M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SURDS – IRRATIONAL NUMBERS

$$\sqrt{2} = 1.41421356237309504880\ ....$$

$$\frac{2}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{2-2\sqrt{5}}{1-5} = \frac{\sqrt{5}-1}{2}$$

$$\left(3-\sqrt{5}\right)^2 = 9 - 6\sqrt{5} + 5 = 14 - 6\sqrt{5}$$

$$\sqrt{75} - \sqrt{12} = \sqrt{25}\sqrt{3} - \sqrt{4}\sqrt{3} = 5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$$

$$\sqrt{2} + \sqrt{32} = \sqrt{2} + \sqrt{16}\sqrt{2} = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$$

$$\sqrt{9+16} = \sqrt{25} = 5$$
$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

$$\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right) = 9 - 2 = 7$$

$$\pi = 3.14159265358979323846\ ......$$

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SURDS

A **surd** is a mathematical expression that includes irrational roots in it. (We are only concerned with square roots at GCSE). If a question asks for an exact answer involving square roots, then it must be left in surd form.

The following rules apply to surds .

- **The square root of one number multiplied by the square root of another number is equal to the square root of their product.**

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Examples (1) : Express the following as the square root of a single number:

i) $\sqrt{2} \times \sqrt{3}$; ii) $\sqrt{2} \times \sqrt{8}$

i) Since $2 \times 3 = 6$, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$.

ii) Similarly $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$. Here, two surds produce an integer when multiplied

- **The square root of one number multiplied by another positive number is equal to the square root of the product of the first number and the *square* of the second one.**

$$\sqrt{a} \times b = \sqrt{ab^2}$$

Example (2a) : Express $\sqrt{2} \times 3$ as the square root of a single number:

$$\sqrt{2} \times 3 = \sqrt{2 \times 3^2} = \sqrt{18}.$$

We must square the 3 before enclosing it in the square root sign; $\sqrt{2} \times 3$ is definitely not $\sqrt{6}$.

We could alternatively re-express 3 as $\sqrt{9}$ and treat the sum as $\sqrt{2} \times \sqrt{9} = \sqrt{18}$.

Examples (2b) : Express the following in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible: i) $\sqrt{48}$; ii) $\sqrt{50}$; iii) $2\sqrt{45}$

This time, it is a matter of finding the largest perfect square factor of the number inside the square root sign.

i) Of the factors of 48, the largest perfect square is 16, so we can re-express as follows :

$$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$$

ii) The largest square factor of 50 is 25, so $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$.

iii) The expression $2\sqrt{45}$ can be simplified further because 45 has a factor of 9:

$$2\sqrt{45} = 2\sqrt{9 \times 5} = 2\sqrt{9}\sqrt{5} = 6\sqrt{5}.$$

- **The square root of one number divided by the square root of another number is equal to the square root of their quotient .**

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Examples (3): Express the following as the square root of a single number:

i) $\dfrac{\sqrt{15}}{\sqrt{5}}$; ii) $\dfrac{\sqrt{27}}{\sqrt{3}}$

i) We replace the two separate square root signs with a single one around the whole fraction:

$\dfrac{\sqrt{15}}{\sqrt{5}} = \sqrt{\dfrac{15}{5}} = \sqrt{3}$;

ii) We do the same here: $\dfrac{\sqrt{27}}{\sqrt{3}} = \sqrt{\dfrac{27}{3}} = \sqrt{9} = 3$. Again the result comes out as an integer.

- **The square root of one number divided by another positive number is equal to the square root of the quotient of the first number and the *square* of the second one. This also holds when the fraction is inverted with a positive numerator.**

$$\frac{\sqrt{a}}{b} = \sqrt{\frac{a}{b^2}} \; ; \quad \frac{a}{\sqrt{b}} = \sqrt{\frac{a^2}{b}}$$

Example (4): Express $\dfrac{\sqrt{20}}{2}$ as the square root of a single number.

$\dfrac{\sqrt{20}}{2} = \sqrt{\dfrac{20}{4}} = \sqrt{5}$

Here, the 2 must first be squared before enclosing it in the square root sign;

$\dfrac{\sqrt{20}}{2}$ is definitely not $\sqrt{10}$. (This case often crops up when solving quadratic equations).

We could alternatively re-express 2 as $\sqrt{4}$ and treat the sum as $\dfrac{\sqrt{20}}{2} = \dfrac{\sqrt{20}}{\sqrt{4}} = \sqrt{\dfrac{20}{4}} = \sqrt{5}$.

Again, we could "spot the square factor" as in Example (2b); $\dfrac{\sqrt{20}}{2} = \dfrac{\sqrt{4 \times 5}}{2} = \dfrac{2\sqrt{5}}{2} = \sqrt{5}$.
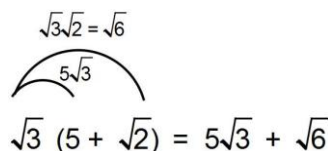
Expansion **of surd expressions.**

Surd expressions can be expanded like other algebraic expressions.
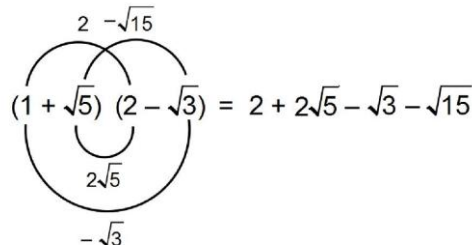
Examples (5): Expand and simplify the following :

i) $\sqrt{3}(5+\sqrt{2})$; ii) $(1+\sqrt{5})(2-\sqrt{3})$; iii) $(1+\sqrt{2})(3+\sqrt{8})$; iv) $(5+\sqrt{2})^2$; v) $(4-\sqrt{7})^2$

i) $\sqrt{3}(5+\sqrt{2})=5\sqrt{3}+\sqrt{6}$.

$$\sqrt{3}\sqrt{2}=\sqrt{6}$$
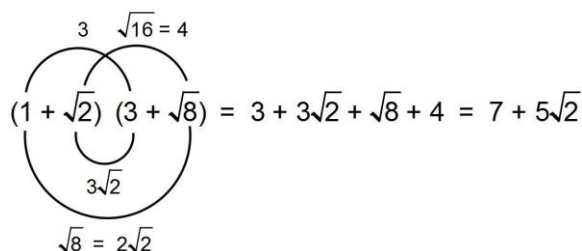$$5\sqrt{3}$$
$$\sqrt{3}\ (5+\ \sqrt{2})\ =\ 5\sqrt{3}\ +\ \sqrt{6}$$

ii) $(1+\sqrt{5})(2-\sqrt{3})=2+2\sqrt{5}-\sqrt{3}-\sqrt{15}$.

Notice that the sum of the two middle terms could not be simplified by collecting.

$$2\quad-\sqrt{15}$$
$$(1+\sqrt{5})\ (2-\sqrt{3})\ =\ 2+2\sqrt{5}-\sqrt{3}-\sqrt{15}$$
$$2\sqrt{5}$$
$$-\sqrt{3}$$

iii) $(1+\sqrt{2})(3+\sqrt{8})=7+5\sqrt{2}$.

This time, it is possible to simplify the final result by various manipulations, as shown in the diagram.

$$3\quad\sqrt{16}=4$$
$$(1+\sqrt{2})\ (3+\sqrt{8})\ =\ 3+3\sqrt{2}+\sqrt{8}+4\ =\ 7+5\sqrt{2}$$
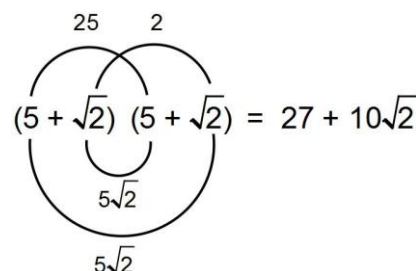$$3\sqrt{2}$$
$$\sqrt{8}\ =\ 2\sqrt{2}$$

iv) $(5+\sqrt{2})^2=27+10\sqrt{2}$.

This perfect square working is analogous to the quadratic result $(a+b)^2 = a^2 + 2ab + b^2$.
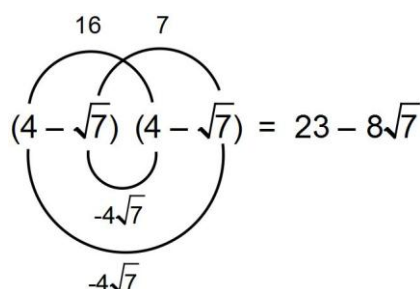Substituting $\sqrt{b}$ for b,
$$\left(a+\sqrt{b}\right)^2 = a^2 + 2a\sqrt{b} + b$$

$$25\quad2$$
$$(5+\sqrt{2})\ (5+\sqrt{2})\ =\ 27+10\sqrt{2}$$
$$5\sqrt{2}$$
$$5\sqrt{2}$$

v) $(4-\sqrt{7})^2=23-8\sqrt{7}$.

This is related to the quadratic result $(a-b)^2 = a^2 - 2ab + b^2$.
Substituting $\sqrt{b}$ for b,

$$16\quad7$$
$$(4-\sqrt{7})\ (4-\sqrt{7})\ =\ 23-8\sqrt{7}$$
$$-4\sqrt{7}$$
$$-4\sqrt{7}$$

$$\left(a - \sqrt{b}\right)^2 = a^2 - 2a\sqrt{b} + b$$

Example (6): Expand and simplify the expression $(3 + \sqrt{5})(3 - \sqrt{5})$.

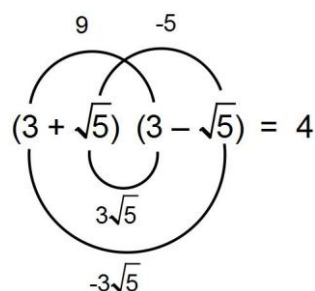$$(3 + \sqrt{5})(3 - \sqrt{5}) = 4$$

This surprisingly simple result is connected to the quadratic "difference of two squares" result

$(a+b)(a-b) = a^2 - b^2$

Substituting $\sqrt{b}$ for b,

$$\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$$

Notice how this manipulation eliminates the surd from the expression, by using the difference of squares. This is of particular importance in Examples (8) below.

It is often desirable to manipulate a fractional surd expression so as to make the denominator a rational number. This is termed **rationalising the denominator**.

Examples (7): Rationalise the denominator in the expressions:

i) $\dfrac{1}{\sqrt{2}}$; ii) $\dfrac{2 + \sqrt{2}}{3\sqrt{3}}$

i) Multiplying the top and bottom by $\sqrt{2}$ gives the result $\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

ii) Here we multiply the top and bottom by $\sqrt{3}$: $\dfrac{2 + \sqrt{2}}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3} + \sqrt{6}}{9}$.

The previous examples showed how to rationalise a simple surd denominator.

When the denominator is of the form $a \pm \sqrt{b}$, we can use the "difference of two squares" method $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$ to rationalise it.

Examples (8): Rationalise the denominator in the expressions: i) $\dfrac{1}{2 + \sqrt{3}}$; ii) $\dfrac{3 - \sqrt{7}}{4 - \sqrt{7}}$

i) The denominator is $2 + \sqrt{3}$, so we rationalise it by multiplying top and bottom by $2 - \sqrt{3}$.

(Recall Example (6)).

$$\frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{2 - \sqrt{3}}{4 - 3} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3}.$$

ii) We need to multiply top and bottom by $4 + \sqrt{7}$ to rationalise the denominator

$$\frac{3 - \sqrt{7}}{4 - \sqrt{7}} \times \frac{4 + \sqrt{7}}{4 + \sqrt{7}} = \frac{(3 - \sqrt{7})(4 + \sqrt{7})}{9} = \frac{12 - 4\sqrt{7} + 3\sqrt{7} - 7}{9} = \frac{5 - \sqrt{7}}{9}$$

Other surd arithmetic.

Sums and differences of surds **cannot** be simplified like products or quotients:

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$, as the counterexample below will show.

$\sqrt{9+16} = \sqrt{25} = 5$; $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

It can still be possible to simplify certain expressions by using the surd laws, however.

The trick here (see Examples (2)) is to look for the largest perfect square factor in the surd expressions, so as to factor out the resulting square root.

Examples (9): Express the following in the form $a + b\sqrt{c}$ where a, b and c are integers

i) $\sqrt{2} + \sqrt{32}$; ii) $\sqrt{75} - \sqrt{12}$

i) $\sqrt{2} + \sqrt{32} = \sqrt{2} + \sqrt{16}\sqrt{2} = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$.

Because 16 is the largest square factor of 32,, we can say $\sqrt{32} = \sqrt{16}\sqrt{2}$ and hence $\sqrt{32} = 4\sqrt{2}$.
We finish by adding the two multiples of √2 using normal algebra.
(Such a case had already cropped up in Example 5(iii)).

ii) $\sqrt{75} - \sqrt{12} = \sqrt{25}\sqrt{3} - \sqrt{4}\sqrt{3} = 5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$

Again, we spot that 25 and 4 are the highest square factors of 75 and 12, respectively. We finish by subtracting multiples of √3.

More on Rational and Irrational Numbers.

A **rational** number is any number which can be written as a fraction whose top and bottom lines are both integers. An integer is a special case of a fraction whose bottom line is 1.

When expressed as a decimal, a rational number may either terminate or go on for ever (recur).
Examples of rational numbers are 4, -2.6, $\frac{3}{4}$ and $\frac{22}{7}$.
Rational numbers therefore include all integers, fractions, terminating decimals and recurring decimals.

An **irrational** number also goes on for ever without giving an exact value, but there is no predictable pattern, and the number cannot be expressed as a fraction.
Examples of irrational numbers are π (the ratio of a circle's circumference to its diameter) and the square roots of most positive integers, such as $\sqrt{2}$.

Adding a rational number to an irrational number, or multiplying an irrational number by a rational number will still result in an irrational one.

The fraction $\frac{22}{7}$ is often used as an approximation for π, and its value is 3.142857142857....
Its decimal value does not terminate, but it is rational because it can be expressed as a fraction.

The decimal value of π begins 3.141592653589... but there is no repeating pattern and so π is irrational.

Example (10): Give two rational numbers and two irrational numbers between 4 and 5.

Two rational numbers are 4.3 and $\frac{9}{2}$.
Two irrational numbers are $\pi + 1$ and $\sqrt{20}$.

The square of 4 is 16 and the square of 5 is 25, therefore the square roots of all the integers in between are irrational. In fact, square roots of non-square integers are a good source of irrational numbers.

Example (11): Give a rational number and an irrational number between π and $\frac{22}{7}$.

Since π = 3.141592... and $\frac{22}{7}$ = 3.142857... , one rational number between the two is 3.142.
Adding a rational number to an irrational one still leaves it irrational, so we can add 0.001 to π and get 3.142592....which is between π and $\frac{22}{7}$.

$\therefore$ 3.142 is a rational number between π and $\frac{22}{7}$, whilst $\pi + 0.001$ is an irrational one.

Example (12): Give two distinct irrational numbers whose product is a rational number.

$\sqrt{2}$ is irrational, but it can be multiplied by $\sqrt{8}$ to give $\sqrt{16}$, which is the rational number 4.

Example (13): Give two distinct irrational numbers whose sum is a rational number.

π is irrational, and so is 4-π , but adding them gives $\pi + 4 - \pi$ which is 4 – a rational number.