DIRECT AND INVERSE PROPORTION

**DIRECT PROPORTION**

- When two quantities are directly proportional, their ratio is constant.
- **Graph:** Linear graph with **y** = **kx**.

**INVERSE PROPORTION**

- When two quantities are inversely proportional, their product is constant.
- **Graph:** Hyperbola with **y** = **k/x**.

**INVERSE SQUARE PROPORTION**

- When one quantity is inversely proportional to the square of another.
- **Graph:** Parabola with **y** = **kx^2**.
DIRECT AND INVERSE PROPORTION.

Relationships between pairs of values.

A driver is travelling at a constant speed on an ‘ideal’ road. His companion records the distance travelled over a series of time intervals.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (km)</td>
<td>18</td>
<td>36</td>
<td>54</td>
<td>72</td>
<td>90</td>
<td>108</td>
</tr>
</tbody>
</table>

Looking at the results, there is a suggested link between the time taken and the distance covered. If the driver covers 18 km in 10 minutes, he will cover 36 km in 20 minutes and 54 km in 30 minutes.

If he spends twice the time driving, he will cover twice the distance; if he drives for three times longer, he will travel three times the distance.

We can say that the time taken and the distance covered are in proportion or direct proportion.

If \(x\) is the time taken and \(y\) is the distance covered, then we say:

\[ y \propto x \text{ or } y = kx. \]

The symbol \(\propto\) stands for “is proportional to” in the first expression; in the second one, \(k\) is a number termed the constant of proportionality.

The linear relationship between \(x\) and \(y\) can be shown graphically, as in the diagram on the right.

Here we can find \(k\) by dividing any of the distances by the corresponding time; for example \(\frac{18}{10}\).
This gives \(k = 1.8\) and hence \(y = 1.8x\) (The constant \(k\) here is the speed in km per minute).
Inverse proportion, \( y \propto 1/x \) or \( y = k/x \).

The previous example was of direct proportion; this one will illustrate inverse proportion.

Another driver conducts a different experiment. This time, he travels for a fixed distance, but varies his speed and his companion records the time taken.

<table>
<thead>
<tr>
<th>Speed (km/h)</th>
<th>120</th>
<th>100</th>
<th>80</th>
<th>60</th>
<th>50</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (min)</td>
<td>10</td>
<td>12</td>
<td>15</td>
<td>20</td>
<td>24</td>
<td>30</td>
</tr>
</tbody>
</table>

This time we can spot a suggested link between the speed and the time taken, but the pattern is different from the last one.

If the driver takes 10 minutes to cover the distance at 120 km/h, he will take 20 minutes to cover the same distance at 60 km/h and 30 minutes to cover it at 40 km/h.

If he halves his speed, the time taken for him to cover the same distance is doubled.
If he travels at one-third of his previous speed, the time taken is tripled.

We therefore say that the speed and the time taken are in inverse proportion.

If \( x \) is the speed and \( y \) the time taken, we say \( y \propto 1/x \) or \( y = \frac{k}{x} \).

\[
\begin{align*}
\text{This time the relationship between } x \text{ and } y \text{ resembles the graph of } y = \frac{1}{x} \text{ for positive } x. \\
\end{align*}
\]

We can find \( k \) by rearranging the formula as \( k = xy \); we multiply any of the time / speed combinations,

e.g. \( 120 \times 10 \), to obtain \( k = 1200 \). Hence here or \( y = \frac{1200}{x} \).

The constant here is the distance travelled (20 km) multiplied by 60. The reason for this is that the times had been quoted in minutes rather than hours.
Example (1): A runner trains on a jogging machine set to a constant speed and the readings are recorded as follows:

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (m)</td>
<td>300</td>
<td>600</td>
<td>900</td>
<td>2400</td>
</tr>
</tbody>
</table>

Find the proportional relationship between the time and the distance travelled.
What is the constant of proportionality, and what does it represent?
Also, fill in the missing values.

It can be seen that when the time is doubled, so is the distance, therefore the distance and time are in direct proportion.

If \( x \) is the time taken and \( y \) is the distance covered, then \( y \propto x \) or \( y = kx \).
Here we can find \( k \) by dividing (say) 300 by 2, giving \( k = 150 \). This is the speed in metres per minute.

Therefore when \( x = 10 \), \( y = 10 \times 150 = 1500 \) – the runner jogs for 1500 metres in 10 minutes.
Similarly when \( y = 2400 \), \( x = \frac{2400}{150} \) or 16 – it take 16 minutes to cover 2400 metres.

Example (2): A tramline building project is running behind schedule, and the contractor needs to increase his workforce to bring the completion date forward, using this table for reference.

<table>
<thead>
<tr>
<th>Workforce (persons)</th>
<th>40</th>
<th>48</th>
<th>60</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Completion time (weeks)</td>
<td>24</td>
<td>20</td>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

Find the proportional relationship between the workforce numbers and the completion time.
Give the constant of proportionality, and fill in the missing values.

As the workforce is doubled from 40 to 80 persons, the completion time is halved from 24 weeks to 12 weeks, and therefore we have inverse proportion.

If \( x \) is the workforce and \( y \) is the completion time, then \( y \propto \frac{1}{x} \) or \( y = \frac{k}{x} \).

We can find \( k \) by re-expressing the relationship as \( xy = k \) and multiplying (for example) 48 by 20, to obtain \( k = 960 \).

When \( x = 60 \) persons, \( y = \frac{k}{x} = \frac{960}{60} = 16 \) days.

When \( y = 10 \) days, \( x = \frac{k}{y} = \frac{960}{10} = 96 \) persons.

Other types of proportional relationship.

The first examples covered direct proportion where \( y = kx \) and inverse proportion where \( y = \frac{k}{x} \).
Those are the main relationships we might come across, but there are several others.
Square relationship, \( y \propto x^2 \) or \( y = kx^2 \).

The relationship between \( x \) and \( y \) now resembles the graph of \( y = x^2 \) for positive \( x \).

The Highway Code uses the following values for the stopping distances of a vehicle at various speeds, as seen in this page from a vintage edition (copyright: HMSO). These distance values are still used today.

(Imperial measures used here.)

What type of proportional relationships can you see for: i) the overall stopping distance  ii) the thinking distance, and iii) the braking distance?

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total stopping distance (ft)</td>
<td>40</td>
<td>75</td>
<td>120</td>
<td>175</td>
<td>240</td>
</tr>
</tbody>
</table>

The values here appear to be directly proportional, but with a non-linear relationship. The total stopping distance at 40mph is not double that at 20 mph, for example.
The stopping distance is obtained by adding together the ‘thinking distance’ and the ‘braking distance’. When those values are tabled separately, the relationships are more obvious.

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thinking dist. (ft)</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>Braking dist. (ft)</td>
<td>20</td>
<td>45</td>
<td>80</td>
<td>125</td>
<td>180</td>
</tr>
<tr>
<td>Total stopping dist. (ft)</td>
<td>40</td>
<td>75</td>
<td>120</td>
<td>175</td>
<td>240</td>
</tr>
</tbody>
</table>

The thinking distance in feet can clearly be seen to be numerically equal to the speed in miles per hour, so we have an obvious direct proportional relationship with the constant of proportionality equal to 1.

With the braking distance, things are slightly more complicated. When the speed (in mph) is doubled from 20 to 40, the braking distance (in feet) is quadrupled from 20 to 80. When the speed is tripled (from 20 to 60 mph), the braking distance is increased from 20 to 180 feet - a factor of 9. This clearly suggests a ‘square’ relationship, i.e. \( y = kx^2 \).

**Example (3):** Find the constant of proportionality relating the braking distance and the speed, and hence find the total stopping distance for a car travelling at 70 mph.

Since \( y = kx^2 \), we can rewrite the relationship as \( k = \frac{y}{x^2} \) and substitute one of the pairs of values in the table, such as \( x = 20, y = 20 \). This gives \( k = \frac{20}{400} = 0.05 \).

When the car is being driven at 70 mph, the thinking distance is 70 feet. The braking distance can be given as \( 0.05 \times 70^2 \) feet, or 245 feet.

\[ 70 \text{ feet} + 245 \text{ feet} = 315 \text{ feet.} \]
Inverse square relationship, \( y \propto 1/x^2 \) or \( y = k/x^2 \).

The relationship between \( x \) and \( y \) resembles the graph of \( y = 1/x^2 \) for positive \( x \).

The intensity of a lighthouse beam is measured at various distances from the source, and the following results obtained, taking the relative intensity at 5 km as 100.

<table>
<thead>
<tr>
<th>Distance (km)</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Intensity</td>
<td>625</td>
<td>100</td>
<td>25</td>
<td>4</td>
</tr>
</tbody>
</table>

**Example (4):** Given that the light intensity falls off with distance according to an inverse square law,

i) find the constant of proportionality relating the intensity of the lighthouse beam and its distance from the source, and hence the relative intensity of the beam at 12.5 km.

ii) At what distance will the intensity of the light beam have fallen to 1% of its value at 5 km?

This time, when the distance is doubled from 5 km to 10 km, the relative intensity is quartered. Similarly, when the distance is multiplied by 5, the relative intensity falls to 4% or \( \frac{1}{25} \) of its previous value.

i) Since \( y = k/x^2 \), we can rewrite the relationship as \( k = x^2y \), where \( x \) is the distance from the source and \( y \) the relative intensity.

Substituting \( x = 5 \) and \( y = 100 \), we therefore have \( k = 2500 \).

When the distance \( x = 12.5 \) km, the relative intensity is \( y = k/x^2 = \frac{2500}{12.5^2} = 16 \).

We make \( x \) the subject of the formula \( k = x^2y \) to obtain \( x = \sqrt{\frac{k}{y}} \) (by rearranging).

The intensity will have fallen to 1% of its value at 5 km when the distance \( x = \sqrt{\frac{2500}{1}} \) km or 50 km.

(We could have also used the values \( x = 5 \), \( y = 100 \). Multiplying the distance by 10 would reduce the light intensity by a factor of \( 10^2 \), or 100.)
Square root relationship, \( y \propto \sqrt{x} \) or \( y = k\sqrt{x} \).

![Graph of square root relationship]

The relationship between \( x \) and \( y \) resembles the graph of \( y = \sqrt{x} \).

**Example (5):** The relationship between an observer’s height above sea level and the distance to the visible horizon is shown as follows:

<table>
<thead>
<tr>
<th>Height above sea level (m)</th>
<th>25</th>
<th>100</th>
<th>400</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance to visible horizon (km)</td>
<td>18</td>
<td>36</td>
<td>72</td>
<td>108</td>
</tr>
</tbody>
</table>

As the height above sea level is increased by a factor of 4, the distance to the visible horizon is doubled, thus confirming the square-root relationship between the sets of data.

i) Find the formula relating the height above sea level and the distance to the visible horizon.

ii) A ship is just visible from the observation deck at Blackpool Tower, at a height of 156.25 m above sea level. How far away is the ship from the Tower?

i) Since \( y = k\sqrt{x} \), \( k = \frac{y}{\sqrt{x}} \). Substituting \( x = 100 \) and \( y = 36 \), we have \( k = 3.6 \) and the formula \( y = 3.6\sqrt{x} \).

ii) Substituting \( x = 156.25 \) (height of Blackpool Tower above sea level) into the formula in i), we have \( y = 3.6\sqrt{156.25} = 3.6 \times 12.5 \text{ km} = 45 \text{ km} \). Hence the ship is 45 km from Blackpool Tower.
Example (6): In an experiment, measurements of \( l \) and \( f \) were taken.

The results were as follows:

\[
\begin{array}{ccc}
\hline
f & 64 & 25 & 16 \\
l & 5 & 8 & 10 \\
\hline
\end{array}
\]

The results are connected by one of these rules:

a) \( l = k\sqrt{f} \) \hspace{1cm} b) \( l = \frac{k}{f} \) \hspace{1cm} c) \( l = \frac{k}{\sqrt{f}} \)

Which rule is the correct one? Give clear reasons for your answer, and hence find the equation connecting \( l \) and \( f \).

Since \( l \) increases as \( f \) decreases, the relationship is one of inverse, rather than direct, proportion. We can therefore rule out option (a).

If we rearrange the formula for option (b), we have \( k = fl \).

However, the values of \( f = 64 \) and \( l = 5 \) would imply \( k = 320 \), but the values of \( f = 25 \) and \( l = 8 \) would imply \( k = 200 \). Since \( k \) is meant to be a constant, we must reject option (b) as well.

This leaves us with option (c), where \( k = l\sqrt{f} \) after rearrangement.

Substituting \( f = 64 \) and \( l = 5 \) would make \( k = 40 \); so would \( f = 25 \) and \( l = 8 \), or \( f = 16 \) and \( l = 10 \).

\[ \therefore \] The relationship between \( f \) and \( l \) is \( l = \frac{40}{\sqrt{f}} \).
Example (7): A container ship has a maximum speed of 28 knots, and the following table is used to estimate daily fuel consumption against speed.

<table>
<thead>
<tr>
<th>Speed in knots, v</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tonnes of fuel used per day, c</td>
<td>128</td>
<td>250</td>
<td>432</td>
</tr>
</tbody>
</table>

i) Which rule represents the relationship between \( v \) and \( c \)? Give reasons for your answer, and so find the constant of proportionality \( k \).

a) \( c = kv \)

b) \( c = kv^2 \)

c) \( c = kv^3 \)

ii) Hence calculate the daily fuel consumption in tonnes if the ship is travelling at its maximum speed of 28 knots.

i) If \( c = kv \), then we would have \( k = \frac{c}{v} \). However, \( \frac{128}{16} = 8 \), \( \frac{250}{20} = 12.5 \) and \( \frac{432}{24} = 18 \).

The resulting values of \( k \) are not constant so we can reject \( c = kv \) as a relationship.

If \( c = kv^2 \), then \( k = \frac{c}{v^2} \). However, \( \frac{128}{16^2} = \frac{1}{2} \), \( \frac{250}{20^2} = \frac{5}{8} \) and \( \frac{432}{24^2} = \frac{3}{4} \).

We therefore eliminate \( c = kv^2 \) as well.

With \( c = kv^3 \), we would have \( k = \frac{c}{v^3} \). Now \( \frac{128}{16^3} = \frac{1}{32} \), \( \frac{250}{20^3} = \frac{1}{32} \) and \( \frac{432}{24^3} = \frac{1}{32} \).

The constant of proportionality, \( k \), is thus \( \frac{1}{32} \).

ii) From the result in part i), the relationship between the speed \( v \) and the fuel consumption \( c \) is given by the formula \( c = \frac{1}{32} v^3 \).

Substituting \( v = 28 \) gives \( c = \frac{1}{32} \times 28^3 = 686 \).

\[ \therefore \text{The ship would consume 686 tonnes of fuel per day if it were travelling at its maximum speed of 28 knots.} \]