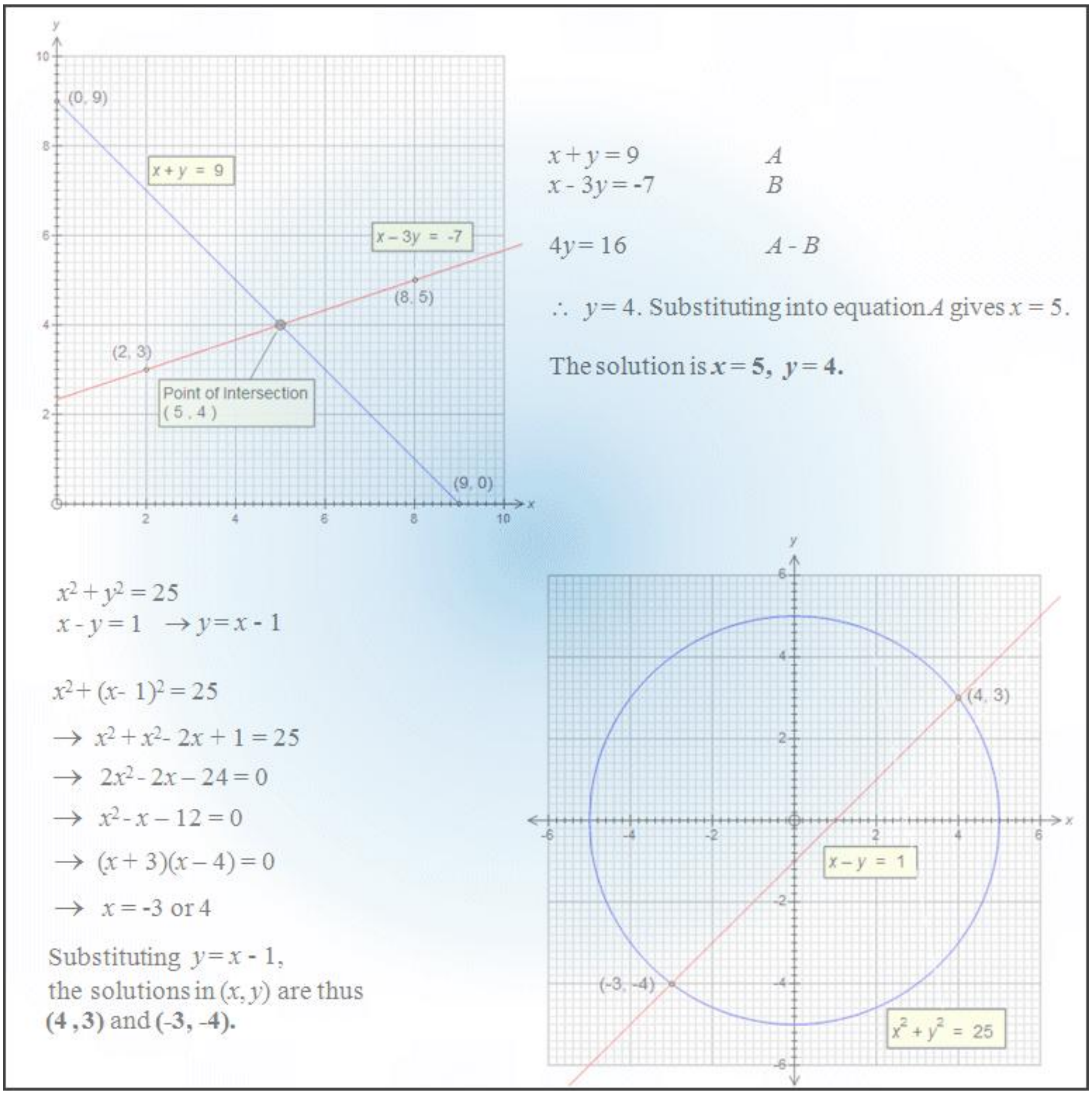
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SIMULTANEOUS EQUATIONS



SIMULTANEOUS EQUATIONS

Introduction.

We begin this section with a 'tea-time' problem.

Example (1) : An apple and a banana cost 50p in total, and the apple costs 10p more than the banana. Find the cost of each fruit.

We need to find two numbers that add to 50, but which also differ by 10.
This requirement can be rewritten in algebraic terms as a pair of equations.

If a is the cost of an apple, and b the cost of a banana, we can say

$a + b = 50$; $a - b = 10$.

What we have here is a pair of **simultaneous equations** in two variables, a and b.

Each equation has an infinite number of solutions by itself, but the combination has only one solution.

There are three methods of solving linear simultaneous equations.

Elimination method.

Returning to Example (1), we have the equations

$a + b = 50$ $\quad A$
$a - b = 10$ $\quad B$

If we were to add equations A and B, we would be left with

$2a = 60$ $\quad A+B$

Since $2a = 60$, $a = 30$, and therefore the apple costs 30p.
Because the apple and the banana cost 50p in total, the banana costs 20p.

What we have done here is eliminate the variable b by adding the two equations and deducing $a = 30$. Then, we substituted $a = 30$ into the equation $a + b = 50$ to obtain $b = 20$.

Example (2): A football league uses a rather peculiar points scoring system, whereby teams obtain w points for a win, d points for a draw and no points for a loss.

Dynamics have won 5, drawn 3 and lost 2 games and are on 31 points.
Locos have won 6, drawn 1 and lost 3 games and are on 32 points.

How many points does this league award for a win, and how many does it award for a draw ?

We begin by forming a pair of simultaneous equations out of the given data. Because a win is worth w points and a draw is worth d points, we can form the equation $5w + 3d = 31$ from Dynamics' statistics. Similarly, we can form the equation $6w + d = 32$ from Locos' statistics. We can ignore the lost games, as a loss is worth zero points here. Therefore :

$5w + 3d = 31$ *A*
$6w + d = 32$ *B*

We can multiply equation *B* by 3, and subtract equation *A* from the result :
$5w + 3d = 31$ *A*
$18w + 3d = 96$ 3*B*

$13w = 65$ 3*B* – *A* $\therefore$ $\boldsymbol{w = 5}$

Substituting 5 for w in the first equation gives $25 + 3d = 31$, and thus $3d = 6$ and $\boldsymbol{d = 2}$.
This league therefore awards 5 points for a win and 2 points for a draw.

Example (3): Use the elimination method to solve the simultaneous equations
$x + 4y = 2$; $x - 3y = -5$

$x + 4y = 2$ *A*
$x - 3y = -5$ *B*

It is possible to eliminate x by subtracting equation *B* from equation *A*.

$7y = 7$ *A*-*B*

This gives $y = 1$, and so the value could be substituted into either of the original equations.
Substituting into equation *A* gives $x + 4 = 2$, therefore $x = -2$.

The solution to these equations is therefore $x = -2$, $y = 1$.

Substitution method.

In this method, take one of the equations and express one variable in terms of the other.

Example (3a): Use the substitution method to solve the simultaneous equations:
$x + 4y = 2$ (A) ; $x - 3y = -5$ (B)

In equation B above,
$x - 3y = -5$ is the same as $x = 3y - 5$, so we substitute for x by replacing all instances of x with the expression $3y - 5$.

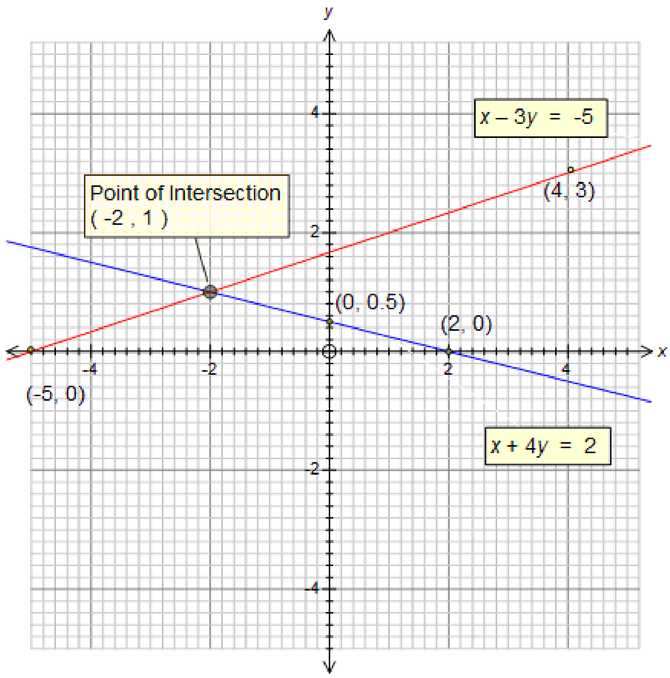
Equation A can thus be rewritten as $(3y - 5) + 4y = 2$, or $7y - 5 = 2$, hence $7y = 7$.

This gives $y = 1$ as in the elimination method, and substitution into equation A gives $x = -2$ as before.

Graphical interpretation.

This brings us to the graphical method of solving linear simultaneous equations, of which an example will be shown later in this document.

Example (3b): Use graphs to solve the simultaneous equations:
$x + 4y = 2$; $x - 3y = -5$



Plot the graphs of the two functions corresponding to the equations. The coordinates of the point of intersection give the solution.

$x + 4y = 2$ can be rewritten as $y = \frac{2 - x}{4}$.

Use points (2, 0) and (0, 0.5) (for example) .

Similarly $x - 3y = -5$ can be rewritten as $y = \frac{x + 5}{3}$.

Use points (-5, 0) and (4, 3) for example.

The point of intersection is (-2, 1), corresponding to $x = -2$, $y = 1$.

Example (4):

Use the elimination method to solve the simultaneous equations
$3x - 4y = 17$; $2x + 5y = -4$

$3x - 4y = 17$ *A*
$2x + 5y = -4$ *B*

We need to multiply both equations by a suitable number so that one of the variables can be eliminated. (The choice of *x* as the variable to eliminate is arbitrary - we could have used *y*.)

Note that the coefficients (i.e. the multiples) of *x* in the two equations are 2 and 3. We need the coefficients of *x* to equal the L.C.M . of 2 and 3, namely 6, in order to eliminate, hence the choice of multipliers. Note also that, because the coefficients have the same sign, we eliminate by subtraction.

$6x - 8y = 34$ 2*A*
$6x + 15y = -12$ 3*B*
$23y = -46$ 3*B* - 2*A*

This makes $y = -2$. Substituting into equation *A* gives $3x - (-8) = 17$, and thus $x = 3$.

$\therefore$ The solution is $\mathbf{x = 3, y = -2}$.

Example (4a): Repeat the last example, but eliminate *y* first.

Had we chosen to eliminate *y*, the working would have been

$3x - 4y = 17$ *A*
$2x + 5y = -4$ *B*

The L.C.M. of the coefficients of *y* is 20 (ignoring the signs), hence the multipliers. Also, since the signs are different, we eliminate by addition.

$15x - 20y = 85$ 5*A*
$8x + 20y = -16$ 4*B*
$23x = 69$ 5*A* + 4*B*

This makes $x = 3$. Substituting into equation *A* gives $9 – 4y = 17$, and thus $y = -2$.

$\therefore$ The solution is $\mathbf{x = 3, y = -2}$.

Example (5):

Use the substitution method to solve the simultaneous equations
$x + 2y = 7$; $3x + y = 1$

The second equation can be rewritten as $y = 1 - 3x$
and hence $1 - 3x$ substituted for y into the first equation, as
$x + 2(1 - 3x) = 7 \rightarrow 2 - 5x = 7 \rightarrow -5x = 5 \rightarrow x = -1$.

Substituting $x = -1$ into the first equation gives $y = 4$.

$\therefore$ The solution is $\mathbf{x = -1,\ y = 4.}$

Example (6):

Use graphs to solve the simultaneous equations
$x - 3y = -7$; $x + y = 9$.

The equations can be rewritten:

$x - 3y = -7 \rightarrow -3y = -7 - x$
$\rightarrow 3y = x + 7 \rightarrow y = \frac{x+7}{3}$.

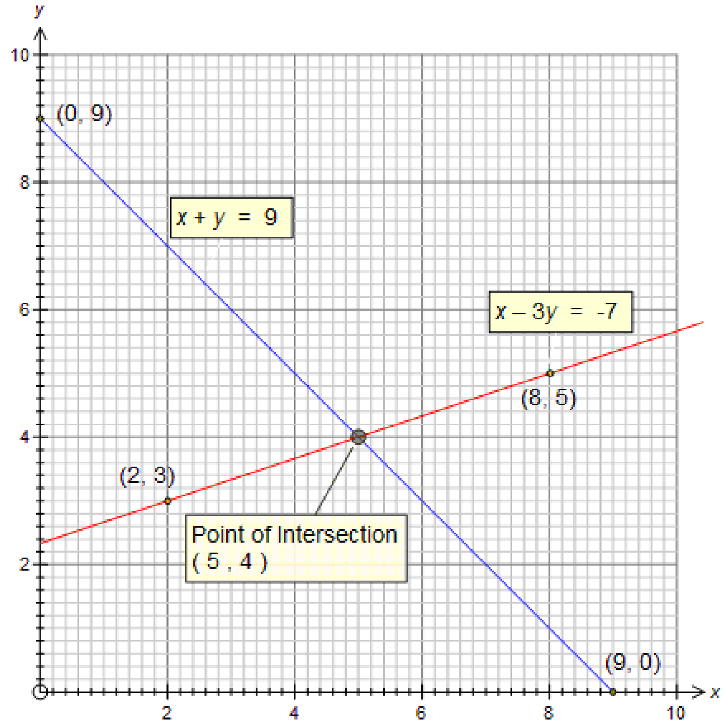
Two points to plot are (2, 3) and (8, 5).

$x + y = 9 \rightarrow y = 9 - x$

Two points to plot are (0, 9) and (9, 0)

The two graphs meet at the point (5, 4) , therefore the solution is

$\mathbf{x = 5, y = 4.}$



Note: Graphical methods are rarely used in solving linear simultaneous equations at higher tier, as they might give only approximated results when we have fractional solutions. By contrast, the methods of elimination and substitution lead to exact solutions.

Linear / Quadratic Simultaneous Equations.

The examples so far dealt with cases where both equations were linear.
Here we will look at what happens where one equation is linear but the other is a quadratic.

Example (7): Solve the simultaneous equations $y = x^2 - 3x + 4$ and $y = x + 1$.

This example is relatively simple because both equations have y solely on the left-hand side. We can use methods learnt when solving quadratic equations, as per the graphical examples of 'solving many quadratic equations from one'.

The original equations are

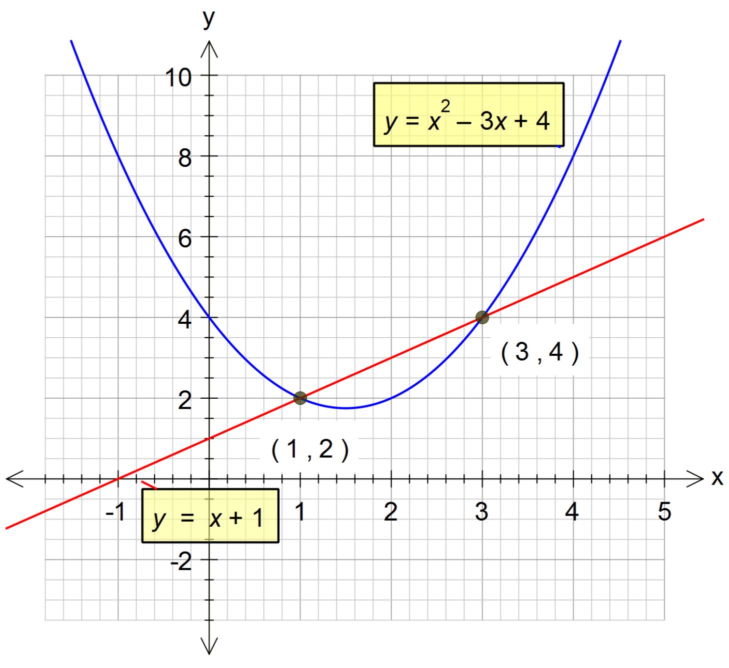
$y = x^2 - 3x + 4$
$y = x + 1$.

They therefore have a solution when $x^2 - 3x + 4 = x + 1$.
This rearranges to give a new quadratic, $x^2 - 4x + 3 = 0$.

Factorising, we have $(x - 3)(x – 1) = 0$
$\therefore$ $x = 3$ or $x = 1$.

To find the values of y corresponding to each value of x, we simply substitute into the linear equation $y = x + 1$. Hence, when $x = 1$, $y = 2$ and when $x = 3$, $y = 4$.

The graphical solution is shown below; the quadratic curve and the line intersect at (1, 2) and (3, 4) – the solutions in (x, y) to the simultaneous equations..



Example (8): Solve the simultaneous equations $x^2 + y^2 = 25$ and $x - y = 1$.

The first thing to notice is that it is impossible to use the elimination method here. You cannot add or subtract multiples of x from x^2 and eliminate x from the equations, neither can you do the same with y.

This leaves only the substitution method as a viable option. It is generally easier to manipulate the linear equation, as in the example below.

$x^2 + y^2 = 25$ *A*
$x - y = 1$ *B*

Rearrange equation *B* to give

$x^2 + y^2 = 25$ *A*
$x = 1 + y$ *B*
$\rightarrow y = x - 1$ *B*

Now substitute (x-1) for y in equation *A* :

$x^2 + (x - 1)^2 = 25$ *A*
$\rightarrow y = x - 1$ *B*

Equation *A* can be simplified:

$x^2 + (x - 1)^2 = 25$

$\rightarrow x^2 + x^2 - 2x + 1 = 25$ (expand)
$\rightarrow 2x^2 - 2x - 24 = 0$ (collect terms)
$\rightarrow x^2 - x - 12 = 0$ (take out common factor of 2)

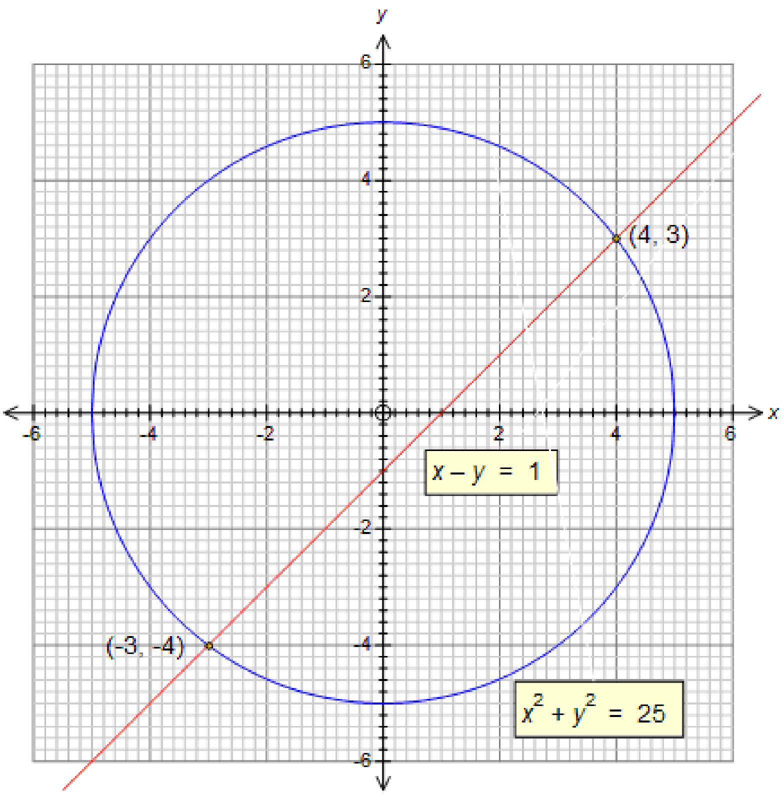
$\rightarrow (x + 3)(x - 4) = 0$ (factorise)

$\therefore x = -3$ or 4.

Having obtained x, the next step is to substitute the x-values back into equation *B* to find the y-coordinates.

This gives $y = -4$ when $x = -3$, and
$y = 3$ when $x = 4$.

The solutions in (x, y) are thus **(4 ,3)** and **(-3, -4).**



The graphical interpretation is shown above.

(N.B. $x^2 + y^2 = 25$ is the equation of a circle centred at the origin and a radius of 5 units.)

Graphical methods of solution are also common in exams. Typically, the quadratic would be shown on the graph, and the student would have to plot the linear graph .

Example (9): Solve the simultaneous equations

$x^2 - 4x - y = 0$; $2x - y = 5$

First we express y in terms of x in the second (linear) equation

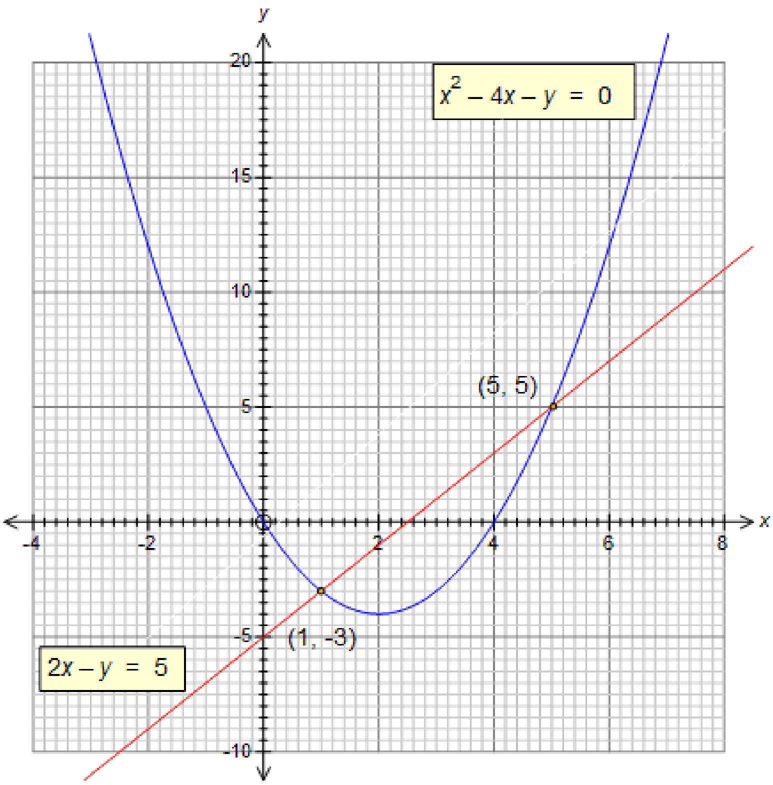
$x^2 - 4x - y = 0$ $\quad A$
$y = 2x – 5$ $\quad B$

Substitute $(2x – 5)$ for y in equation A:

$x^2 - 4x – (2x – 5) = 0$ $\quad A$
$\rightarrow x^2 - 6x + 5 = 0$ $\quad A$
$y = 2x – 5$ $\quad B$

Factorise the quadratic equation A:
$x^2 - 6x + 5 = 0 \rightarrow (x - 5)(x-1) = 0$
$\therefore$ $x = 1$ or $x = 5$.

Substituting for x in equation B, $y = -3$ when $x = 1$, and, $y = 5$ when $x = 5$.
$\therefore$ The solutions to the simultaneous equations are **(x, y)= (1, -3) and (5, 5).**



The solutions to the equations are shown where the straight line intersects the parabola - here it cuts it at the two points $(x, y) = (1, -3)$ and $(x, y) = (5, 5)$.

It also illustrates a way of graphically solving many quadratics from one.

The following examples will highlight two other possible scenarios.

Example (10): Solve the simultaneous equations $x^2 - 4x - y = 0$; $2x - y = 9$.

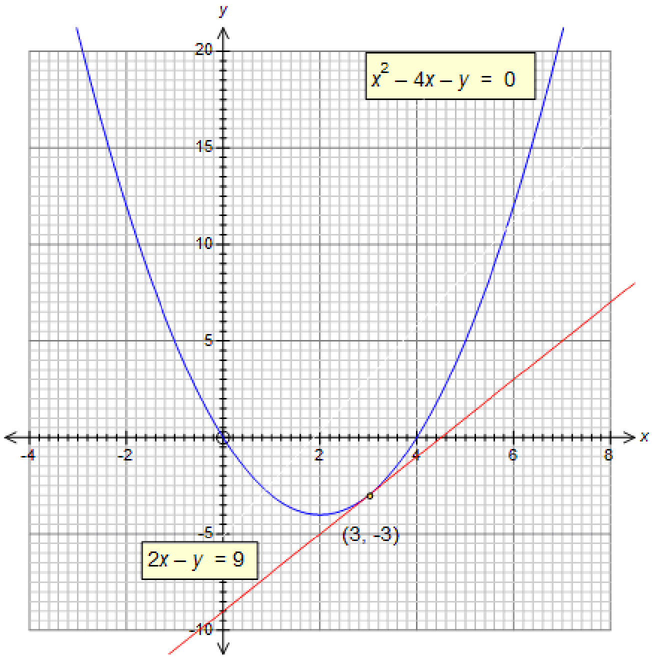
The working is similar to Example (8), but now $y = 2x – 9$.

$x^2 - 4x – (2x – 9) = 0$ *A*
$\rightarrow x^2 - 6x + 9 = 0$ *A*
$y = 2x – 9$ *B*

The resulting quadratic can also be factorised:
$x^2 - 6x + 9 = 0 \rightarrow (x - 3)^2 = 0 \therefore x = 3$.

Substituting for x in equation *B* , $y = -3$ when $x = 3$.
$\therefore$ The single solution to the simultaneous equations is $(x, y) =$ **(3, -3).**

The line $2x - y = 9$ now touches the parabola at only one point, the tangent at (3, -3).



Example (11): Solve the simultaneous equations $x^2 - 4x - y = 0$; $2x - y = 12$, using the general quadratic formula. What happens here?

The working is again similar to Example (10), but now $y = 2x – 12$.

$x^2 - 4x – (2x – 12) = 0$ *A*
$\rightarrow x^2 - 6x + 12 = 0$ *A*
$y = 2x – 12$ *B*

The clue in this question is that the quadratic cannot be factorised.

Substituting $a = 1$, $b = -6$ and $c = 12$ into the general formula gives $b^2 = 36$ and $4ac = 48$.

Crucially, $b^2 - 4ac < 0$ (i.e. $b^2 < 4ac$), and therefore the quadratic has no real roots and the simultaneous equations have no solution.

Graphically, the line $2x - y = 12$ misses the parabola completely.