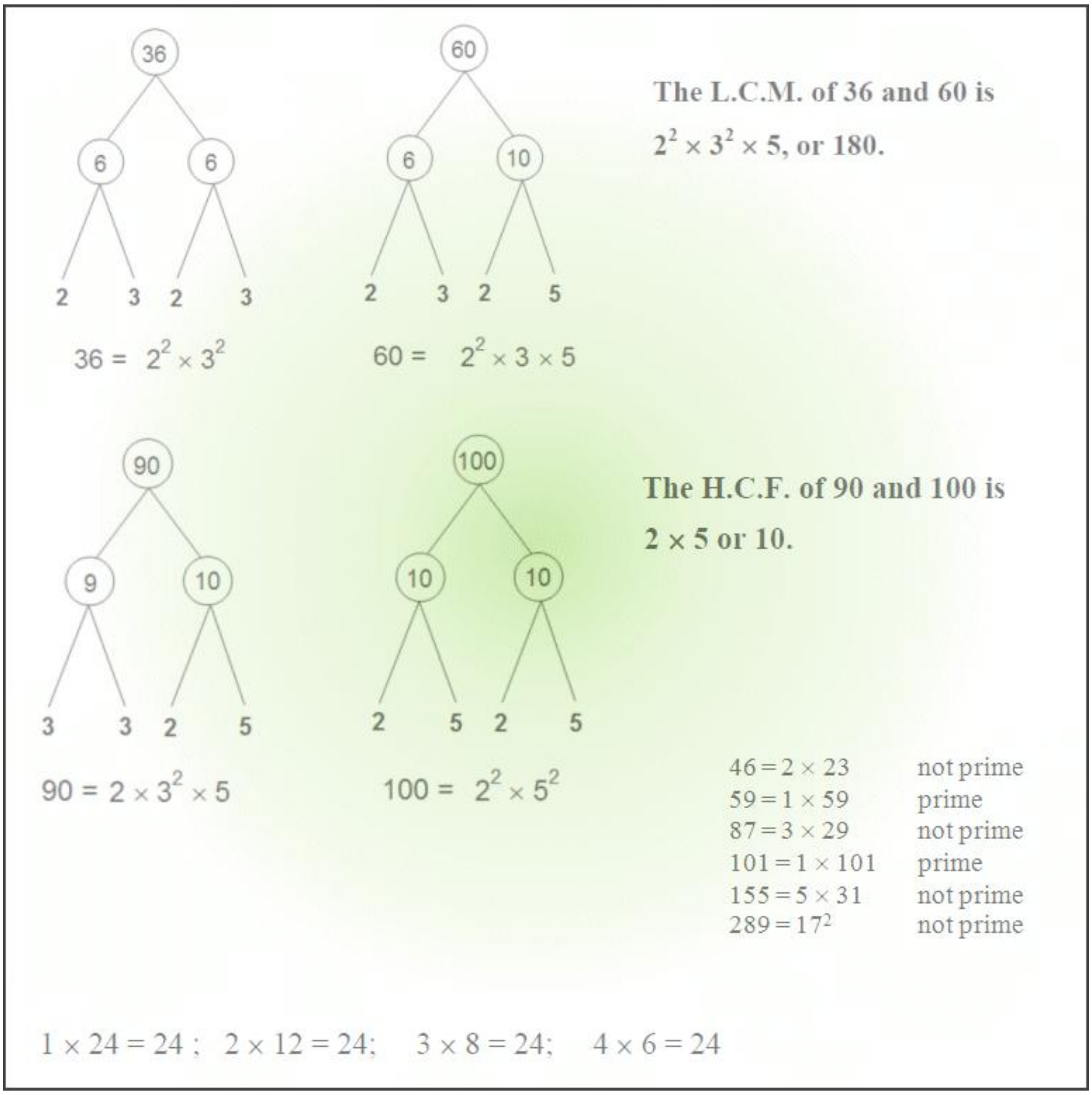
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

FACTORS, PRIME NUMBERS, H.C.F. AND L.C.M.



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FACTORS AND MULTIPLES.

(These terms relate only to positive integers.)

A **factor** of a number is any number that divides into it without remainder. For example, the factors of 18 are 1, 2, 3, 6, 9 and 18.

A **multiple** of a number is that number multiplied by another positive integer. Therefore the first five multiples of 12 are 12, 24, 36, 48 and 60.

A number is said to be **prime** if it has no other factors than itself and 1. The first five prime numbers are 2, 3, 5, 7 and 11. (1 is not considered a prime number).

Example 1. Find all the factors of 24.

The number 24 can be expressed as a product of two numbers in various ways:

$1 \times 24 = 24$; $2 \times 12 = 24$; $3 \times 8 = 24$; $4 \times 6 = 24$

We can stop here, because the next highest factor of 24, namely 6, has already been included. Once we reach a possible factor greater than the square root of the number, the process is finished, as the products would only reappear in reverse order.

$\therefore$ The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

When the number is larger, some trial and error might be needed using a calculator.

Example 2. Find all the factors of 252.

The square root of 252 is 15.87.... so there is no need to look for factors over 15.

$1 \times 252 = 252$
$2 \times 126 = 252$
$3 \times 84 = 252$
$4 \times 63 = 252$
(5 does not divide exactly into 252, giving an answer of 50.4)
$6 \times 42 = 252$
$7 \times 36 = 252$
(8 does not divide exactly into 252, giving an answer of 31.5)
$9 \times 28 = 252$
(10 and 11 do not divide into 252)
Also, if a number is not a factor, neither is any of its multiples. We found that 5 was not a factor of 252; therefore 10, being a multiple of 5, cannot be a factor either.
$12 \times 21 = 252$
(13 does not divide into 252)
$14 \times 18 = 252$
(15 does not divide into 252)

$\therefore$ The factors of 252 are 1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126 and 252.

Example 3. Find all the factors of 225.

The square root of 225 is exactly 15, so we do not need to look for any factors above it.
Because 15 is a factor, so are 3 and 5; also, since 225 is odd, we can forget about any even factors.
$1 \times 225 = 225$
$3 \times 75 = 225$
$5 \times 45 = 225$
(7 does not divide exactly into 225)
$9 \times 25 = 225$
(11 does not divide exactly into 225)
(13 does not divide exactly into 225)
$15 \times 15 = 225$
$\therefore$ The factors of 225 are 1, 3, 5, 9, 15, 25, 45, 75 and 225.

Factor / divisibility tests for numbers (factors up to 12).
There are many short cuts to test if one number divides exactly into another. Although they are not part of the syllabus, some of them can be quite useful time-savers, even in calculator exams.

All numbers ending in 2, 4, 6, 8 or 0 (even numbers) are divisible by 2.
36 is divisible by 2, but 37 is not.

All numbers whose digit sum is a multiple of 3 are divisible by 3.
51 is divisible by 3, since its digits, 5+1, add up to 6; 58 is not since its digits sum to 13,which is not a multiple of 3.

All numbers which are still even after halving are divisible by 4.
248 is divisible by 4 since half of it, 124, is still even; 250 is not because 125 isn't.

All numbers ending in 0 or 5 are divisible by 5.
125 is divisible by 5; 122 is not.

All even numbers whose digit sum is a multiple of 3 are divisible by 6.
246 is even, and its digits add up to 12, a multiple of 3.

Sorry – there are no quick tests for divisibility by 7 !

All numbers which are even after halving them twice running are divisible by 8.
232 is a multiple of 8, as halving it to 116 and then to 58 still leaves an even number.
252 is not, as halving it twice gives 63, an odd number.

All numbers whose digit sum is a multiple of 9 are divisible by 9.
189 is divisible by 9, since its digits, 1+8+9, add up to 18, a multiple of 9; 840 is not since its digits sum to 12,which is not a multiple of 9. Notice, that if you take the digit sum of 18 and add *its* digits, you will end up with 9.

All numbers ending in 0 are multiples of 10. Trivial !

All numbers whose alternate digit sums are equal or differ by a multiple of 11, are divisible by 11.
This is easier than it sounds – 594 is divisible by 11 because the sum of the 'odd-placed' digits (5 and 4) equals the 'even-placed' digit, here 9.
576 is not, because the sum of the 'odd' digits (5 and 6) = 11 and the 'even' digit is equal to 7. The difference between 'odds' and 'evens' is 4.

3091 is also divisible by 11 – the 'odd' digits (3 and 9) add to 12 and the 'even' digits (0 and 1) add to 1 – a difference of 11.

All numbers divisible by both 3 and 4 are divisible by 12.
276 can be halved to give the even value of 138, and its digits add up to 15, a multiple of 3.

Testing for prime numbers.
To determine if a number is prime or not, we need to try dividing it by every prime number up to its square root. If none of those primes divide into it, then the number is a prime. If there is one counter-example, then the number is not prime.

Example (4). Which, if any, of the following numbers are prime ?

46, 59, 87, 101, 155, 289 ?

46 is not a prime because it is even and greater than 2.

The square root of 59 lies between 7 and 8, so we need to test if it is divisible by 2, 3, 5 or 7.
It is odd (not a multiple of 2), its digits add up to 14 (so it's not a multiple of 3), it does not end in 5 or 0 (indivisible by 5), and it leaves a remainder after dividing by 7. Therefore 59 is prime.

The digits of 87 sum to 15, a multiple of 3, so 87 is a multiple of 3 – 87 is thus not prime

101 has a square root less than 11, it is odd, its digits add up to 2 (indivisible by 3), it doesn't end in 5 or 0, and it leaves a remainder after dividing by 7. 101 is thus prime.

155 ends in 5, and is therefore divisible by 5 and hence not prime.

289 has a square root of exactly 17, therefore it is not prime.

The Sieve of Eratosthenes.

This method of finding prime numbers was demonstrated over two thousand years ago, and we shall use it in the next example.

Example (5): Use the 'Sieve' to find all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Start with all the numbers 1-100

1	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

Get rid of all the even numbers after 2

1	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

Do the same with the multiples of 3

1	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			

... and with the multiples of 5

1	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

... and finally the multiples of 7.

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

THE END - note that 1 is not a prime !

The first grid shows all the numbers from 1 to 100.

The first prime number is 2, so we keep it, but eliminate, or "sieve out" all the other even numbers, as they all have a factor of 2 as well as 1 and the number itself.

Next, we continue in the same vein by getting rid of all the multiples of 3 excluding 3 itself.

We do the same with the multiples of 5 and 7, and finally remove 1 from the grid as it is not technically a prime.

Because the square root of 100 is 10, there is no need to test for multiples of 11 and over, as any such numbers less than 101 would have been thrown out in earlier stages.

For example, $77 = 7 \times 11$ would have been removed at the 'multiples of 7' stage.

Prime Factors.

Because a number can possibly be shown as a product of various factors, such as 24 = 3 × 8 or 24 = 4 × 6, it is usual practice to express a number as a unique product of its **prime** factors.

Using the fact that 24 = 4 × 6, we can separate 4 and 6 into products of prime factors, namely 2 × 2 and 2 × 3.

Therefore **24 = 2 × 2 × 2 × 3**, or $\mathbf{2^3 \times 3}$.

(such representations are generally sorted in ascending order, as shown above).

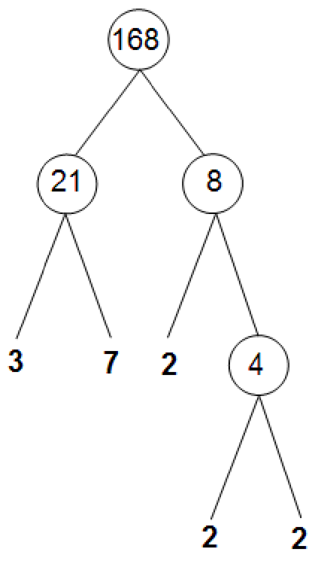
A 'tree' diagram is often helpful when separating numbers into prime factors – start by splitting the number into one pair of factors, and keep splitting each factor until you are left only with prime numbers at the end of each 'branch'.

Example (5). Separate 168 into its prime factors.

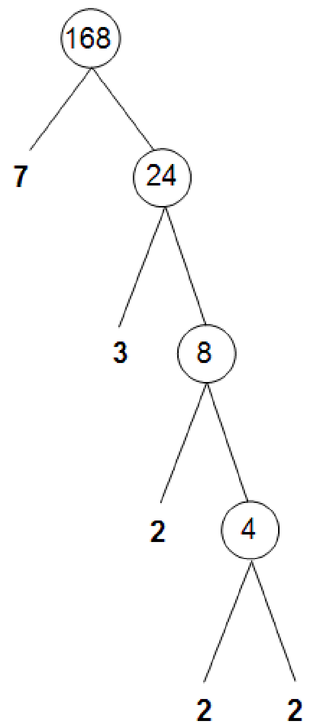
We start by seeing that 168 is equal to 21× 8. As we move down the tree, we then split 21 into 3 × 7 and 8 into 4 × 2. Now, 3, 7 and 2 are already prime (shown in bold), so there is no need to go further. However, 4 is still not prime, so we must split it into 2 × 2. Only then do we have the prime factors of 168, shown in bold at the 'leaves' of the tree, as per the diagram below left.

Sorted in ascending order they are **2 × 2 × 2 × 3 × 7**, or $\mathbf{2^3 \times 3 \times 7}$.

The prime factors are unique, even though the stages in the factorisation process do not need to be. Another arrangement is shown below right. .



$168 = 2^3 \times 3 \times 7$



$168 = 2^3 \times 3 \times 7$

The Highest Common Factor, or H.C.F.

The highest common factor of two or more numbers is the largest number that can divide into all of them without a remainder.

Example (6) : Andy has bought two large packs of sweets to give to neighbouring children for Trick or Treat. One pack contains 32 lollipops, and the other contains 48 chocolate bars. He wants to ensure that each child has exactly the same number of lollipops and chocolate bars, without any of either left over.

What is the largest number of children that could be provided for in this way, and how many lollipops and chocolate bars would each child receive ?

Andy thinks: "Well, there are eight children on our road, and 8 goes into both 32 and 48. That way, each child can have four lollipops and six chocolate bars."

In other words, 8 is a common factor of both 32 and 48.

He then realises that four lollipops and six chocolate bars can still be split equally between two people since 4 and 6 are both even, and thus he works out that the 32 lollipops and 48 chocolate bars can be split equally between 16 people and not just 8, with one person receiving two lollipops and three chocolate bars.

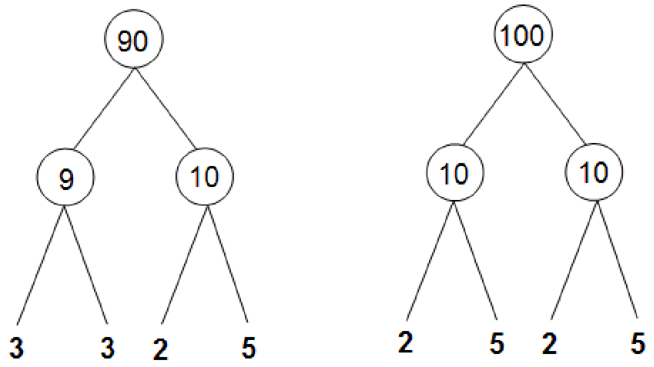
The **highest common factor**, or H.C.F., of 32 and 48 is therefore 16.

If the H.C.F. cannot be found easily, then separate each number into prime factors and find those that are common to both.

The **lower** power of each prime factor becomes a factor of the H.C.F. Any prime factors occurring in only one of the numbers are absent from the H.C.F.

Example (7). Find the H.C.F. of 90 and 100.

This H.C.F. can be found to be 10 by inspection, but the formal method is shown below.
Separating into prime factors we have



$90 = 2 \times 3^2 \times 5$ $100 = 2^2 \times 5^2$

Therefore 90 = $2 \times 3 \times 3 \times 5$ or $2 \times 3^2 \times 5$, and 100 = $2 \times 2 \times 5 \times 5$ or $2^2 \times 5^2$.

Going through each prime factor in turn:

2 occurs once (as 2^1) in 90, but as 2×2 or 2^2 in 100, and so must therefore occur as the lower power, namely 2^1 or simply 2, in the H.C.F.

3 occurs as 3×3 or 3^2 in 90, but not at all in 100, so it cannot be present in the H.C.F.

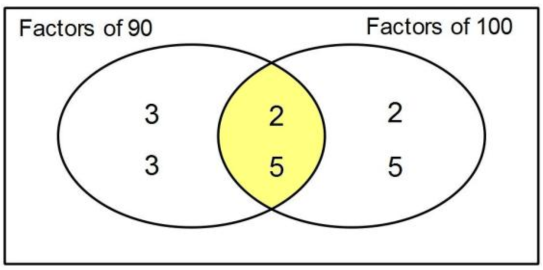
5 occurs once (as 5^1) in 90, but as 5×5 or 5^2 in 100, and so it must occur as 5^1, i.e. 5, in the H.C.F.

This result can also be shown by expressing 90 and 100 as prime factor products in "long" non-index form and highlighting the factors common to both:

$90 = 2 \times 3 \times 3 \times 5$
$100 = 2 \times 2 \times 5 \times 5$

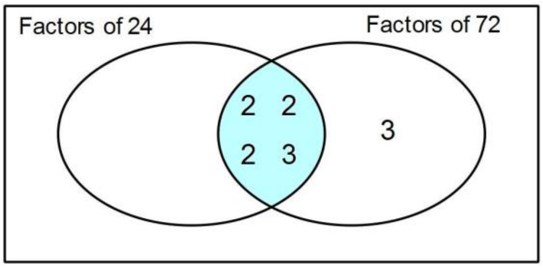
The H.C.F. of 90 and 100 is therefore 2×5 or 10.

This result can also be shown using a **Venn diagram** with two overlapping regions. All the prime numbers inside the "Factors of 90" region multiply together to give 90, and all those inside the "Factors of 100" region multiply together to give 100. The factors inside the area of overlap give the H.C.F. when multiplied.



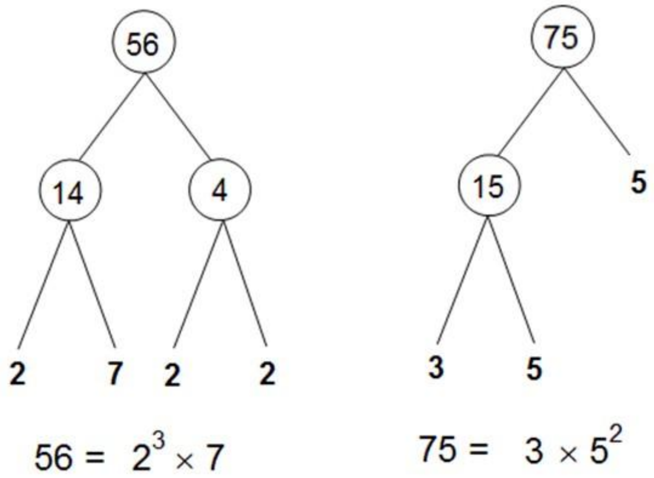
If one of the two numbers is a factor of the other, their H.C.F. is simply the smaller number.

For example, the H.C.F. of 24 and 72 is 24.



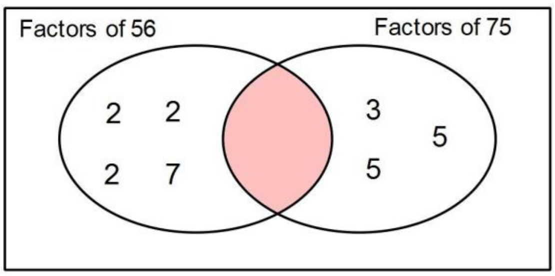
Two numbers are said to be mutually prime is they have no common factors between them, i.e. an H.C.F. of 1.

Example (8). Show that 56 and 75 are mutually prime.



56 = $2 \times 2 \times 2 \times 7$ or $2^3 \times 7$, and 75 = $3 \times 5 \times 5$ or 3×5^2.

56 has powers of 2 and 7 as its factors but 75 has no occurrences of either; 75 has powers of 3 and 5 as its factors but this time 56 has neither. Therefore 56 and 75 have no common factors and are mutually prime.



The H.C.F. of more than two numbers is found in a similar way.

Harder Example (9). Find the H.C.F. of 252, 336 and 420, given that:

$252 = 2^2 \times 3^2 \times 7$
$336 = 2^4 \times 3 \times 7$
$420 = 2^2 \times 3 \times 5 \times 7$

The factor of 2 occurs as 2^2 in 252 and 420,
and as 2^4 in 336.

The lowest power of 2 is its square, so we have 2^2 in the H.C.F.

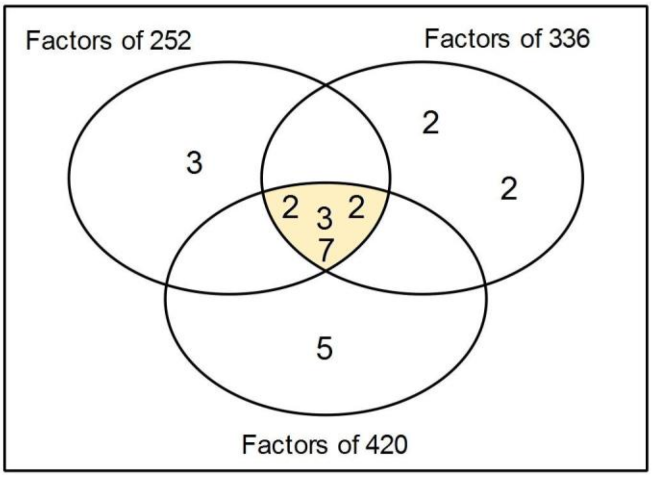
The factor of 3 occurs as 3^2 in 252 , and as
3^1 , or simply 3, in 336 and 420.

The lowest power of 3 is 3 itself, so we have 3 in the H.C.F.

The factor of 5 occurs only in 420, and is therefore absent in the H.C.F.

The factor of 7 occurs as 7^1 , or simply 7, in all three numbers, and therefore it occurs as 7 in the H.C.F.
.

The H.C.F. of 252, 336 and 420 is therefore $2^2 \times 3 \times 7$, or 84.



The Lowest Common Multiple, or L.C.M.

The lowest common multiple of two or more numbers is the smallest number into which all of them can be divided without a remainder.

Example (10): Karen is planning a barbecue and wants to buy equal numbers of burgers and buns. Burgers are sold in packs of 12; buns, in packs of 10. What is the smallest total number of burgers and buns she can buy without having any of either left over, and how many packs of each is that ?

Karen thinks: "Well, $12 \times 10 = 120$, and so is 10×12 . This means that if I buy 10 packs of burgers and 12 packs of buns, then I'll have enough to make 120."

In other words, 120 is a common multiple of both 10 and 12.

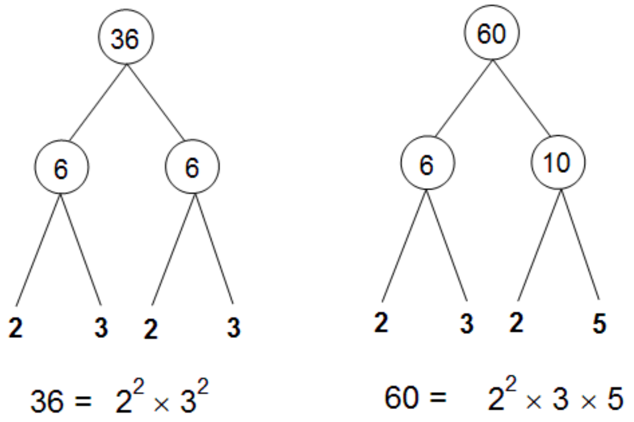
She then realises that she might not need 120 burgers and buns after all, since she reckons that 60 could also be divided by 10 and 12. In other words, she buys 5 packs of 12 burgers and 6 packs of 10 buns, making 60 burgers and 60 buns in all.

The **lowest common multiple, or L.C.M.,** of 10 and 12 is therefore 60.

If the L.C.M. cannot be found easily, then separate each number into prime factors.

By contrast with the H.C.F, the L.C.M. must have the **highest** power of each prime factor to be found in each number. If a factor is present in one number but absent in the other, then it must be included in the L.C.M.

Example (11). Find the L.C.M. of 36 and 60.

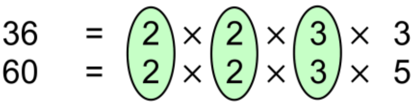


$36 = 2 \times 2 \times 3 \times 3$ or $2^2 \times 3^2$, and $60 = 2 \times 2 \times 3 \times 5$ or $2^2 \times 3 \times 5$.

Both 36 and 60 have 2^2 as a factor, therefore it must be in the L.C.M.
Again, both numbers have 3 as a factor, but 36 has the higher power (3^2), so that must be in the L.C.M.
The factor of 5 is absent in 36, but since it is included in 60, it has to be in the L.C.M. as well.

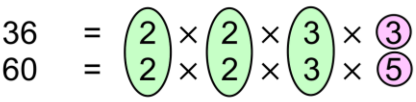
The L.C.M. of 36 and 60 is therefore $2^2 \times 3^2 \times 5$, or 180.

This result can also be shown by expressing 36 and 60 as prime factor products in "long" non-index form and firstly highlighting the factors common to both (thus finding the H.C.F. into the bargain):



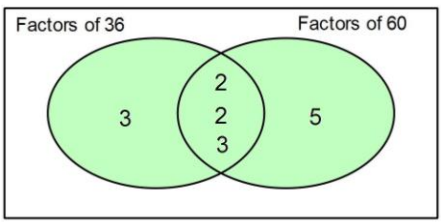
The H.C.F. of 36 and 60 is $36 = 2 \times 2 \times 3$ or 12, and we will use this result later.

We are still left with unmatched factors of 3 and 5 though.



We need to multiply the intermediate H.C.F. of 12 by these unmatched factors of 3 and 5.
The L.C.M. of 36 and 60 is therefore $12 \times 3 \times 5$, or 180.

Venn diagram illustration :

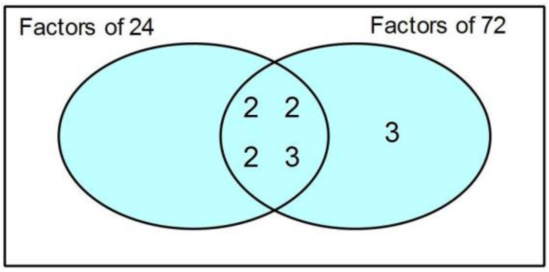


We multiply together every factor inside the Venn circles, thus $2 \times 2 \times 3 \times 3 \times 5 = 180$.

If one of the two numbers is a factor of the other, their L.C.M. is simply the larger number.

For example, the L.C.M. of 24 and 72 is 72.

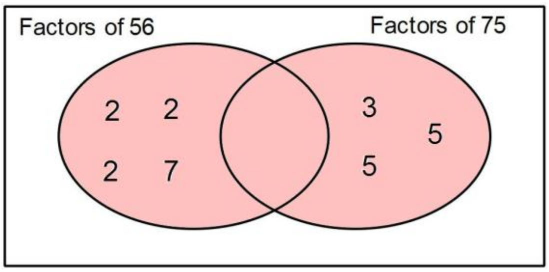
If two numbers are mutually prime (see under H.C.F.), then their L.C.M. is simply their product.



Example (12). Using the result from Example 8, find the L.C.M. of 56 and 75.

Since 56 and 75 are mutually prime (in other words, they have no common factors) , their L.C.M. is simply their product.

Hence the L.C.M. of 56 and 75 is 56×75 or 4200.

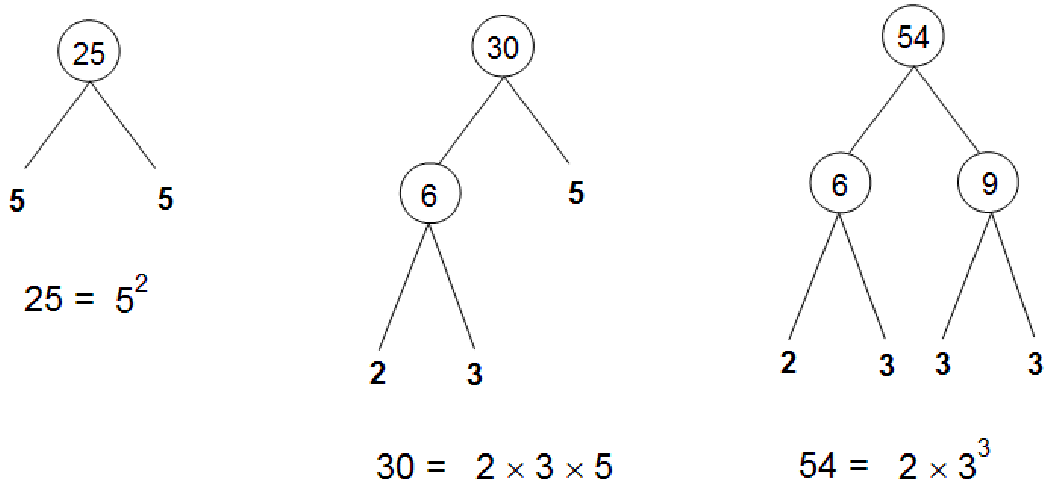


Again, the problem of finding the L.C.M. can be generalised for three or more numbers.

The list can be shortened if any number is a factor of another in the list.

Example (13). Find the L.C.M. of 15, 25, 30 and 54.

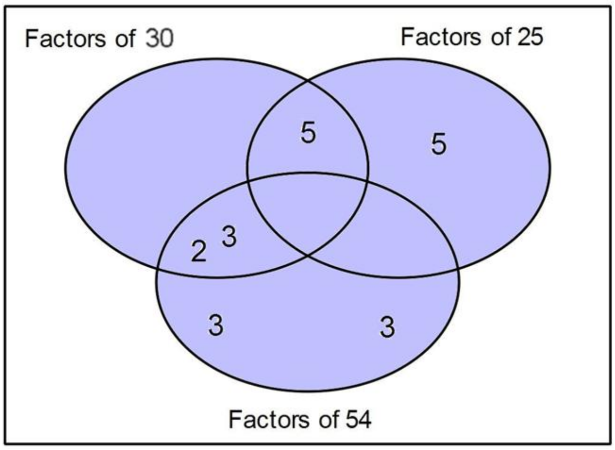
15 can be removed from the list, as it is already a factor of 30, and hence any multiple of 30 is automatically a multiple of 15.



The prime factors of the remaining numbers are:

$25 = 5 \times 5$ or $\mathbf{5^2}$
$30 = \mathbf{2} \times 3 \times 5$
$54 = \mathbf{2} \times 3 \times 3 \times 3$ or $\mathbf{2 \times 3^3}$

The L.C.M will therefore have powers of 2, 3 and 5 as its factors. The highest-occurring powers are shown in bold, so the L.C.M. is $\mathbf{2 \times 3^3 \times 5^2 = 1350}$.

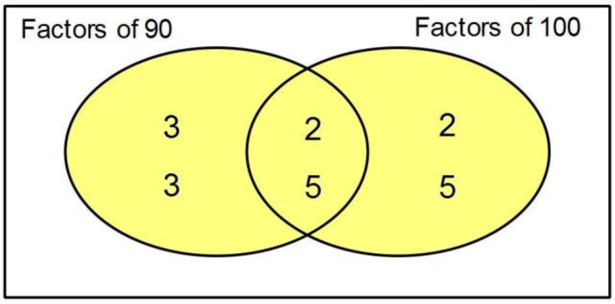


Euclid's Algorithm.

The L.C.M. of two numbers can be found by multiplying them together, and then dividing by their H.C.F.

Example (14): Find the L.C.M. of 90 and 100, given that their H.C.F. is 10 from Example (7).

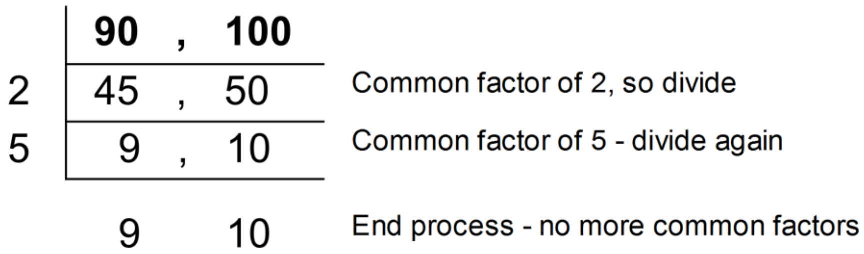
The L.C.M. of 90 and 100 = $\frac{90 \times 100}{10} = 900.$



Finding the H.C.F and L.C.M. of two numbers using the "Ladder" method.

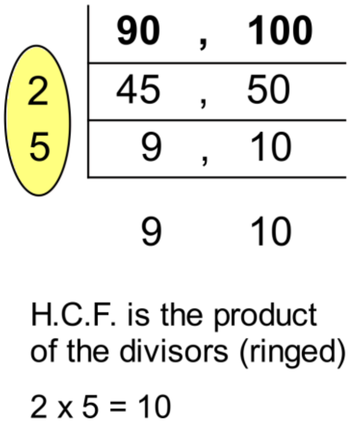
This is another method of finding the H.C.F. and L.C.M. of two numbers in one sum, with a layout recalling the process of division.

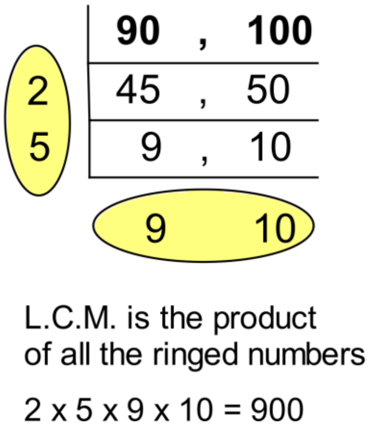
Example (15) : Find the H.C.F. and L.C.M. of 90 and 100 using the "Ladder" method.



We see that 90 and 100 have a common factor of 2, so we first divide by 2 to obtain 45 and 50. Next, we see that 45 and 50 still have a common factor of 5, so we divide by 5 to get 9 and 10.

The numbers 9 and 10 have no common factors, so the process ends here.





We were able to divide both 90 and 100 by 2 and then by 5, so the H.C.F. of 90 and 100 is 2×5 or 10, namely the product of just the two divisors.

The final quotients of 9 and 10 can then be multiplied to give 90, and that result multiplied by the H.C.F. of 10 to give the L.C.M. of 90 and 100, i.e. $2 \times 5 \times 9 \times 10$ or 900.

N.B. We could have divided 90 and 100 by 10 in one step, and still obtained the same results.

We finish with two "real-life" problems :

Example (16): Stan works in a garden centre and finds eight plant canes 1.8 metres long, and another five canes 2.4 metres long.

He wants to cut all the canes up into shorter sections of equal length without any wastage, whilst trying to keep each cut section as long as possible.

.
Work out the length of each cut piece in centimetres, and hence calculate the total number of pieces Stan could make out of the original canes.

We first convert the lengths of the canes into centimetres, and proceed to find the H.C.F. of the two values.

The prime factors of 180 and 240 are:

180 = **2** × **2** × 3 × **3** × **5** or $2^2 \times 3^2 \times 5$
240 = **2** × **2** × 2 × 2 × **3** × **5** or $2^4 \times 3 \times 5$.

The H.C.F. is thus $2^2 \times 3 \times 5 = 60$ (choosing lower powers of each factor), or by highlighting the common factors and multiplying them together.

The cut pieces of cane therefore have a length of **60 cm**.
Now, 180 = 3 × 60 and 240 = 4 × 60, so Stan can make three cut pieces out of each 1.8 m cane and four pieces out of each 2.4 m cane.

There are eight of the 1.8 m canes, so Stan can cut 8 × 3, or 24 pieces from them.
Similarly, Stan can cut 5 × 4, or 20 pieces, from the five 2.4 m canes.

He can therefore cut **44 pieces of 60 cm** from all the canes.

Example (17): Buses run from Bury Interchange to Ramsbottom every 15 minutes, and to Tottington every 18 minutes.

The two bus services leave Bury together at 10:00.
i) When do the departure times from Bury coincide again ?
ii) State the other times that the buses leave simultaneously, up to 17:00.

i) We could write a list of departure times for the two buses by counting 15 and 18 minute intervals:

To Ramsbottom: 10:00, 10:15, 10:30, 10:45, 11:00, 11:15, **11:30**, 11:45 ....
To Tottington: 10:00, 10:18, 10:36, 10:54, 11:12, **11:30**, 11:48 ....

∴ The next time buses to Ramsbottom and Tottington depart from Bury at the same time is **11:30**.

Alternatively, we could have found the L.C.M. of 15 and 18:

Now, $15 = 3 \times 5$ and $18 = 2 \times 3^2$, so the L.C.M. of the two numbers is $2 \times 3^2 \times 5 =$ **90**.

∴ The next time the two buses leave together is 90 minutes, or 1 hour 30 minutes, after 10:00, namely at **11:30**.

ii) Since the simultaneous departures occur every 90 minutes, or 1 hour 30 minutes, the Ramsbottom and Tottington departures will leave Bury together again at **13:00**, then at **14:30**, and finally at **16:00**.