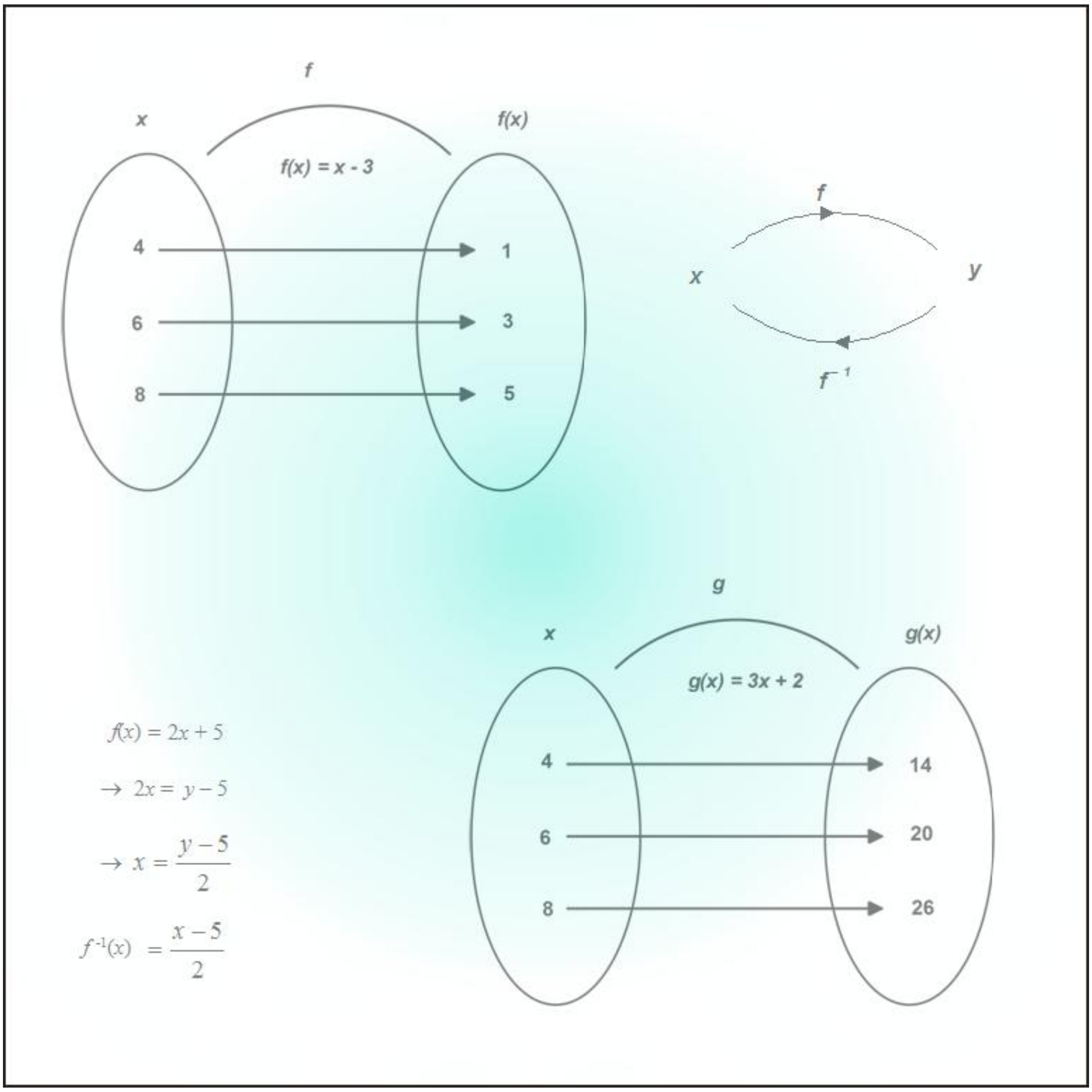
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

FUNCTIONS



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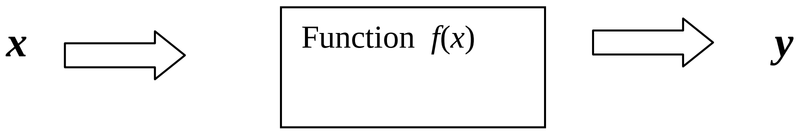
FUNCTIONS

A function can be thought of as being a rule which takes each member x of a set of values, and then assigns it to a value y.

In short, a function **maps** each value of x to an **image** value of y.



Functions are usually denoted by the letters f, g or h.

A function f which squares a number x can therefore be expressed in the following ways :

$y = x^2$ (if the result is assigned the variable y)

$f(x) = x^2$

$f: x \mapsto x^2$

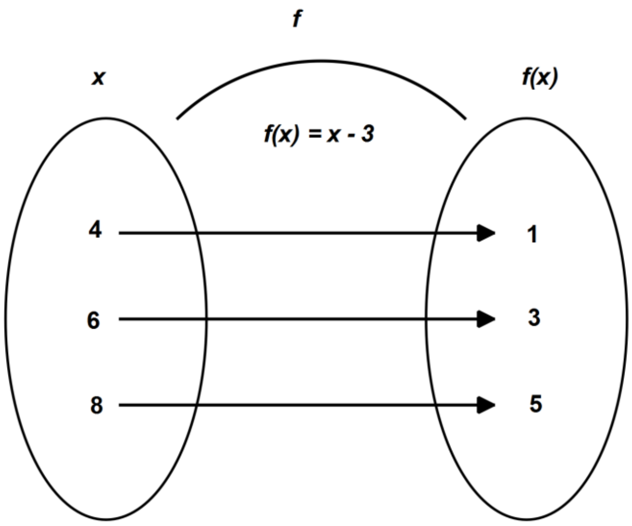
Here, f maps 4 to 16, and therefore we say that $f(4) = 16$.

Example (1) : A function f maps x to y by the following rule; subtract 3 from x. Express this in function notation, and write out $f(4)$.

The function can be denoted by $f(x) = x - 3$ or $f: x \mapsto x - 3$

Hence $f(4) = 1$.

Functions are also often illustrated by **mapping diagrams**.

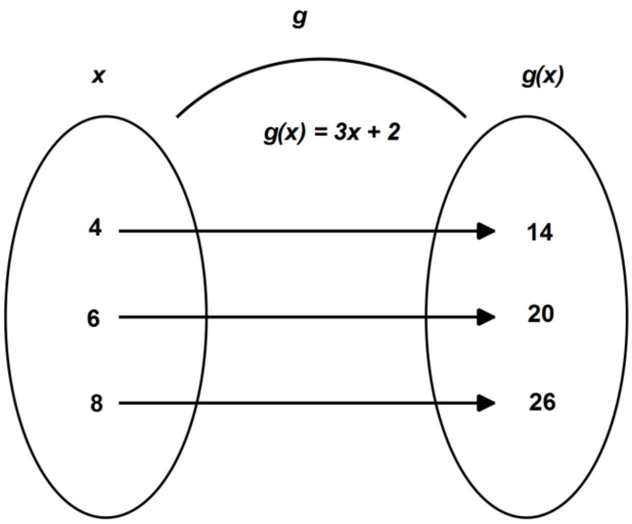


Mapping diagrams are a useful tool for visualising functions, although they can only display a limited set of mappings.

Example (2) : A function *g* maps *x* to *y* by the following rule; multiply *x* by 3, and then add 2. Express this in function notation write out $g(8)$, and draw a mapping diagram featuring three values of *x* and their images in *g*.

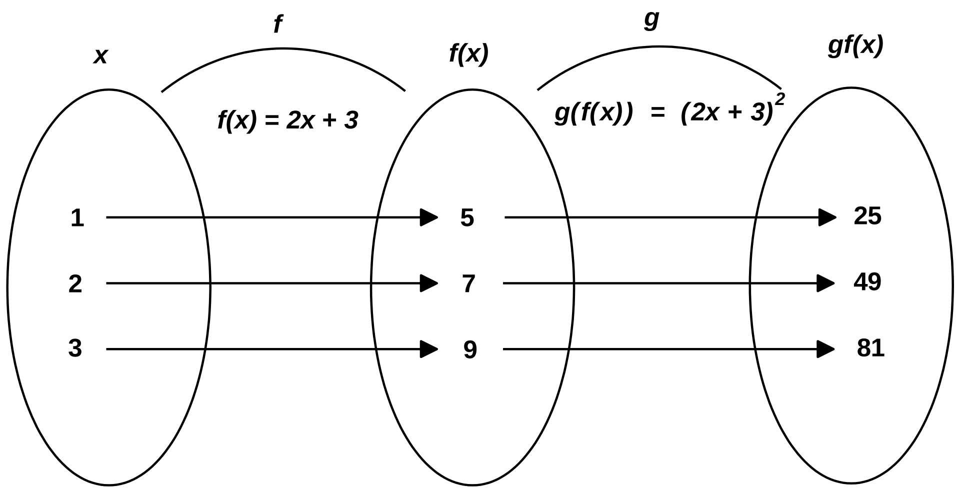
The function can be denoted by $g(x) = 3x + 2$ or $g: x \mapsto 3x + 2$

Hence $g(8) = 26$.



Composition of functions.

Example (3): Take two functions $f(x) = 2x + 3$ and $g(x) = x^2$.

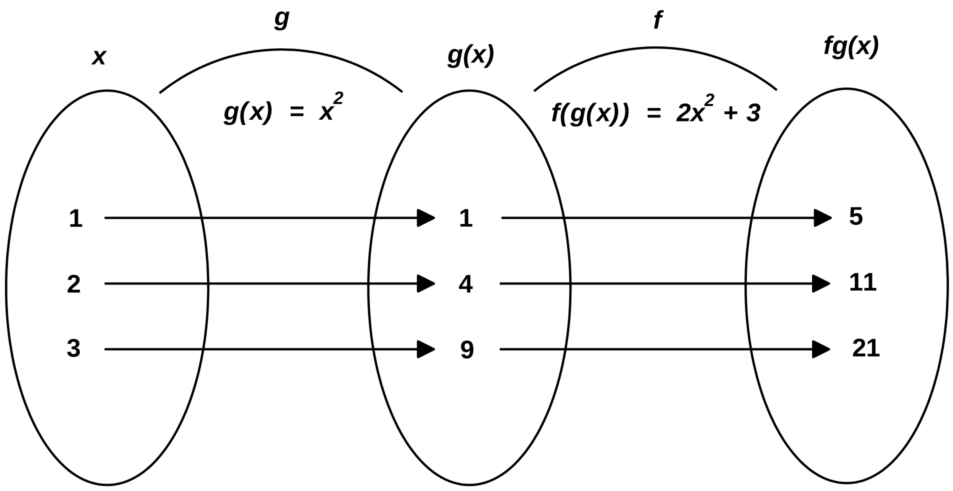


Here the function *f* is applied to *x* which doubles *x* and then adds 3. Next, the function *g* is applied to square that result.

This is written as $gf(x)$ or $g(f(x))$, and this new function is equivalent to $(2x + 3)^2$.

Note that we perform the function which is closest to the *x* first.

The order in which the functions are applied is important ! The diagram below shows what happens when *g* is performed first. This time, *g* is applied to *x* first, squaring it. Then *f* doubles the result and adds 3 to it. This is written as $fg(x)$ or $f(g(x))$, and this new function is equivalent to $2x^2 + 3$.



Generally, for any two functions *f* and *g*, the composite functions *fg* and *gf* will be different.

Example (4): Two functions are defined as follows :

$f(x) = x^2 + 4$; $g(x) = 1 - 2x$

i) Find expressions for $fg(x)$ and $gf(x)$, and hence work out $fg(3)$ and $gf(3)$.
ii) Solve $gf(x) = -15$.
iii) Solve $fg(x) = 53$.

iv) Find expressions for $ff(x)$ and $gg(x)$, and hence work out $ff(2)$ and $gg(2)$.
v) Solve $ff(x) = 29$.
vi) Solve $gg(x) = 23$.

i) To find $fg(x)$, we take the expression $f(x) = x^2 + 4$ and replace the variable x with the expression for $g(x)$, i.e. $1 - 2x$. Hence $fg(x) = f(1 – 2x) = (1 - 2x)^2 + 4$, or $4x^2 - 4x + 5$. Substituting $x = 3$, $fg(3) = 29$.

To find $gf(x)$, we take $g(x) = 1 - 2x$ and replace the x with the expression for $f(x)$, or $x^2 + 4$. Therefore $gf(x) = 1 – 2f(x) = 1 – 2(x^2+4)$, or $-2x^2 - 7$. Substituting $x = 3$, $gf(3) = -25$.

ii) To solve $gf(x) = -15$, write as $-2x^2 – 7 = -15$ and rearrange as
$8 - 2x^2 = 0 \rightarrow 4 - x^2 = 0 \rightarrow (2 + x)(2 – x) = 0 \rightarrow$ $\boldsymbol{x = \pm 2}$**.**

iii) We solve $fg(x) = 53 \rightarrow 4x^2 - 4x + 5 = 53 \rightarrow 4x^2 - 4x - 48 = 0$
$\rightarrow x^2 - x - 12 = 0 \rightarrow (x + 3)(x – 4) = 0 \rightarrow$ $\boldsymbol{x = 4, x = -3}$**.**

Functions can also be combined with themselves, as in the next sections:.

iv) $ff(x) = f(x^2 + 4) = (x^2 + 4)^2 + 4$. Also, $gg(x) = g(1 - 2x) = 1- 2(1 – 2x) = 4x – 1$.
Hence $ff(2) = 68$ and $gg(2) = 7$.

v) As $ff(x) = (x^2 + 4)^2 + 4$ we solve $(x^2 + 4)^2 + 4 = 29 \rightarrow (x^2 + 4)^2 = 25 \rightarrow x^2 + 4 = 5$
$\rightarrow x^2 = 1 \rightarrow$ $\boldsymbol{x = \pm 1}$**.**

(We excluded the case $x^2 + 4 = -5 \rightarrow x^2 = -9$ when taking the square root, because this equation has no solution.)

vi) As $gg(x) = 4x – 1$, we solve $4x – 1 = 23 \rightarrow 4x = 24 \rightarrow$ $\boldsymbol{x = 6}$**.**

The inverse of a function.

The inverse of a function f is another function that reverses whatever f does. This inverse function is usually written f^{-1}. Not all functions have an inverse, but that need not concern us at GCSE.

It therefore follows that for a function $f(x)$ with an inverse (see later), $f^{-1}(f(x)) = x$ and $f(f^{1}(x)) = x$.

Example (5): Find the inverses of the following functions:

i) $f(x) = x + 4$; ii) $g(x) = 3x$; iii) $h(x) = 2x + 5$; iv) $fg(x)$ (using the examples in i) and ii); v) $gf(x)$.

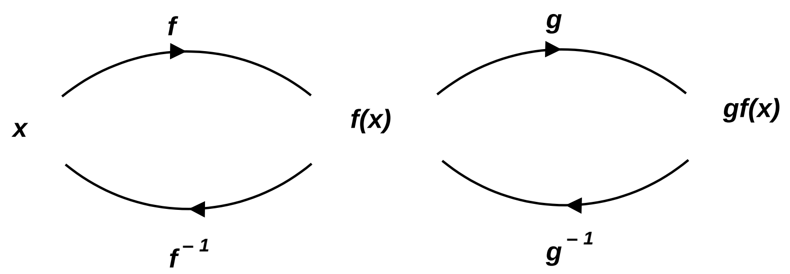
In i), the function f adds 4 to x ; the inverse function f^{-1} subtracts 4 from it,
$\therefore \quad f^{-1}(x) = x - 4$.

In ii), the function g multiplies x by 3 ; the inverse function g^{-1} divides it by 3,
$\therefore \quad g^{-1}(x) = \frac{x}{3}$.

In iii), the function h can be seen as a two-step process; first double x and then add 5 to the result. The inverse function must undo the processes, but in the reverse order to the original function.

To undo h, we must subtract 5 first and then halve the result.
$\therefore \quad h^{-1}(x) = \frac{x-5}{2}$.

The result from iii) can be generalised in the diagram below: **the inverse of gf, $(gf)^{-1}$, is $f^{-1}g^{-1}$.**

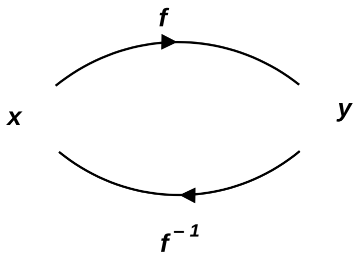


For iv), we already have the inverses of f and g. The inverse of $fg(x)$ is thus $g^{-1}f^{-1}(x)$ or $\frac{x-4}{3}$.

In v), the inverse of $gf(x)$ is thus $f^{-1}g^{-1}(x)$ or $\frac{x}{3} - 4$.

Finding the inverse of a function – formal method

In Example (5), we found the inverses of various linear functions mentally, but we can use a method very similar to “changing the formula subject” for more complicated examples.



The easiest way of finding the inverse of a function is to write it in the form $y = f(x)$ and then manipulate it to put x on the left-hand side, i.e. to express it in the form $x = f^{-1}(y)$.

The inverse function is then re-written in terms of x by exchanging the x and y.

Example (6): Find the inverse of the function (using the method above) $f(x) = 2x + 5$..

Write the function as $y = 2x + 5$.

Turn it round to bring x to the left-hand side:
$\rightarrow\ 2x + 5 = y$

Make x the subject by algebraic manipulation:

$\rightarrow\ 2x = y - 5$

$$\rightarrow\ x = \frac{y-5}{2}$$

As $x = f^{-1}(y)$, the inverse function is here defined in terms of y.

The inverse function $y = f^{-1}(x)$ is therefore $y = \dfrac{x-5}{2}$ (final exchange of x and y)

Hence $f^{-1}(x) = \dfrac{x-5}{2}$.

Example (7): Find the inverse of the function

$g(x) = \frac{5}{x-2}$, valid for all x except 2.

Write the function as $y = \frac{5}{x-2}$

Multiply both sides by $x - 2$ to get rid of fractions : $y(x - 2) = 5$

Divide both sides by y $\rightarrow$ $x - 2 = \frac{5}{y}$

Add 2 to both sides: $\rightarrow$ $x = \frac{5}{y} + 2$

Exchange x and y to give the inverse function : $\rightarrow$ $y = \frac{5}{x} + 2$

The inverse function, $g^{-1}(x) = \frac{5}{x} + 2$, valid for all non-zero x.

The restrictions on x in the case of both $g(x)$ and $g^{-1}(x)$, are necessary because division by zero is undefined.

More formally, we can say $g(x) = \frac{5}{x-2},\ x \neq 2$ and $g^{-1}(x) = \frac{5}{x} + 2,\ x \neq 0$