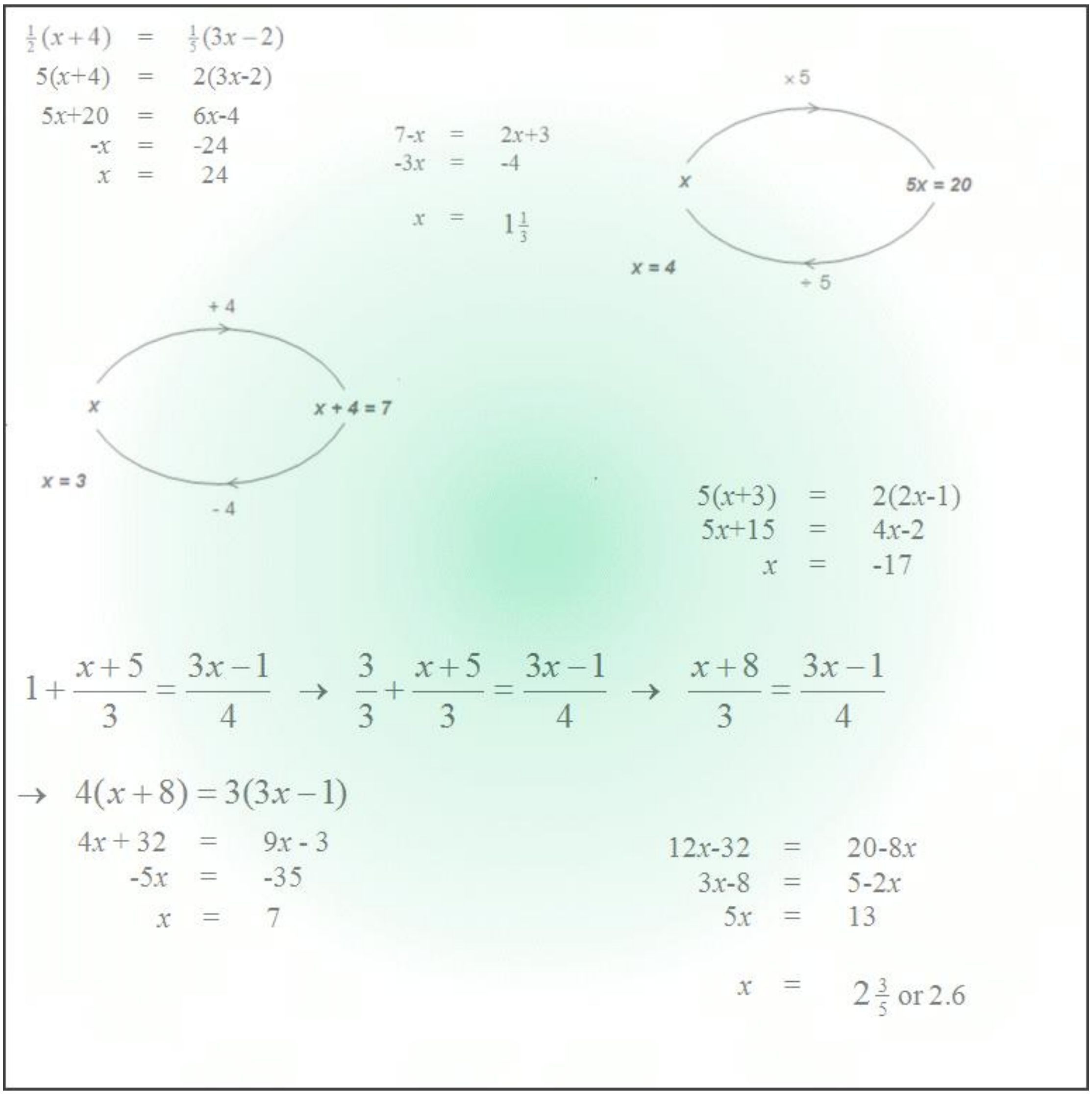
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SIMPLE EQUATIONS



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An **equation** connects two algebraic expressions, and always includes an equality sign.

A few examples of equations are: $2A + 3B = 4A – B$; $7x = 42$; $y^2 = 4ax$; $2x^2 + 5x - 3 = 0$

An **identity** is an equation which holds true regardless of what value the variables can take.

Thus the statement $2x^2 = 32$ is an equation,. as it has only two solutions for x, namely 4 and -4, and is not true when x takes any other value.

The statement $2x + 10 = 2(x+5)$ is an identity, because it holds true for **all** values of x.
We could also say that the expressions $2x + 10$ and $2(x+5)$ are equivalent, and , to emphasise the point, we have the sign ≡ ("is equivalent to") for identities.

Pedantically, we should say $2x + 10 \equiv 2(x+5)$, but most written material generally just uses the = sign, even for identities..

Setting up an equation from a mathematical statement.

Example (1).
Write the equation corresponding to the instructions
"Take a number x, double it and then add 5. Store the result in y."

The equation is $y = 2x + 5$.

Example (2).
Write the equation corresponding to the instructions
"Take a number a, square it and then subtract 7. Store the result in b."

The equation is $b = a^2 - 7$.

Example (3).
Emma has x beads, Fiona has three times as many as Emma, and Gina has four fewer beads than Fiona. All three have 80 beads between them.

Form an equation connecting the total number of beads.

If Emma has x beads, Fiona will have $3x$ beads, and Gina will have $3x – 4$ beads. The expression for the total number of beads is $x + 3x + 3x – 4$, or $7x – 4$. As the total number of beads is 80, the corresponding equation is $7x – 4 = 80$.

Linear Equations.

If an equation contains no power of a variable greater than the first, it is termed a **linear** equation. Thus the equations $2x + 5 = 11$ and $8 - y = 5$ are linear, but $x^2 - 2x + 1 = 5$ is not, because of the presence of the x^2 term.

The key to solving linear equations is to "balance" them by performing the same arithmetic on each side of the equality sign so that the variable ends up by itself on one side.

The simplest equations have the expression with the variable on the left, and a number on the right.

Take the equation $x + 3 = 12$. We can add 2 to both sides and end up with the equation $x + 5 = 14$, or we can double it and end up with $2x + 6 = 24$, but the modified equations still have the same solution as the original. Even better, we can subtract 3 from both sides and end up with the equation $x = 9$. In other words, we have solved the equation $x + 3 = 12$, giving the solution $x = 9$. .

What we had done was to change the expression $x + 3$ into x by performing the inverse operation of subtracting 3 from it.

Similarly, the expression $x - 5 = 6$ can have 5 added to both sides, leading to the solution $x = 11$. (Addition is the inverse of subtraction)

Because $5x$ is x multiplied by 5, we can divide both sides of the equation $5x = 35$ by 5, leading to the solution $x = 7$. (Division is the inverse of multiplication and vice versa)

Finally. $\frac{x}{3}$ is x divided by 3, so we multiply both sides of the equation $\frac{x}{3} = 8$ by 3, leading to the solution $x = 24$.

Examples (4): Solve the equations i) $5x = 20$; ii) $x-7 = -1$; iii) $2x - 3 = 0$; iv) $4(x - 3) = 7$.

5*x* = 20

$5x$	=	20	
x	=	4	Divide both sides by 5

***x*-7 = -1**

$x-7$	=	-1	
x	=	6	Add 7 to both sides

2*x* – 3 = 0

$2x-3$	=	0	
$2x$	=	3	Add 3 to both sides
x	=	$1\frac{1}{2}$ or 1.5	Divide both sides by 2

4(*x* – 3) = 7

$4(x-3)$	=	7	
$4x-12$	=	7	Expand brackets
$4x$	=	19	Add 12 to both sides
x	=	$4\frac{3}{4}$ or 4.75	Divide both sides by 4

Correct use of the equality sign.

The solution to equation i) in the last example could have been written in shorter form as follows:

	$5x = 20, \therefore\ x = 4$	using the sign for 'therefore'
or	$5x = 20$, hence $x = 4$	in words
or	$5x = 20 \rightarrow x = 4$	using arrow symbol as shorthand for ' if $5x = 20$, then $x = 4$'

The last statement is one of **implication**.

There is a more correct symbol for implication, namely $\Rightarrow$, but the simpler arrow $\rightarrow$ is acceptable at GCSE.

There is however no doubt about this incorrect way of writing down the solution.

$5x = 20 \quad = \quad x = 4$ **Completely unacceptable !**

We have two equations here: $5x = 20$ and $x = 4$. The second one is the first one divided by 5, and so they are not equal to each other. They have the same *solutions* for *x*, however.

In addition, the faulty 'equation' $5x = 20 = x = 4$ implies that all three of the statements "$5x = 20$", "$20 = x$" and "$x = 4$" are satisfied, which is clearly impossible.

On the other hand a statement such as $2x + 6 = 2(x+3)$ is mathematically correct, because we have *expressions* on each side of the equals sign, not *equations*.
Sometimes a linear equation might have the variable on both sides of the equality sign.

In such cases, we must bring the variable onto one side of the equation (separate the variable).

Examples (5): Solve the equations i) $9-x = 2x+3$; ii) $12x-32 = 20-8x$

$9-x = 2x+3$

The aim here is to bring the x to one side of the equation and any numbers on to the other.

$9-x$	=	$2x+3$	
$9-3x$	=	3	Subtract $2x$ from each side
$-3x$	=	-6	Subtract 9 from each side (this isolates x)
x	=	2	Divide both sides by –3

We could have avoided the division by a negative if we had brought x to the right-hand-side.

$9-x$	=	$2x+3$	
9	=	$3x+3$	Add x to each side
6	=	$3x$	Subtract 3 from each side (this isolates x)
2	=	x	Divide both sides by 3

The statements $2 = x$ and $x = 2$ are equivalent, of course.

With practice, the two subtractions can be carried out in one go:

$9-x$	=	$2x+3$	
6	=	$3x$	Add $x - 3$ to each side (separating the x-term from the number).
2	=	x	Divide both sides by 3

Adding $x - 3$ to each side of the equation separates the x-term and the number term in one step.

$12x-32 = 20-8x$

$12x-32$	=	$20-8x$	
$3x-8$	=	$5-2x$	Divide both sides by the common factor of 4
$5x-8$	=	5	Add $2x$ to each side
$5x$	=	13	Add 8 to each side
x	=	$2\frac{3}{5}$ or 2.6	Divide both sides by 5

Again, the two additions can be carried out in one go:

$12x-32$	=	$20-8x$	
$3x-8$	=	$5-2x$	Divide both sides by the common factor of 4
$5x$	=	13	Add $2x + 8$ to each side (separate x term from number)
x	=	$2\frac{3}{5}$ or 2.6	Divide both sides by 5

Examples (6): Solve the equations i) $5(x+3) = 2(2x-1)$; ii) $\frac{x+4}{2} = \frac{3x-2}{5}$

i) **$5(x+3) = 2(2x-1)$**

$5(x+3)$	=	$2(2x-1)$	
$5x+15$	=	$4x-2$	Expand both sides
x	=	-17	Subtract $(4x+15)$ from each side (separate terms)

ii) $\frac{x+4}{2} = \frac{3x-2}{5}$

When an equation includes fractional terms, it is best to get rid of the fractions by using a suitable multiplier – i.e. the lowest common multiple of the denominators.

$\frac{x+4}{2}$	=	$\frac{3x-2}{5}$	
$\frac{10(x+4)}{2}$	=	$\frac{10(3x-2)}{5}$	Eliminate the fractions by multiplying both sides by 10 (the LCM of the denominators)
$5(x+4)$	=	$2(3x-2)$	With practice, the two last steps can be combined mentally.
$5x+20$	=	$6x-4$	Expand both sides
20	=	$x-4$	Subtract $5x$ from each side
24	=	x	Add 4 to each side (separate the terms) (We brought x to the RHS to avoid dividing by a negative).

Solution is $x = 24$.

Example (7).
Emma has x beads, Fiona has three times as many as Emma, and Gina has four fewer beads than Fiona. All three have 80 beads between them. Form an equation connecting the total number of beads and thus work out how many beads they each have. (This is Example (3) with the extra requirement of solving the resulting equation.)

If Emma has x beads, Fiona will have $3x$ beads, and Gina will $3x – 4$ beads. The expression for the total number of beads is $7x – 4$, and the required equation is $7x – 4 = 80$.

$7x – 4 = 80$

$7x-4$	=	80	
$7x$	=	84	Add 4 to both sides
x	=	12	Divide both sides by 7

Emma has x or 12 beads, Fiona has $3x$ or 36, and Gina has $3x – 4$, or 32.

Sometimes an equation may be written in the form $\frac{a}{b} = \frac{c}{d}$ where *a*, *b*, *c* and *d* are algebraic expressions, and *b* and *d* are **not zero.**

The expression can be rewritten as $ad = bc$ - a process known as **cross-multiplication.**

(The = sign has been replaced below to illustrate the 'cross'.)

$$\frac{a}{b} \times \frac{c}{d} \rightarrow ad = bc .$$

This works because $\frac{a}{b} = \frac{c}{d}$ can be rewritten as

$\frac{a(bd)}{b} = \frac{c(bd)}{d}$ (multiplying both sides by the product of the denominators) , and finally

$\frac{a(\cancel{b}d)}{\cancel{b}} = \frac{c(b\cancel{d})}{\cancel{d}}$ (cancelling common factors) $\rightarrow ad = bc.$

The next example is ridiculous and trivial, but it does illustrate an important point.

Example (7): Solve $2x = x$.

2*x* = *x*

$2x$	=	x	
x	=	0	Subtract *x* from both sides

This is the correct solution of the equation, but see what happens if we try the alternative below.

2*x* = *x*

$2x$	=	x	
2	=	1	Divide both sides by *x*

We have ended up with the statement 2 = 1, which is clearly nonsense.

The problem here is that we have divided both sides of the equation by a ***variable***, *x*, rather than by a ***non–zero constant***. Because the only solution of the trivial equation is $x = 0$, we have effectively divided both sides by zero. This also holds true when 'cancelling out' expressions on dividing.

In mathematics, division by zero is an **undefined** operation – you just can't do it !

Example (8): Solve the equation $\frac{x+4}{2} = \frac{3x-2}{5}$**.** (This is identical to part ii of Example 6.)

Cross-multiplying, $$\frac{x+4}{2} \quad \frac{3x-2}{5} \rightarrow 5(x+4) = 2(3x+2).$$

The process continues as in Example 6 ii).

5(x+4)	=	2(3x-2)	With practice, the two last steps can be combined mentally.
5x+20	=	6x-4	Expand both sides
20	=	x -4	Subtract 5x from each side
24	=	x	Add 4 to each side (separate the terms) (We brought x to the RHS to avoid dividing by a negative).

Although cross-multiplication is a useful tool, care must be taken if an equation is not exactly of the form $\frac{a}{b} = \frac{c}{d}$.

Example (9): Solve the equation $1 + \frac{x+5}{3} = \frac{3x-1}{4}$.

(Without cross-multiplication)

$12 + 4(x + 5)$	=	$3(3x-1)$	Eliminate the fractions by multiplying both sides by 12 (the LCM of the denominators)
$4x + 32$	=	$9x - 3$	Expand both sides
$-5x$	=	-35	Subtract $(9x + 32)$ from each side to separate the terms
x	=	7	Divide both sides by –5

(With cross-multiplication)

We **cannot** cross-multiply straight away to get something like $1 + 4(x + 5) = 3(3x -1)$, since the left-hand side is not a simple fraction.

The LHS would need to be converted to a simple fraction to put the equation into $\frac{a}{b} = \frac{c}{d}$ form:

$$1 + \frac{x+5}{3} = \frac{3x-1}{4} \rightarrow \frac{3}{3} + \frac{x+5}{3} = \frac{3x-1}{4} \rightarrow \frac{x+8}{3} = \frac{3x-1}{4}$$

Both sides of the equation are now of the correct form for us to be able to cross-multiply:

$$\frac{x+8}{3} = \frac{3x-1}{4} \rightarrow 4(x+8) = 3(3x-1)$$

$4x + 32$	=	$9x - 3$	Expand both sides
$-5x$	=	-35	Subtract $(9x + 32)$ from each side to separate the terms
x	=	7	Divide both sides by –5

On the whole, it is better not to cross-multiply unless the equation is already in the $\frac{a}{b} = \frac{c}{d}$ form.

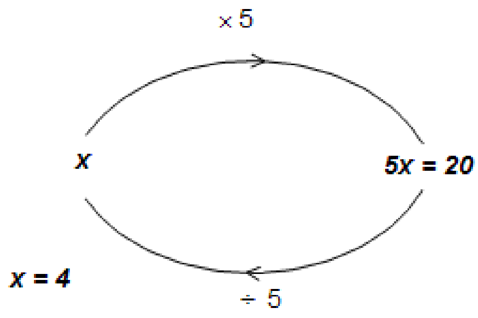
Using L.C.M's of the denominators is safer !

Inverse Operations.

The last examples demonstrated the idea of **inverse** operations.

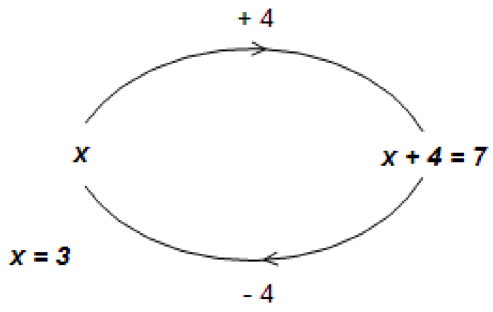
When solving $5x = 20$, we realised that we had to **multiply** x by 5 to give 20.

The opposite process of finding x from 20 involved **dividing** by 5.



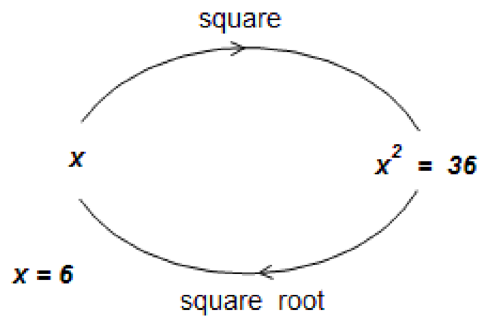
Similarly to solve an equation like $x + 4 = 7$, we must have **added** 4 to x to give 7.

Therefore to find x we **subtract** 4 from 7 to get 3.

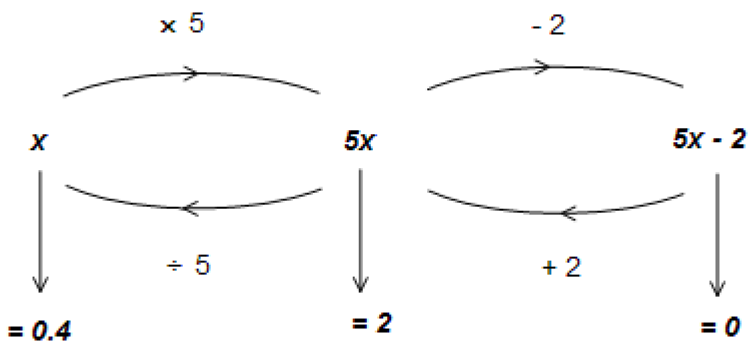


The equation $x^2 = 36$ is not linear, but we can see that we took the **square** of x to get 36.

Therefore we take the **square root** of 36 to get the original x.



When the solution of an equation takes more than one step, the order of inverse operations is also reversed.



When solving $5x-2 = 0$, we had to perform two operations; firstly to multiply x by 5 and then to subtract 2 from the result.

The inverse process is to add 2 and then divide by 5 to get the solution $x = 0.4$.

From the examples above, it can be seen that addition and subtraction are inverse operations, as are multiplication and division.

Other examples of inverse pairs are taking squares and square roots, cubes and cube roots.

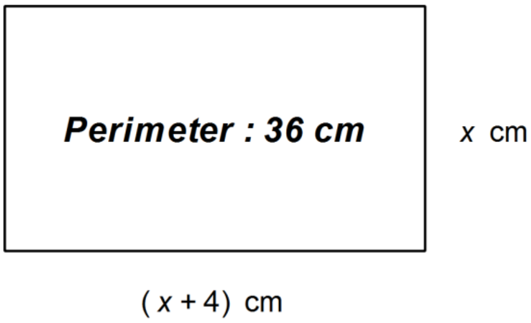
Some operations are self-inverse, such as taking the reciprocal of a number (i.e. dividing it into 1). Thus $1 \div 2 = \frac{1}{2}$ and $1 \div \frac{1}{2} = 2$.

Linear equations with a "shape and space" tie-in.

The following examples require us to form equations from geometrical facts, and then to solve it.

N.B. **None of the diagrams in those examples are drawn accurately.**

Example (10): The long sides of a rectangle are 4 cm longer than the short sides. Given that the perimeter of the rectangle is 36 cm, find the lengths of the sides of the rectangle.



The length of a short side is x cm, and that of a long side is $(x + 4)$ cm.

The perimeter is 36 cm, so we can form this equation from the formula for the perimeter of a rectangle:

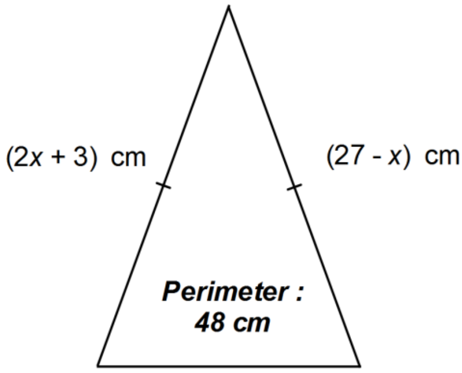
$x + (x + 4) + x + (x + 4) = 36$

Tidying up, we have the equation $4x + 8 = 36$, and then we solve it:

$4x + 8 = 36 \rightarrow 4x = 28 \rightarrow x = 7$. (Intermediate steps not shown in detail)

Since $x = 7$, the short sides are x cm, or **7 cm** long and the long sides are $(x + 4)$ cm, or **11 cm** long.

Example (11): An isosceles triangle has a perimeter of 48 cm. The lengths, in centimetres, of the two equal sides are given as $2x + 3$ and $27 – x$. Find the length of the third side.



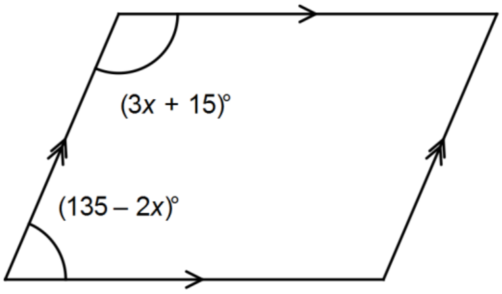
From the given data, we can form the linear equations:

$\mathbf{2x + 3 = 27 - x}$ (given the two equal sides)

and then use the resulting solution in x and the perimeter formula to find the third side.

$2x + 3 = 27 – x \rightarrow 3x + 3 = 27 \rightarrow 3x = 24 \rightarrow x = 8$.

By substituting $x = 8$ into either $2x + 3$ or $27 – x$, we find that the two equal sides are each 19 cm long. As the perimeter is 48 cm, we subtract twice 19 from 48, giving the length of the third side as **10 cm**.



Example (12): An obtuse angle of a parallelogram is $(3x + 15)°$ and an acute one is $(135 – 2x)°$. Find the sizes of both angles.

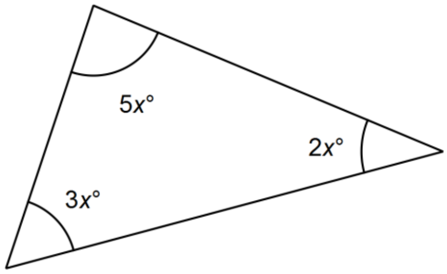
The two labelled angles are allied (co-interior) and therefore their sum is 180°.

We can therefore form the equation
$\mathbf{3x + 15 + 135 - 2x = 180}$ and solve it.

We simplify this as $\mathbf{x + 150 = 180}$, and thus $x = 30$.

Substituting $x = 30$ into $3x + 15$ gives us the obtuse angle of **105°.**
The acute angle is thus **75°** by the laws of allied angles.

(Check: when $x = 30$, $135 – 2x = 75$).



Example (13): The angles in a triangle are $3x°$, $2x°$ and $5x°$.

What type of triangle is it ? Justify the answer with full working.

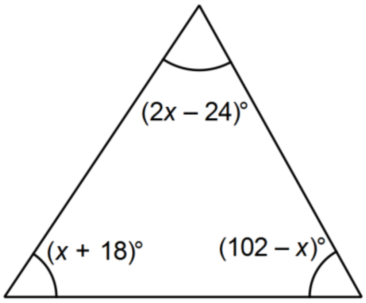
The angle sum of a triangle is 180°, so we form the equation $\mathbf{3x + 2x + 5x = 180}$ and solve it.

This simplifies to $10x = 180 \rightarrow x = 18$.

The angles in the triangle are $\mathbf{3x° = 54°, 2x° = 36°}$ **and** $\mathbf{5x° = 90°}$**.**

This particular triangle is therefore **right-angled.**

Example (14): The angles in another triangle are $(x + 18)°$, $(102 - x)°$ and $(2x – 24)°$.



Find x, and hence all the angles, showing your working. What type of triangle is it ?

The angle sum of a triangle is 180°, giving us the equation

$(x + 18) + (102 - x) + (2x - 24) = 180$ to solve.

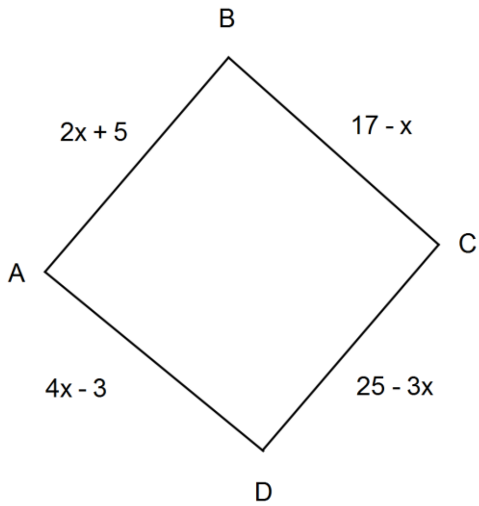
This simplifies to

$2x + 96 = 180 \rightarrow 2x = 84 \rightarrow x = 42$.

The angles in the triangle are **$(x + 18)° = 60°$, $(102-x°) = 60°$ and $(2x-24)° = 60°$.**

This particular triangle is thus **equilateral**.

Example (15): A quadrilateral *ABCD* is shown below, and we are given that sides *AB* and *BC* are equal.



Lengths are in centimetres.

What type of quadrilateral is it ? Justify the answer with full working.

We begin by solving the equation **$2x + 5 = 17 – x$.**

$2x + 5 = 17 – x \rightarrow 3x + 5 = 17 \rightarrow 3x = 12$

$\rightarrow$ **$x = 4$.**

We then substitute $x = 4$ into all the side length expressions:

$2x + 5 = 13$, so $AB = 13$ cm, as is $17 – x$, so $BC = 13$ cm as well. (It should be, as we've just solved this equation !)

Now $AD = 4x – 3 = 13$ cm, and $DC = 25 – 3x = 13$ cm.

All four sides of the quadrilateral *ABCD* are equal, hence *ABCD* is a **rhombus.**

(We have no details of the angles, so we cannot assume it's a square.)