (i.e. multiply by $\frac{5}{3}$ ) |