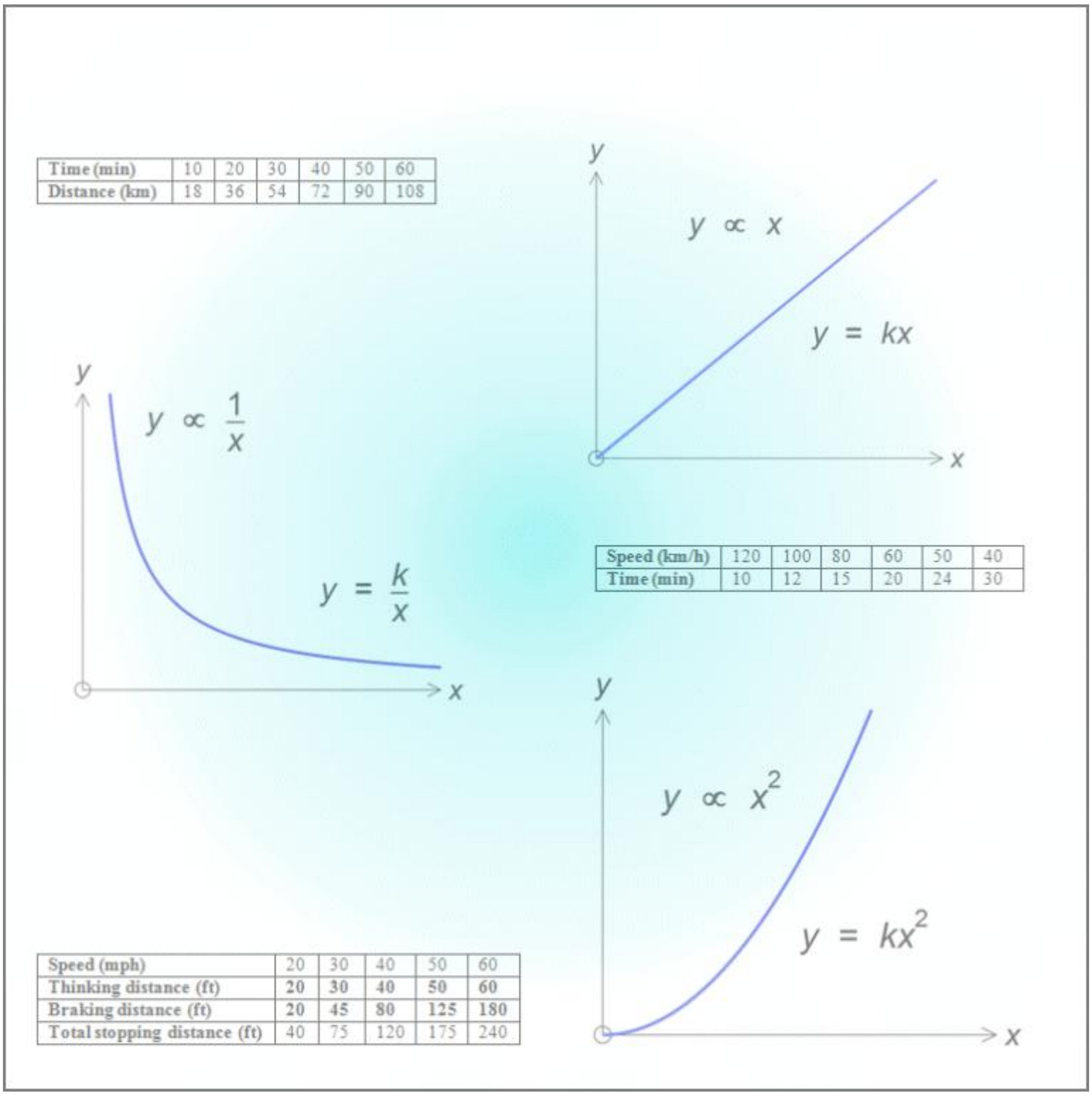
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

DIRECT AND INVERSE PROPORTION



Version: 3.1 Date: 20-09-2015

DIRECT AND INVERSE PROPORTION.

Relationships between pairs of values.

A driver is travelling at a constant speed on an 'ideal' road.
His companion records the distance travelled over a series of time intervals.

Time (min)	10	20	30	40	50	60
Distance (km)	18	36	54	72	90	108

Looking at the results, there is a suggested link between the time taken and the distance covered.
If the driver covers 18 km in 10 minutes, he will cover 36 km in 20 minutes and 54 km in 30 minutes.

If he spends **twice** the time driving, he will cover **twice** the distance; if he drives for **three times** longer, he will travel **three times** the distance.

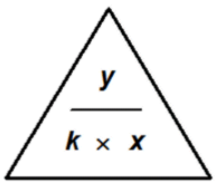
We can say that the time taken and the distance covered are in **proportion** or **direct proportion**.

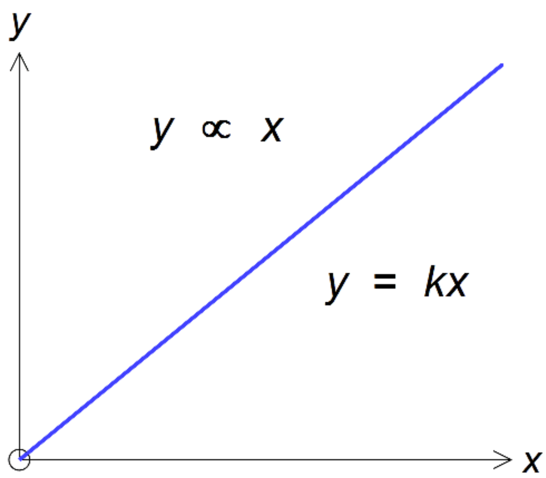
If x is the time taken and y is the distance covered, then we say:

$$\boldsymbol{y \propto x} \quad \text{or} \quad \boldsymbol{y = kx}.$$

The symbol $\propto$ stands for "is proportional to" in the first expression; in the second one, k is a number termed the **constant of proportionality.**

The linear relationship between x and y can be shown graphically, as in the diagram on the right.





Here we can find k by dividing any of the distances by the corresponding time; for example $\frac{18}{10}$.

This gives k = 1.8 and hence $\boldsymbol{y = 1.8x}$ (The constant k here is the speed in km per minute).

Inverse proportion, $y \propto 1/x$ or $y = k/x$.

The previous example was of direct proportion; this one will illustrate **inverse proportion**.

Another driver conducts a different experiment. This time, he travels for a fixed distance, but varies his speed and his companion records the time taken.

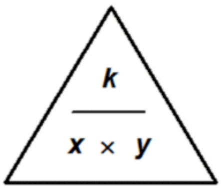
Speed (km/h)	120	100	80	60	50	40
Time (min)	10	12	15	20	24	30

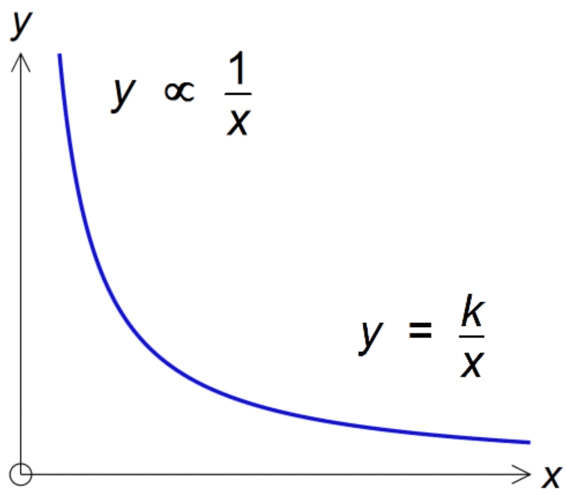
This time we can spot a suggested link between the speed and the time taken, but the pattern is different from the last one.
If the driver takes 10 minutes to cover the distance at 120 km/h, he will take 20 minutes to cover the same distance at 60 km/h and 30 minutes to cover it at 40 km/h.

If he **halves** his speed, the time taken for him to cover the same distance is **doubled**.
If he travels at **one-third** of his previous speed, the time taken is **tripled**.

We therefore say that the speed and the time taken are in **inverse proportion**.

If x is the speed and y the time taken, we say $y \propto \dfrac{1}{x}$ or $y = \dfrac{k}{x}$.





This time the relationship between x and y resembles the graph of $y = 1/x$ for positive x.

We can find k by rearranging the formula as $k = xy$; we multiply any of the time / speed combinations,

e.g. 120×10, to obtain $k = 1200$. Hence here or $y = \dfrac{1200}{x}$.

The constant here is the distance travelled (20 km) multiplied by 60. The reason for this is that the times had been quoted in minutes rather than hours.

Example (1): A runner trains on a jogging machine set to a constant speed and the readings are recorded as follows:

Time (min)	2	4	6	10	
Distance (m)	300	600	900		2400

Find the proportional relationship between the time and the distance travelled.
What is the constant of proportionality, and what does it represent ?
Also, fill in the missing values.

It can be seen that when the time is doubled, so is the distance , therefore the distance and time are in direct proportion.

If x is the time taken and y is the distance covered, then $\boldsymbol{y \propto x}$ or $\boldsymbol{y = kx}$**.**
Here we can find k by dividing (say) 300 by 2, giving k = 150. This is the speed in metres per minute.

Therefore when x = 10, y = 10 × 150 = 1500 – the runner jogs for 1500 metres in 10 minutes.
Similarly when y = 2400, $x = \frac{2400}{150}$ or 16 – it take 16 minutes to cover 2400 metres.

Example (2): A tramline building project is running behind schedule, and the contractor needs to increase his workforce to bring the completion date forward, using this table for reference.

Workforce (persons)	40	48	60	80	
Completion time (weeks)	24	20		12	10

Find the proportional relationship between the workforce numbers and the completion time.
Give the constant of proportionality, and fill in the missing values.

As the workforce is **doubled** from 40 to 80 persons, the completion time is **halved** from 24 weeks to 12 weeks, and therefore we have inverse proportion.

If x is the workforce and y is the completion time, then $\boldsymbol{y \propto \frac{1}{x}}$ or $\boldsymbol{y = \frac{k}{x}}$**.**

We can find k by re-expressing the relationship as $\boldsymbol{xy = k}$ and multiplying (for example) 48 by 20, to obtain k = 960.

When x = 60 persons, $y = \frac{k}{x} = \frac{960}{60} = 16$ days.

When y = 10 days, $x = \frac{k}{y} = \frac{960}{10} = 96$ persons.

Example (3a): Rick is an enthusiast for recorded music on vinyl and has been testing several turntables for mechanical accuracy, especially correct playback speed, as well as sound quality.

Rick tested one turntable by playing a piece and comparing the playback time with the stated time on the sleeve notes. He found that a piece that should have played for 4 minutes and 33 seconds at a playing speed of $33\frac{1}{3}$ revolutions per minute actually played for 4 minutes and 20 seconds.

i) What type of proportional relationship exists between the playing speed and the playing time ?
ii) Hence calculate the actual playing speed of the turntable.

i) If we increase the playing speed of a turntable, the time taken to play back the same amount of music will be reduced. The relationship is therefore one of inverse proportion.

ii) If x is the playing time and y is the turntable speed, then $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$, i.e. $k = xy$.

When the speed of the turntable is the correct one of $33\frac{1}{3}$ revolutions per minute, the playing time is 4 minutes and 33 seconds or 273 seconds. Hence $x = 273$, $y = 33\frac{1}{3}$ and $k = xy = 9100$.

The actual playing time was 4 minutes and 20 seconds, or 260 seconds.

Hence when $x = 260$ seconds, $y = \frac{k}{x} = \frac{9100}{260} = 35$.

$\therefore$ The turntable rotated at **35 r.p.m.**

Example (3b): Rick is also a piano tuner, and used his skills to test the speed of another turntable by playing along to a recorded piano piece, where the note A was tuned to 440 Hz. The turntable, however, played back that note of A at 422.4 Hz .

(The hertz (Hz) is a unit of musical pitch, or the frequency of a wave. In the case of sound, the greater the frequency, and the higher-pitched the sound.)

i) What type of proportional relationship exists between the playing speed and the note pitch ?
ii) Hence calculate the actual playing speed of the turntable.

i) If we decrease the playing speed of a turntable, the pitch of the note at playback will also be reduced. The relationship is therefore one of direct proportion.

ii) If x is the note pitch and y is the turntable speed, then $y \propto x$, or $y = kx$, i.e. $k = \frac{y}{x}$.

At the correct turntable speed of $33\frac{1}{3}$ revolutions per minute, the note A is played back at 440 Hz.

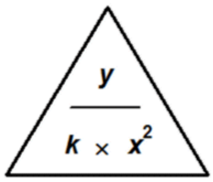
Hence $x = 440$, $y = 33\frac{1}{3}$ and $k = \frac{y}{x} = \frac{5}{66}$.

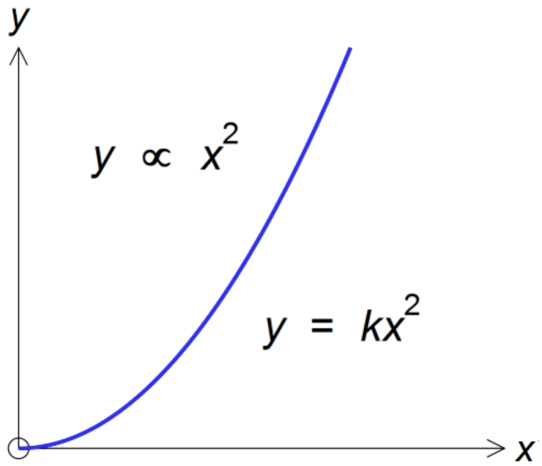
The actual pitch of the note A as played on the turntable was 422.4 Hz,
so here $y = kx = 422.4 \times \frac{5}{66} = 32$.

$\therefore$ The turntable rotated at **32 r.p.m.**

Other types of proportional relationship.

Square relationship, $y \propto x^2$ or $y = kx^2$.





The relationship between x and y now resembles the graph of $y = x^2$ for positive x.

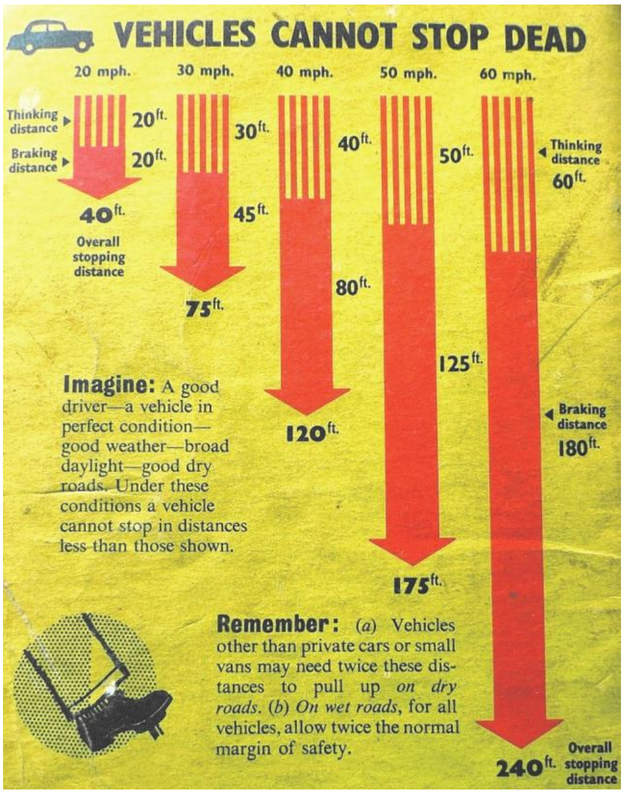
Example (4a):

The Highway Code uses the following values for the stopping distances of a vehicle at various speeds, as seen in this page from a vintage edition (copyright: HMSO). These distance values are still used today.

(Imperial measures used here.)

What type of proportional relationship can you see for:

i) the total overall stopping distance ?
ii) the thinking distance ?
iii) the braking distance ?



i) Total overall stopping distance

Speed (mph)	**20**	**30**	**40**	**50**	**60**
Total stopping distance (ft)	**40**	**75**	**120**	**175**	**240**

The values here appear to be directly proportional, since an increase in speed causes an increase in the total stopping distance. This relationship is not linear, though, because the total stopping distance at 40mph is not double that at 20 mph, for example.

The stopping distance is obtained by adding together the 'thinking distance' and the 'braking distance'. When those values are tabled separately, the relationships are more obvious.

ii) Thinking distance

Speed (mph)	20	30	40	50	60
Thinking distance (ft)	**20**	**30**	**40**	**50**	**60**

The thinking distance in feet can clearly be seen to be numerically equal to the speed in miles per hour, so we have an obvious direct proportional relationship with the constant of proportionality equal to 1.

iii) Braking distance

Speed (mph)	20	30	40	50	60
Braking distance (ft)	**20**	**45**	**80**	**125**	**180**

With the braking distance, things are slightly more complicated. When the speed (in mph) is doubled from 20 to 40, the braking distance (in feet) is quadrupled from 20 to 80. When the speed is tripled (from 20 to 60 mph), the braking distance is increased from 20 to 180 feet - a factor of 9.
This clearly suggests a 'square' relationship, i.e. $y = kx^2$.

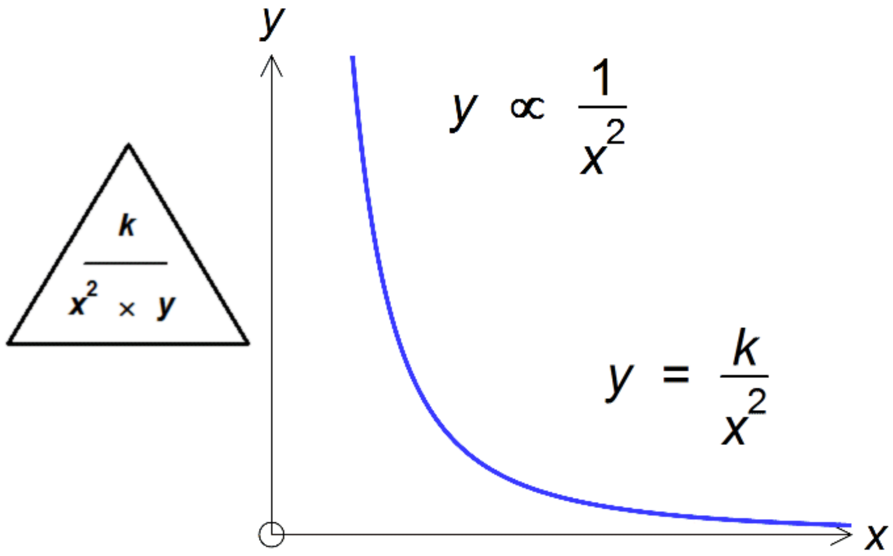
Example (4b): Find the constant of proportionality relating the braking distance and the speed, and hence find the **total** stopping distance for a car travelling at 70 mph.

Since $y = kx^2$, we can rewrite the relationship as $k = \frac{y}{x^2}$ and substitute one of the pairs of values in the table, such as $x = 40$, $y = 80$. This gives $k = \frac{80}{1600} = \frac{1}{20}$.

When the car is being driven at 70 mph, the thinking distance is 70 feet.
The braking distance can be given as $\frac{1}{20} \times 70^2 = 245$ feet.

$\therefore$ The total stopping distance for a car driven at 70 mph is (70 + 245) feet or **315 feet.**

Inverse square relationship, $y \propto 1/x^2$ or $y = k/x^2$.



The relationship between x and y resembles the graph of $y = 1/x^2$ for positive x.

The intensity of a lighthouse beam is measured at various distances from the source, and the following results obtained, taking the relative intensity at 5 km as 100.

Distance (km)	2	5	10	25
Relative Intensity	625	100	25	4

Example (5): Given that the light intensity falls off with distance according to an inverse square law,

i) find the constant of proportionality relating the intensity of the lighthouse beam and its distance from the source, and hence the relative intensity of the beam at 12.5 km.

ii) At what distance will the intensity of the light beam have fallen to 1% of its value at 5 km ?

This time, when the distance is **doubled** from 5 km to 10 km, the relative intensity is **quartered.** Similarly, when the distance is multiplied by 5, the relative intensity falls to 4% or $\frac{1}{25}$ of its previous value.

i) Since $y = k/x^2$**,** we can rewrite the relationship as $k = x^2y$, where x is the distance from the source and y the relative intensity.

Substituting $x = 5$ and $y = 100$, we therefore have $k = 2500$.

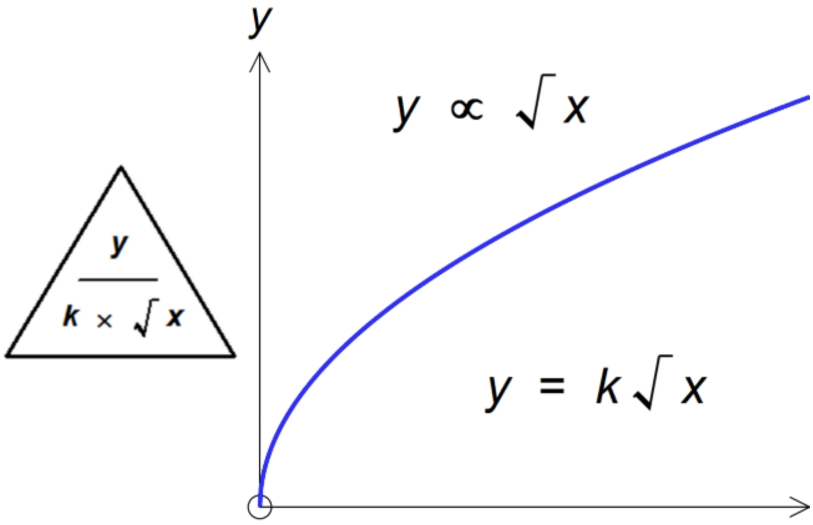
When the distance $x = 12.5$ km, the relative intensity is $y = k/x^2 = \dfrac{2500}{12.5^2} = 16$.

We make x the subject of the formula $k = x^2y$ to obtain $x = \sqrt{\dfrac{k}{y}}$ (by rearranging).

The intensity will have fallen to 1% of its value at 5 km when the distance $x = \sqrt{\dfrac{2500}{1}}$ km or 50 km.

(We could have also used the values $x = 5$, $y = 100$. Multiplying the distance by 10 would reduce the light intensity by a factor of 10^2, or 100.)

Square root relationship, $y \propto \sqrt{x}$ or $y = k\sqrt{x}$.



The relationship between x and y resembles the graph of $y = \sqrt{x}$.

Example (6): The relationship between an observer's height above sea level and the distance to the visible horizon is shown as follows :

Height above sea level (m)	25	100	400	900
Distance to visible horizon (km)	18	36	72	108

As the height above sea level is increased by a factor of 4, the distance to the visible horizon is doubled, thus confirming the square-root relationship between the sets of data.

i) Find the formula relating the height above sea level and the distance to the visible horizon.

ii) A ship is just visible from the observation deck at Blackpool Tower, at a height of 156.25 m above sea level. How far away is the ship from the Tower ?

i) Since $y = k\sqrt{x}$, $k = \frac{y}{\sqrt{x}}$. Substituting $x = 100$ and $y = 36$, we have $k = 3.6$ and the formula $y = 3.6\sqrt{x}$.

ii) Substituting $x = 156.25$ (height of Blackpool Tower above sea level) into the formula in i), we have $y = 3.6\sqrt{156.25} = 3.6 \times 12.5$ km = 45 km. Hence the ship is 45 km from Blackpool Tower.

Example (7): In an experiment, measurements of l and f were taken.

The results were as follows :

f	64	25	16
l	5	8	10

The results are connected by one of these rules.

a) $l = k\sqrt{f}$ b) $l = \dfrac{k}{f}$ c) $l = \dfrac{k}{\sqrt{f}}$

Which rule is the correct one ? Give clear reasons for your answer, and hence find the equation connecting l and f.

Since l increases as f decreases, the relationship is one of inverse, rather than direct, proportion. We can therefore rule out option (a).

If we rearrange the formula for option (b) , we have $k = fl$.
However, the values of $f = 64$ and $l = 5$ would imply $k = 320$, but the values of $f = 25$ and $l = 8$ would imply $k = 200$. Since k is meant to be a constant, we must reject option (b) as well.

This leaves us with option (c), where $k = l\sqrt{f}$ after rearrangement.
Substituting $f = 64$ and $l = 5$ would make $k = 40$; so would $f = 25$ and $l = 8$, or $f = 16$ and $l = 10$.

$\therefore$ The relationship between f and l is $l = \dfrac{40}{\sqrt{f}}$.

Example (8): A container ship has a maximum speed of 28 knots, and the following table is used to estimate daily fuel consumption against speed.

Speed in knots, *v*	16	20	24
Tonnes of fuel used per day, *c*	128	250	432

i) Which rule represents the relationship between *v* and *c* ? Give reasons for your answer, and so find the constant of proportionality *k*.

a) $c = kv$ b) $c = kv^2$ c) $c = kv^3$

ii) Hence calculate the daily fuel consumption in tonnes if the ship is travelling at its maximum speed of 28 knots.

i) If $c = kv$, then we would have $k = \frac{c}{v}$. However, $\frac{128}{16} = 8$, $\frac{250}{20} = 12.5$ and $\frac{432}{24} = 18$.

The resulting values of *k* are not constant so we can reject $c = kv$ as a relationship.

If $c = kv^2$, then $k = \frac{c}{v^2}$. However, $\frac{128}{16^2} = \frac{1}{2}$, $\frac{250}{20^2} = \frac{5}{8}$ and $\frac{432}{24^2} = \frac{3}{4}$.

We therefore eliminate $c = kv^2$ as well.

With $c = kv^3$, we would have $k = \frac{c}{v^3}$. Now $\frac{128}{16^3} = \frac{1}{32}$, $\frac{250}{20^3} = \frac{1}{32}$ and $\frac{432}{24^3} = \frac{1}{32}$.

The constant of proportionality, *k*, is thus $\frac{1}{32}$.

ii) From the result in part i), the relationship between the speed *v* and the fuel consumption *c* is given by the formula $c = \frac{1}{32}v^3$.

Substituting $v = 28$ gives $c = \frac{1}{32} \times 28^3 = 686$.

$\therefore$ The ship would consume 686 tonnes of fuel per day if it were travelling at its maximum speed of 28 knots.

Multiple-stage direct and inverse proportion.

The next example is a little more tricky, as it combines both direct and inverse proportion in the same question. Such problems are best tackled in stages.

Example (9): If 12 checkout staff at a supermarket can serve 48 customers in 20 minutes,

i) how long would it take 15 checkout staff to serve 84 customers ?

ii) how many checkouts need to be open to serve 240 customers in one hour ?

If we increase the number of checkout staff, we increase the number of customers being served in the same time interval. Hence the checkout staffing level and the number of customers are in direct proportion when time is unchanged.

More checkouts = more customers (in same time)

Also, an increase in the number of checkout staff would lead to a decrease in the time taken to serve the same number of customers. This time, the checkout staffing level and the time taken to serve the customers are in inverse proportion when the number of customers is unchanged.

More checkouts = less time (for same number of customers)

Finally, if the number of checkout staff remains constant, an increase in the number of customers would lead to an increase in the time taken to serve them. In other words, the number of customers and the time taken to serve them are in direct proportion when the number of checkout staff is unchanged.

More customers = more time (for same number of checkouts)

i) We can see that the number of checkout staff has increased from 12 to 15, or in the ratio 15 : 12, simplifying to 5 : 4. Therefore 15 checkout staff can serve $\frac{5}{4} \times 48$, or 60 customers, in 20 minutes.
We now have the correct number of checkout staff, but we still need to increase the number of customers to 84, or in the ratio 84 : 60, simplifying to 7 : 5.
Since customer numbers and the time are in direct proportion, the time taken is $\frac{7}{5} \times 20$, **or 28 minutes**.

	12 checkouts;	48 customers;	20 minutes	Original problem
$\rightarrow$	**15 checkouts;**	**60 customers;**	20 minutes	Customers / checkouts in direct proportion, so increase both in ratio 5 : 4 (i.e. multiply by $\frac{5}{4}$)
$\rightarrow$	15 checkouts;	**84 customers;**	**28 minutes**	Customers / time in direct proportion, so increase both in ratio 7 : 5 (i.e. multiply by $\frac{7}{5}$)

ii) This time, the number of customers has increased from 48 to 240, a ratio of 5 : 1, and so the time taken will increase by the same ratio, from 20 minutes to 5×20 or 100 minutes.

The question however asks us to work out the checkouts per hour, and so we have to decrease the time from 100 to 60 minutes, or multiply it by $\frac{60}{100}$ or $\frac{3}{5}$.

Because the time taken and the number of checkouts are inversely proportional, we have to **multiply** the number of checkouts by $\frac{5}{3}$, giving us $\frac{5}{3} \times 12$, **or 20 checkouts**.

	12 checkouts;	48 customers;	20 minutes	Original problem
→	12 checkouts;	**240 customers;**	**100 minutes**	Customers / time in **direct** proportion, so increase both in ratio 5 : 1
→	**20 checkouts;**	240 customers;	**60 minutes**	Checkouts / time in **inverse** proportion, so **decrease** time in ratio 5 : 3 (i.e. multiply by $\frac{3}{5}$)and **increase** checkouts in ratio 5 : 3 (i.e. multiply by $\frac{5}{3}$)