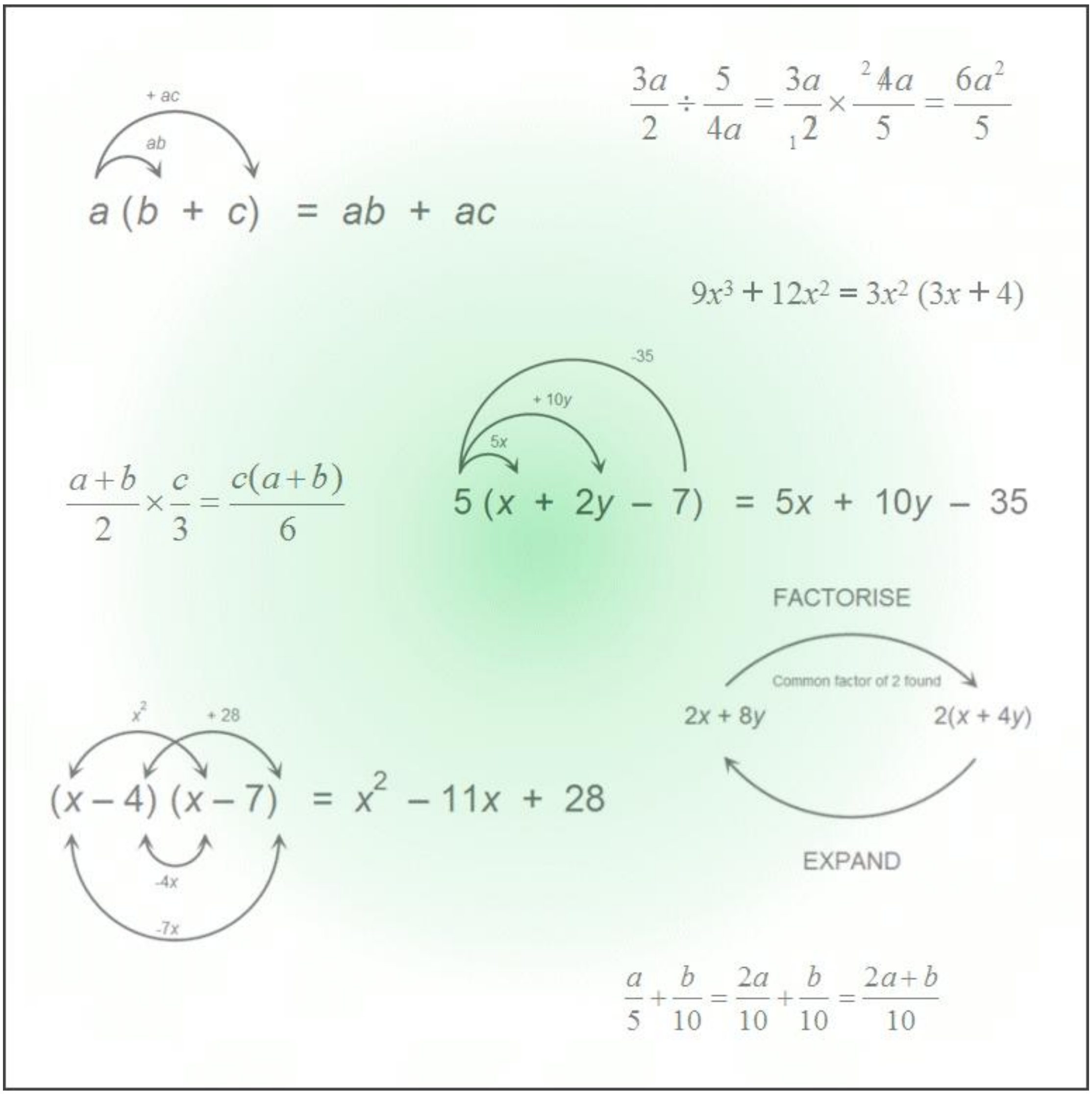
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

ALGEBRAIC MANIPULATION



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BASICS OF ALGEBRA

First, a few definitions:

A **term** is an arrangement of algebraic symbols, which could include numbers, letters or both.
The symbols in a term can be products or quotients.

Letters are used to represent unknown quantities.
Examples of terms are $2x$, $4xy$, $-2x^2$, y^3 and $\frac{x}{y}$.

It is usual not to include the × sign to indicate products, so we write $2x$ rather than $2 \times x$.
Neither do we put 1 in front of a term, so we say x and not $1x$, and $-x$ instead of $-1x$.

Also, products are always written with numbers (if any exist), followed by letters, preferably in alphabetical order.

Thus, we say $2ab$ and never $a2b$; also $3xy$ is preferable to $3yx$.

Quotients are also usually shown as fractions rather than by using the ÷ sign,
so we write $\frac{x}{y}$ in preference to $x \div y$.

An **expression** is any arrangement of terms, connected by + and – signs.
The sign refers to the term immediately after it, and if the first term is not signed, it is taken to be positive, i.e. have an invisible + sign at the start. .

Therefore $2x^2 - 6x$ means the sum of the terms '$+2x^2$ ' and '$-6x$'.

Expressions and quantities in algebra are handled as they are in arithmetic:

$3A + A = 4A$; (remember $A = 1A$); $7x \times 2x = 14x^2$ (recall index laws); $y^3 \div y^2 = y$ (y is the same as y^1).

Collecting like terms

When algebraic terms are all multiples of the **same power of a variable** (or variables) , they can be simplified by collecting like terms. Numbers (constants) can similarly be collected.

Examples (1):

$3a + 5 + 4a - 1$	can be simplified to $7a + 4$
$6a - b$	cannot be simplified as there are no like terms. (a and b are different variables)
$x^2 + 3x$	cannot be simplified by collecting, as the powers of x are different.
$x^2 - 5xy + y^2$	cannot be simplified by collecting, as the powers of x and y are different.
$3n - 2n - 4n$	can be simplified to $-3n$.
$5a + 4b + 3a - 6b$	can be simplified to $8a - 2b$ (Add the a's first, then the b's, noting that the – sign goes with the $6b$)
$3yx + xy$	can be simplified to $4xy$, since xy and yx are the same.
$5n^2 + 3n - 2n + n^3 - 2n^2$	can be simplified to $n^3 + 3n^2 + n$, by collecting all the like powers of n. It is also usual to write the result in descending order of powers of n.

Algebraic expansion ("multiplying brackets out").

Multiplying letters, numbers and brackets.
Expressions are often simplified by multiplying them together. Thus:

Examples (2):

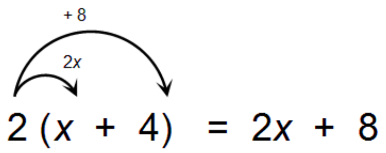
$3a \times 4b$	$=$	$3 \times 4 \times a \times b$	$=$	$12ab$
$5a \times 3b \times 2c$	$=$	$(5 \times 3 \times 2) \times (a \times b \times c)$	$=$	$30abc$

Other examples are to be found in the document "Laws of Indices".

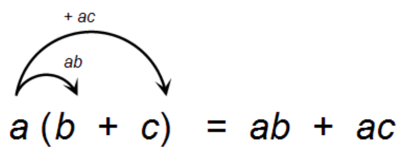
Multiplying out single brackets.
Everything inside the bracket must be multiplied by everything outside it, a process called **expanding.**

Examples (3): Expand the following: i) $2(x+4)$; ii) $a(b+c)$; iii) $x(x-2)$; iv) $5(x + 2y – 7)$

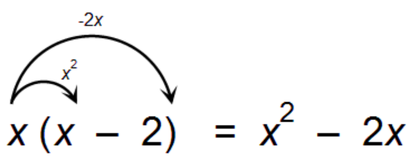
i) $2(x+4) = 2x + 8$



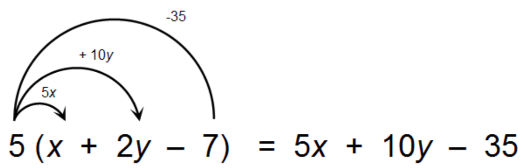
ii) $a(b+c) = ab + ac$



iii) $x(x-2) = x^2 – 2x$



iv) $5(x + 2y – 7) = 5x + 10y - 35$



If the term *outside* the bracket is negative, all the signs of the terms *inside* the brackets must be reversed when expanding.

Examples (4): Expand the following: i) $-3(4p + 5q)$; ii) $-3(1 - 2a)$; iii) $-2x(3x - 1)$

i) $-3(4p + 5q) = -12p - 15q$

$$-3(4p + 5q) = -12p - 15q$$

ii) $-3(1 - 2a) = -3 + 6a = 6a - 3$

$$-3\,(1 - 2a) = -3 + 6a = 6a - 3$$

If an expression contains a mixture of positive and negative terms, it is more usual to express it with positive term first, i.e. $1 - 2a$ rather than $-2a + 1$.

iii) $-2x(3x - 1) = -6x^2 + 2x$

$$-2x(3x - 1) = -6x^2 + 2x = 2x - 6x^2$$

Examples (5) : Evaluate: i) $2(a + 3) + 3(a + 1)$; ii) $5(a + b) - 2(a + 2b)$, simplifying the results.

i) $2(a + 3) + 3(a + 1) = 2a + 6 + 3a + 3 = 5a + 9$.

ii) $5(a + b) - 2(a + 2b) = 5a + 5b - 2a - 4b = 3a + b$.

Multiplying out double brackets.

Each term in the first bracket is multiplied by each term in the second bracket. All the examples here involve expressions of two terms being multiplied together.

Examples (6): Expand and simplify; i) $(x+3)(x+5)$; ii) $(x-4)(x-7)$; iii) $(x+4)^2$; iv) $(x+5)(x-5)$

i) $(x+3)(x+5) = x^2 + 8x + 15$.
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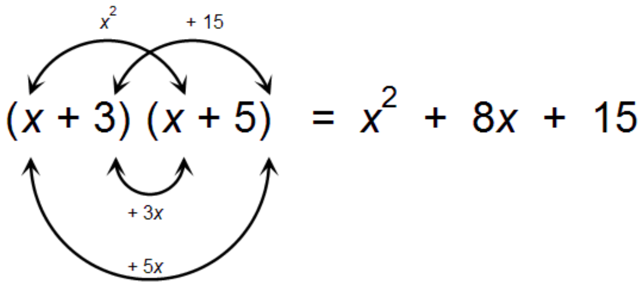
There are two terms in x in the expansion, $3x$ and $5x$. They can be collected to give $8x$.

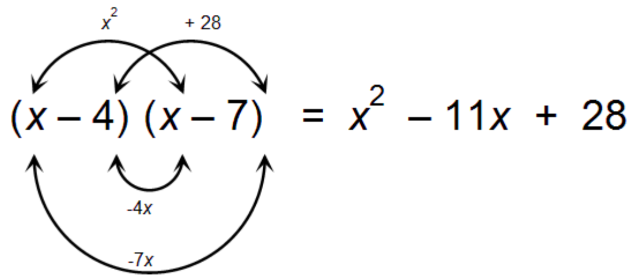


Alternatively, we could work in one line as follows :

$(x+3)(x+5) = x(x + 5) + 3(x + 5)$

$= x^2 + 5x + 3x + 15$ (expand)
$= x^2 + 8x + 15$ (collect)

ii) $(x-4)(x-7) = x^2 - 11x + 28$.

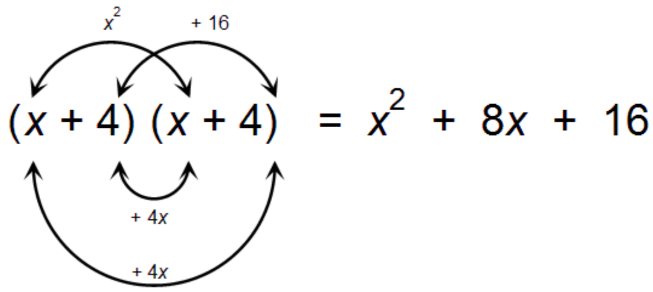


Working without diagram:

$(x-4)(x-7) = x(x - 7) - 4(x - 7)$
$= x^2 - 7x - 4x + 28$ (expand)
$= x^2 -11x + 28$ (collect)

(Watch the minus signs !)

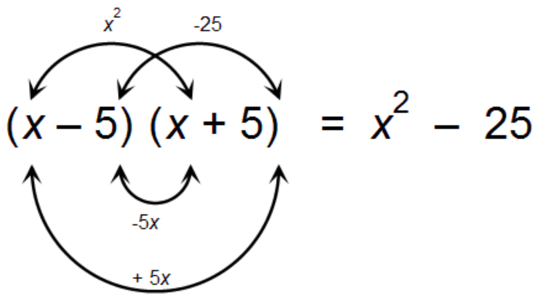
iii) $(x+4)^2 = (x+4)(x+4) = x^2 + 8x + 16$.
We have had to rewrite this squared expression.



Working without diagram:

$(x+4)(x+4) = x(x + 4) + 4(x + 4)$
$= x^2 + 4x + 4x + 16$ (expand)
$= x^2 + 8x + 16$ (collect)

iv) $(x+5)(x-5) = x^2 - 25$.
This time, the terms in x cancel out to zero.



Working without diagram:

$(x+5)(x-5) = x(x - 5) + 5(x - 5)$
$= x^2 -5x + 5x -25$ (expand)
$= x^2 -25$ (collect)

Multiplying out triple brackets. (From 2017).

This is an extension to multiplying out double brackets, and is best performed in linear form.

Examples (6): Expand and simplify; i) $(x+2)(x+3)(x+5)$; ii) $(x-6)(x-4)(x-7)$;
iii) $(x-4)(2x+3)(5x+6)$

i) We begin with $(x+3)(x+5) = x(x + 5) + 3(x + 5)$

$= x^2 + 5x + 3x + 15$ (expand)
$= x^2 + 8x + 15$ (collect)

Next, we multiply this product by $(x+2)$:

$(x+2)(x+3)(x+5)$

$= x(x^2 + 8x + 15) + 2(x^2 + 8x + 15)$

$= x^3 + 8x^2 + 15x + 2x^2 + 16x + 30$ (expand)
$= x^3 + 10x^2 + 31x + 30$ (collect)

$\therefore$ $\mathbf{(x+2)(x+3)(x+5) = x^3 + 10x^2 + 31x + 30}$.

ii) Start with $(x-4)(x-7) = x(x - 7) - 4(x - 7)$

$= x^2 - 7x - 4x + 28$ (expand)
$= x^2 -11x + 28$ (collect)

Then we multiply by $(x–6)$:

$(x-6)(x-4)(x-7)$

$= x(x^2 -11x + 28) – 6(x^2 -11x + 28)$
$= x^3 -11x^2 + 28x - 6x^2 + 66x – 168$ (Be careful with the minus signs !)
$= x^3 - 17x^2 + 94x – 168$

$\therefore$ $\mathbf{(x-6)(x-4)(x-7) = x^3 - 17x^2 + 94x – 168}$.

iii) Firstly, $(x-4)(2x+3) = x(2x+3) – 4(2x+3)$

$= 2x^2 + 3x - 8x - 12$ (expand)
$= 2x^2 -5x - 12$ (collect)

Multiplying by $(5x+6)$ we then have

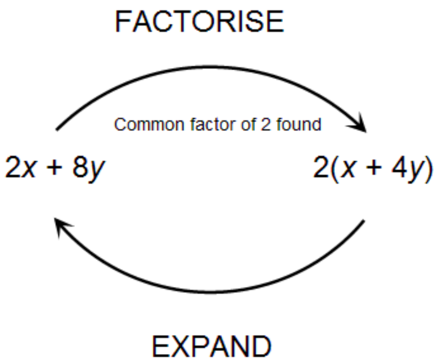
$(x-4)(2x+3)(5x+6)$

$= 5x (2x^2 -5x - 12) + 6(2x^2 -5x - 12)$
$= 10x^3 - 25x^2 - 60x + 12x^2 - 30x – 72$
$= 10x^3 - 13x^2 - 90x – 72$

$\therefore$ $\mathbf{(x-4)(2x+3)(5x-6) = 10x^3 - 13x^2 - 90x – 72}$.

Factorisation (putting brackets in).

This is the reverse process of expanding brackets. An expression is put into brackets by taking out common factors.



Thus to factorise $2x + 8y$, we need to a) recognise that 2 is a factor of each term, b) take 2 out as a common factor and c) complete the expression within the bracket by dividing each term by 2, so that the result is equivalent to $2x + 8y$ when multiplied out.

Examples (8): Factorise: i) $8x – 16$; ii) $3x + 18$; iii) $5x^2 + x$; iv) $6x^2 + 4x$; v) $9x^3 + 12x^2$

i) $8x – 16 = 8(x – 2)$. Checking the numbers first, we find that 8 and 16 have a common factor of 8. Since only one of the terms has an x in it, we cannot take x out.

ii) $3x + 18 = 3(x + 6)$. The common factor is 3.

iii) $5x^2 + x = x(5x + 1)$. This time, there are no common factors among the numbers, but x occurs in both – as its square in $5x^2$ and by itself in x. Thus x is a common factor and can be taken out, whilst all the powers of x inside the bracket are reduced by 1.

iv) $6x^2 + 4x = 2x(3x + 2)$. Here we have a common factor of 2 among the numbers, and of x among the powers of x. We therefore take out a common factor of $2x$.

v) $9x^3 + 12x^2 = 3x^2 (3x + 4)$. The common factor among the numbers is 3, and the powers of x in each term are 3 and 2 respectively. The lowest power of x is its square, so we also take out x^2 . The common factor to go outside the bracket is therefore $3x^2$.

What we are effectively doing here is finding the H.C.F. of all the terms in question.

To find this H.C.F, we first find the H.C.F. of the number parts of each term.
Then we go through the letters, and check out the lowest power of any particular letter. If any letter is absent in some terms but present in others, then that letter will be absent from the H.C.F.

When we factorised $6x^2 + 4x$, we first checked the number parts, and noticed that 4 and 6 had an H.C.F. of 2, so that 2 was taken out of the brackets.

Then, we checked out the letter x, finding x^2 and x there. The lowest power of x was x itself, and so that went outside the brackets.

Hence $2x$ is the H.C.F. of $6x^2$ and $4x$, and the expression can be rewritten $2x(3x + 2)$ after dividing each term by $2x$.

Example (9): Factorise $20a^2b^2c+ 15a^3b^3 – 10a^2b$.

Checking the numbers: the H.C.F. of 20, 15 and 10 is 5.
Checking a: the powers found are 2, 3 and 2, of which the lowest is 2, or a^2 , so a^2 is in the H.C.F.
Checking b: the powers found are 2, 3 and 1, of which the lowest is 1, or b, so b is in the H.C.F.
Checking c: this letter is found in the first term only, so cannot occur in the H.C.F.

The H.C.F. of all the terms is therefore the product of 5, a^2 and b, or $5a^2b$.
Dividing each term by $5a^2b$ and bracketing gives a final result of $5a^2b(4bc + 3ab^2 + 2)$.

We can always check the result of a factorisation by expanding out the brackets again.

Algebraic fractions.

Algebraic fractions are handled in the same way as fractions in regular arithmetic.

When adding and subtracting, we need to find the L.C.M. of the denominators (bottom lines) so as to create equivalent fractions. Letters are handled as numbers, so that the L.C.M. of $3x$ and x^2 is $3x^2$.

Other examples: the L.C.M. of $4a^2b^2$ and $6ab^3$ is $12a^2b^3$.

We begin with the numbers: the L.C.M. of 4 and 6 is 12. With the letters, the L.C.M. is simply the highest power of each letter, so the L.C.M. of a^2 and a is a^2, whilst the L.C.M. of b^2 and b^3 is b^3.

Examples (10): Evaluate i) $\frac{a}{5}+\frac{b}{10}$; ii) $\frac{2a}{5}-\frac{4}{a}$; iii) $\frac{2}{3x}+\frac{5}{y}$; iv) $\frac{5}{a^2}-\frac{3}{a}$

i) The L.C.M. of the denominators is 10, so we need to convert $\frac{a}{5}$ to the equivalent form $\frac{a}{5}\times\frac{2}{2}=\frac{2a}{10}$.

Hence $\frac{a}{5}+\frac{b}{10}=\frac{2a}{10}+\frac{b}{10}=\frac{2a+b}{10}$.

ii) The L.C.M. here is $5a$, so we say $\frac{2a}{5}\times\frac{a}{a}=\frac{2a^2}{5a}$ and $\frac{4}{a}\times\frac{5}{5}=\frac{20}{5a}$.

Thus $\frac{2a}{5}-\frac{4}{a}=\frac{2a^2}{5a}-\frac{20}{5a}=\frac{2a^2-20}{5a}$.

(With practice, the conversion into equivalent forms can be done mentally.)

iii) This time the L.C.M. is $3xy$ so we continue with $\frac{2}{3x}+\frac{5}{y}=\frac{2y}{3xy}+\frac{15x}{3xy}=\frac{15x+2y}{3xy}$.

iv) The. L.C.M. is a^2 here: $\frac{5}{a^2}-\frac{3}{a}=\frac{5}{a^2}-\frac{3a}{a^2}=\frac{5-3a}{a^2}$

Multiplication and division are equally straightforward – take out common factors from the result if necessary.

Examples (11) : Evaluate the following: i) $\frac{a+b}{2}\times\frac{c}{3}$; ii) $\frac{3a}{5b}\times\frac{2b}{9}$; iii) $\frac{3a}{2}\div\frac{5}{4a}$

i) $\frac{a+b}{2}\times\frac{c}{3}=\frac{c(a+b)}{6}$

ii) $\frac{3a}{5b}\times\frac{2b}{9}=\frac{\cancel{3}a}{5\cancel{b}}\times\frac{2\cancel{b}}{\cancel{9}_3}=\frac{2a}{15}$ (We have cancelled out both the number 3 and the letter b.)

iii) $\frac{3a}{2}\div\frac{5}{4a}=\frac{3a}{{}_1\cancel{2}}\times\frac{{}^2\cancel{4}a}{5}=\frac{6a^2}{5}$ (We have cancelled out 2.)

It must be emphasised that only **factors** can be cancelled out in this manner, *never* **terms**.

The working $\frac{(x+4)(2x-7)}{(x+4)(x-4)}=\frac{(2x-7)}{(x-4)}$ is correct, because $(x + 4)$ is a factor of both the top and bottom lines of the left-hand expression.

By contrast, the working $\frac{(5x+2y+4)}{(2y+7)}=\frac{5x+4}{7}$ is nonsense, because, although $2y$ is present in both the top and bottom lines of the left-hand side, it occurs as a term only, and is not a factor.

We **cannot** cancel here !

The index laws recalled.

Also see “Laws of Indices”.

Examples (12): Evaluate the following, giving answers in their simplest form :

i) $2a \times 3a^2$; ii) $\dfrac{12a^4b^5}{2a^2b}$; iii) $(4a^2)^3$; iv) $\sqrt{25a^4b^6}$; v) $(7a^3)^{-2}$

i) $2a \times 3a^2$ = $6a^3$ (Note that *numbers* are multiplied, but *powers* added !)

ii) $\dfrac{12a^4b^5}{3a^2b}$ = $4a^2b^4$ (Numbers divided, but powers subtracted)

iii) $(4a^2)^3$ = $64a^6$ (The number 4 has been cubed, but power of *a* has been tripled !)

iv) $\sqrt{25a^4b^6}$ = $5a^2b^3$ (Halve the powers of *a* and *b* , but take square root of 25)

v) $(7a^3)^{-2}$ = $7^{-2}a^{-6} = \dfrac{1}{49}a^{-6} = \dfrac{1}{49a^6}$ (Take reciprocal of square of 7)