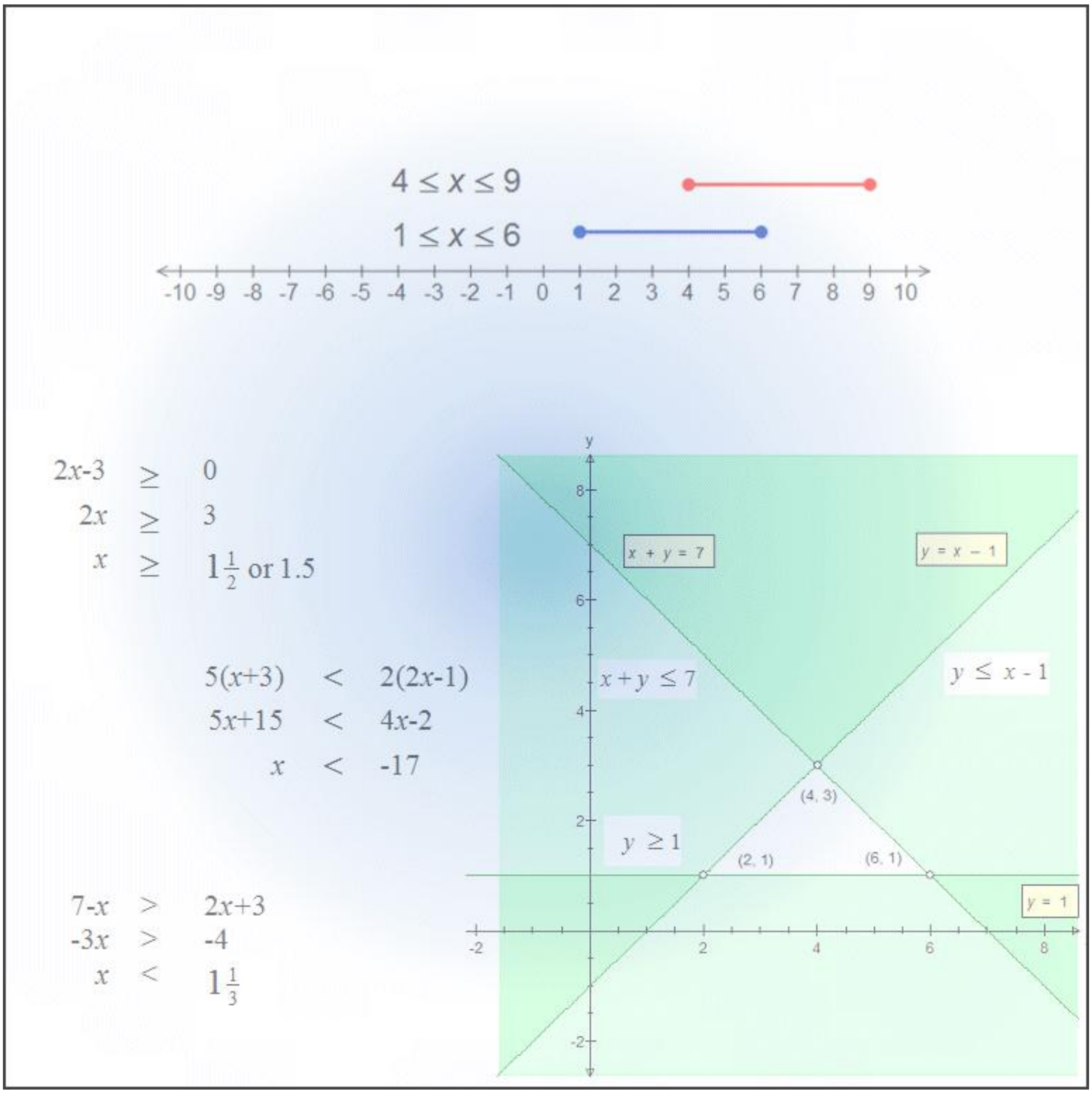


M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

INEQUALITIES



Version: 3.5 Date: 22-10-2015

Linear Inequalities.

Inequalities, like equations, connect two expressions involving definite unknown values.

An inequality must always include one of the following signs:
> (greater than)
< (less than)
≥ (greater than or equal to)
≤ (less than or equal to).

Inequalities involving < and > signs are known as **strict** inequalities.

Inequalities are solved in the same way as equations, but with an important difference in detail. The **sign must be reversed** when multiplying or dividing by a negative number, or when turning the inequality around.

<u>Inequalities in one variable.</u>

Examples (1): Solve the inequalities i) $5x < 20$; ii) $x-7 \leq -1$; iii) $2x - 3 \geq 0$

$5x < 20$

$5x$	$<$	20	
x	$<$	4	Divide both sides by 5

$x-7 \leq -1$

$x-7$	$\leq$	-1	
x	$\leq$	6	Add 7 to both sides

$2x - 3 \geq 0$

$2x-3$	$\geq$	0	
$2x$	$\geq$	3	Add 3 to both sides
x	$\geq$	$1\frac{1}{2}$ or 1.5	Divide both sides by 2

Examples (2):

Solve the inequalities i) $5(x+3) \geq 2(2x- 1)$; ii) $7-x > 2x+ 3$

$5(x+3) \geq 2(2x- 1)$

$5(x+3)$	$\geq$	$2(2x-1)$	
$5x+15$	$\geq$	$4x-2$	Expand both sides
x	$\geq$	-17	Subtract $15 + 4x$ from each side

$7-x > 2x+ 3$

The final step here involves dividing both sides of the inequality by a negative number.

$7-x$	$>$	$2x+3$	
$-3x$	$>$	-4	Subtract $(2x + 7)$ from each side
x	$<$	$1\frac{1}{3}$	Divide both sides by -3 and **REVERSE THE INEQUALITY SIGN**

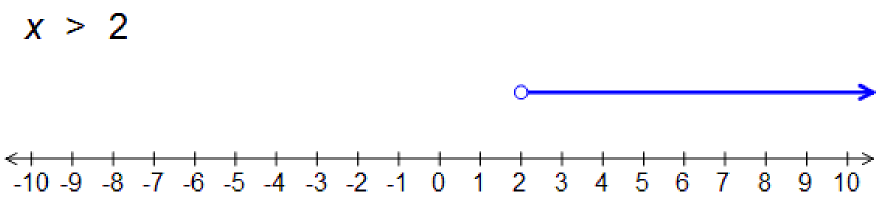
Inequalities on the number line.

Inequalities can be illustrated on the number line as follows:

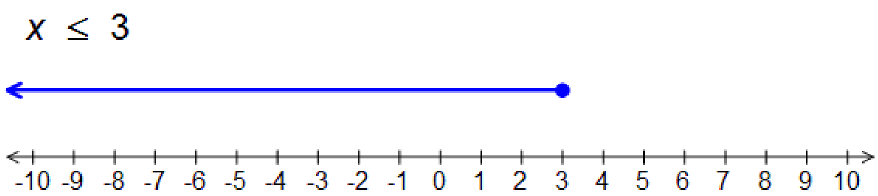
Examples (3): Illustrate the following inequalities on the number line: i) $x > 2$; ii) $x \leq 3$; iii) $-5 < x < 6$.

i) The inequality $x > 2$ is shown below.

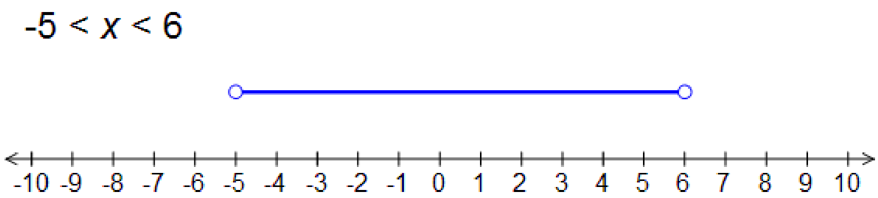
The arrow at the end is to signify that the range of x extends to $+\infty$; the **outline** circle at $x = 2$ is to show that the inequality is a strict one, and that $x = 2$ does not satisfy it..



ii) For $x \leq 3$, there are two differences from the last case. The arrow at the end goes towards $-\infty$; also the circle at $x = 3$ is shown as a **solid dot,** to show that the inequality is not a strict one, so that $x = 3$ does satisfy it.

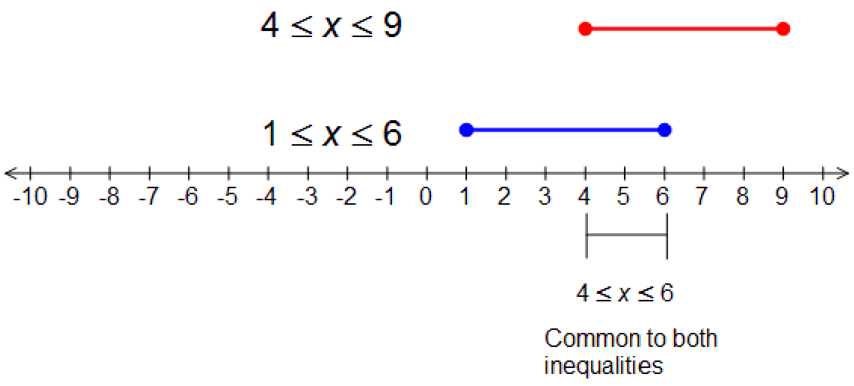


iii) The last two inequalities were open-ended, extending to infinity in one direction.
For $-5 < x < 6$, we now have two ends to the inequality, at –5 and 6. (This is a strict inequality, so the circles are shown outlined).



Example (4): Show the inequalities $1 \le x \le 6$ and $4 \le x \le 9$ on the same number line. Do any values of x satisfy both inequalities ?.

There is a region of overlap; when x is between 4 and 6 inclusive, both inequalities are satisfied. Note the solid dots; the inequality is not strict.



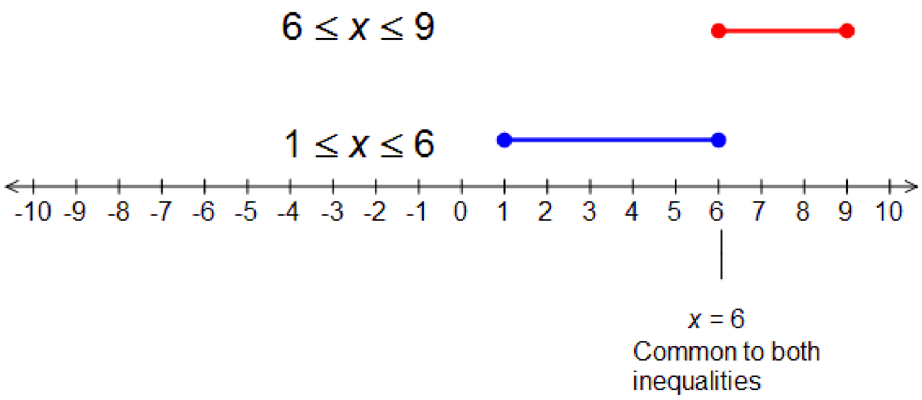
The set of values satisfying both inequalities is therefore $4 \le x \le 6$.

Example (5):

i) Show the inequalities $1 \le x \le 6$ and $6 \le x \le 9$ on the same number line. Do any values of x satisfy both inequalities ?. What if either inequality had been strict ?

ii) State all the integer values of x satisfying the inequality $6 \le x \le 9$.

i) The inequalities overlap at one point here, namely at $x = 6$, which is the only value of x satisfying both .
Had either inequality been strict, there would have been **no** value satisfying both, as at least one of them would not have included 6 in the allowable values for x.

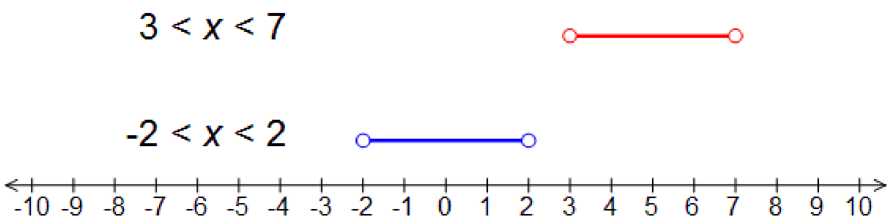


ii) The integer values of x satisfying the inequality $6 \le x \le 9$ are 6, 7, 8 and 9.

Example (6):

i) Show the inequalities -2 < x < 2 and 3 < x < 7 on the same number line. Do any values of x satisfy both inequalities ?

ii) State all the integer values of x satisfying **either** inequality.



The inequalities do not overlap at all , so no value of x satisfies both .

The integers satisfying -2 < x < 2 are -1, 0 and 1; those satisfying 3 < x < 7 are 4, 5 and 6.

The next example uses inequalities to solve a real-life problem.

Example (7): Two taxi firms operating from Manchester Airport offer the following tariffs :

Taxi-Fly offer a rate of £1.40 per every mile travelled.
Air-Cabs offer a rate of £1 per mile travelled plus a flat fee of £6 at the start.

i) Over what travel distance is Taxi-Fly cheaper than Air-Cabs ?
ii) What would be the distance and fare where both taxi firms offer the same price ?

i) First we set up the formulae and the inequality:
Taxi-Fly's rate (in £) can be expressed as $1.4x$, where x is the number of miles travelled.
Air-Cabs' rate (in £) can be expressed as $x + 6$, again where x is the number of miles travelled.

For Taxi-Fly's rate to be cheaper than Air-Cabs', we must solve the inequality $1.4x < x + 6$.

$1.4x < x + 6$

$1.4x$	$<$	$x + 6$	
$0.4x$	$<$	6	Subtract x from both sides
x	$<$	15	Divide both sides by 0.4

Taxi-Fly is therefore cheaper than Air-Cabs for journeys of less than 15 miles.

ii) We change the inequality into the equation $1.4x = x + 6$, and its solution is $x = 15$. The two taxi firms charge the same fare for journeys of exactly 15 miles, and substituting into either fare formula gives a fare of £ (1.4×15), or £$(6 + 15)$, namely £21.

Inequalities in two variables.

In earlier sections, we saw how inequalities in one variable were illustrated by line segments along the number line.

When we have two variables, inequalities are represented by **regions** in the *x*-*y* plane.
The method is to plot the graph of the corresponding equation, and then eliminate the region on the side of the line *not* satisfying the corresponding inequality.

When the inequality is strict (i.e. with > and < signs), then the graph of the equation is best shown as a dotted line. With non-strict inequalities (≥ and ≤ signs), then show the equation as a solid line graph.

Note **: In the examples below, the region *not* satisfying the inequality is shown shaded out**.

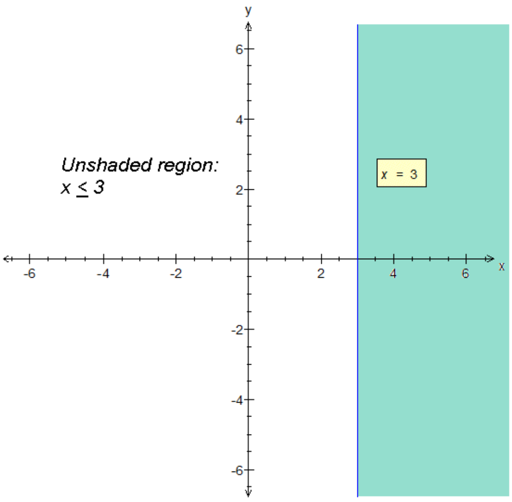
Some questions in exams might however use the opposite practice and shade the region that ***does*** satisfy the inequality. Always read the question carefully !

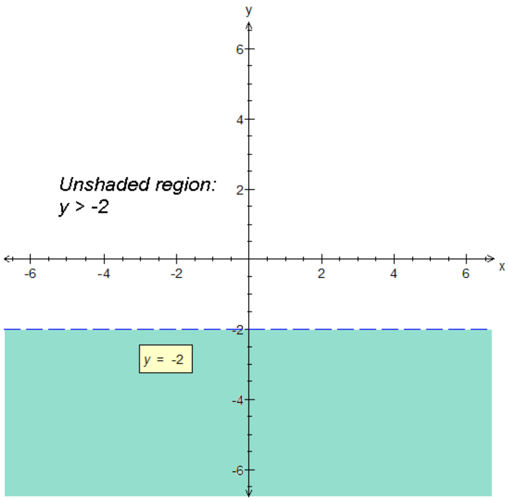
Examples (8): Sketch the following inequalities, by shading out the regions that do *not* satisfy them:

i) $x \leq 3$; ii) $y > -2$; iii) $-2 \leq x \leq 3$; iv) $1 \leq y \leq 4$ v) $y < x$; vi) $y \geq 2x + 3$.

i) Since all points to the right of the line $x = 3$ do not satisfy the inequality $x \leq 3$, that entire region can be shaded out and removed. The whole plane to the left of the line $x = 3$, and the line itself, satisfy the inequality (See diagram below left).

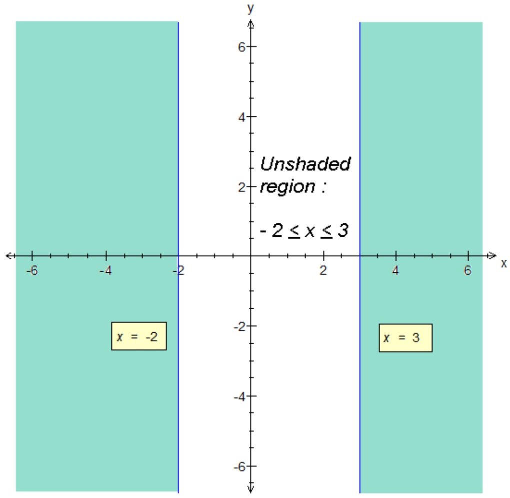
ii) Since all points below the line $y = -2$ do not satisfy the inequality $y > -2$, that entire region can be shaded out and removed. The whole plane above that line satisfies the inequality, but the line itself does not, for this is a strict inequality. (See diagram below right).

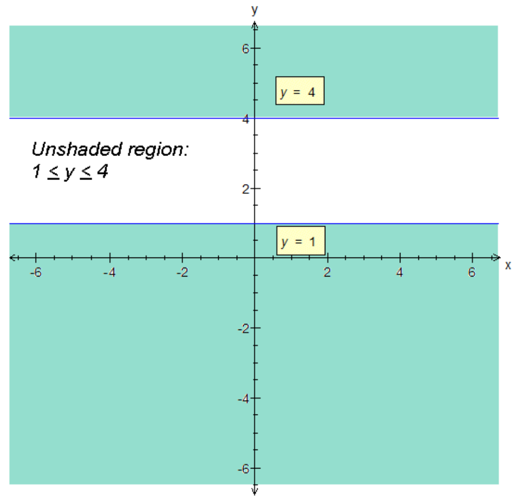




iii) This time the inequality is in a two-part form, and so we need to draw two lines, namely x = -2 and x = 3. The required region, also known as the feasible region, lies between the two vertical lines and includes them. (See diagram below left).

iv) Here we draw the lines y = 1 and y = 4. The feasible region lies between the two horizontal lines and includes them. (See diagram below right).



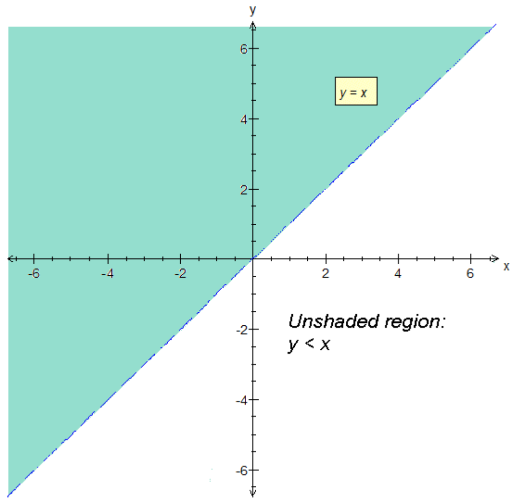


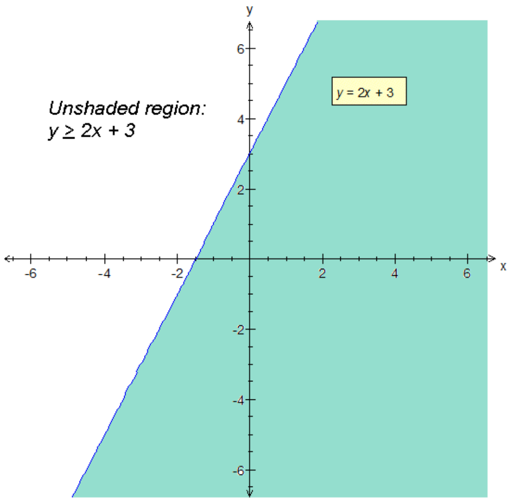
The next two cases involve lines which are not parallel to either of the axes, so a little more care is needed.

To determine which side of the line is to be rejected, we need to choose a random point on either side of the line. We then substitute the x and y coordinates of that point into the corresponding inequality. If the inequality statement holds true, then that point is in the region that we want to keep (the feasible region); if the inequality is false, then that point is in the region that we want to reject.

In v), the line $y = x$ is the boundary of the strict inequality $y < x$, and is shown as a dotted line. To determine which 'side' to reject, we choose a point on one side – say the point (2, 0). Since $0 < 2$, it means that the inequality $y < x$ holds true on that side of the graph. We therefore keep that side unshaded, and subsequently reject the area above and left of the line $y = x$, including the line itself.

In vi), the boundary is the line $y = 2x + 3$ and is shown solid here, since the corresponding inequality $y \geq 2x + 3$ is not strict. Choosing the origin as a test point and substituting $(x, y) = (0.0)$ into the inequality, we have $0 \geq 3$, which is a false statement. Hence we reject the area containing the origin, i.e. to the lower right of the line $y = 2x + 3$. (The line itself satisfies the inequality.)

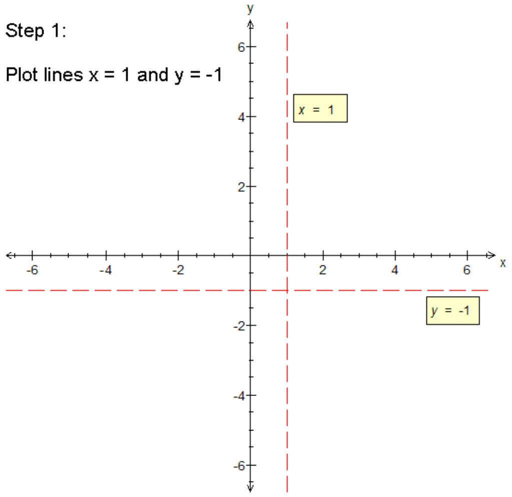


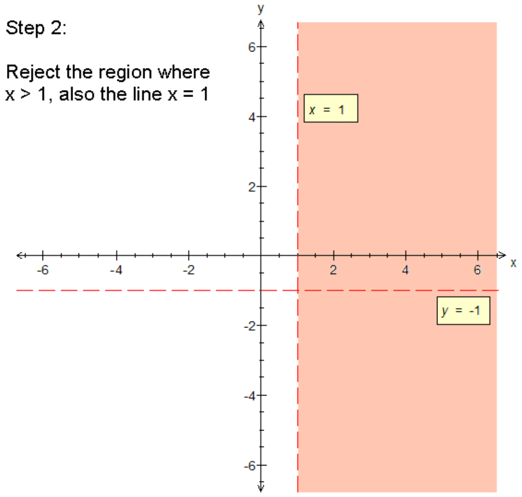


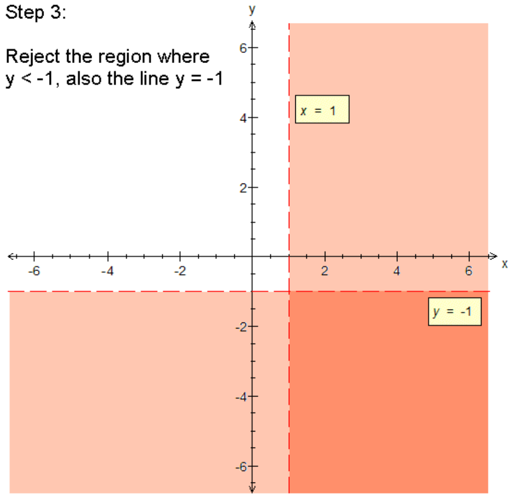
Inequalities can also be combined, as the following examples show.

Examples (9): Sketch the following inequalities, by shading out the regions that do ***not*** satisfy them:
i) $x < 1$ combined with $y > -1$; ii) $-1 \le x \le 2$ combined with $2 \le y \le 4$.

i) $x < 1$ combined with $y > -1$







Note the following:

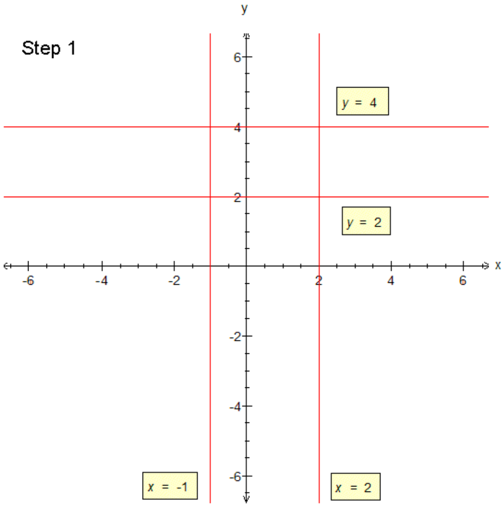
Since we are dealing with strict inequalities, lines $x = 1$ and $y = -1$ are shown dotted rather than solid.

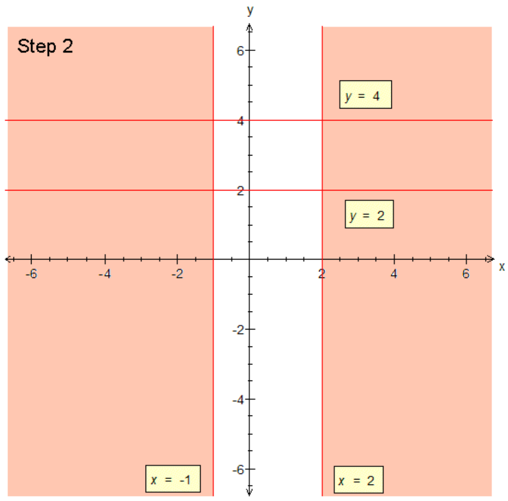
The order in which the regions are rejected is not important – we could have rejected the horizontal region first. This is true for all such exercises.

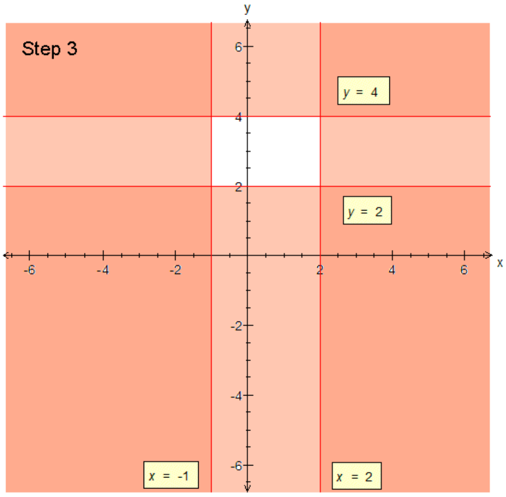
The region where **both** inequalities ‘fail’ is shown with darker shading, but this is not usually important for this type of question.

The feasible region satisfying **both** inequalities is still infinite in extent.

ii) $-1 \le x \le 2$ combined with $2 \le y \le 4$.







Step 1: Plot the lines
x = -1, x = 2, y = 2, y = 4

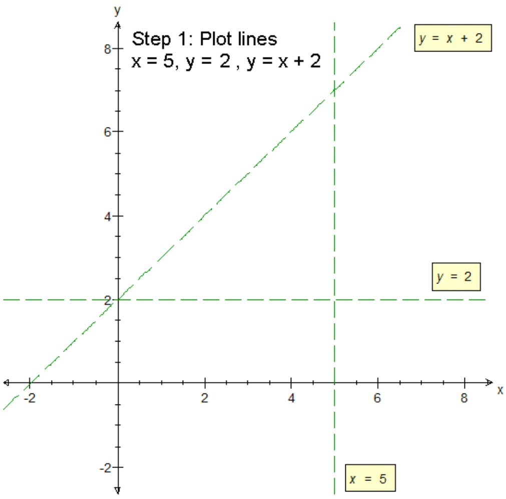
Step 2: Reject regions where
x < -1 and where x > 2

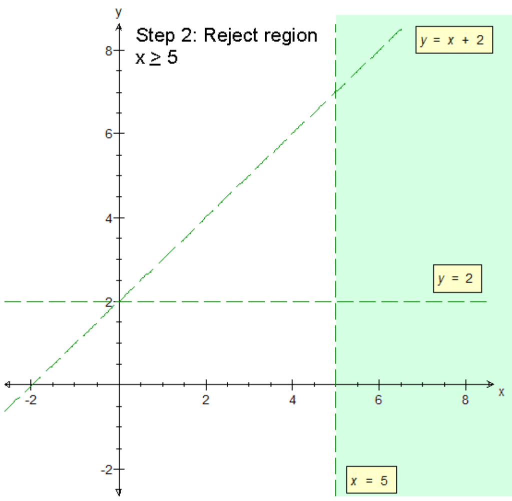
Step 3: Reject regions where
y < 2 and where y > 4

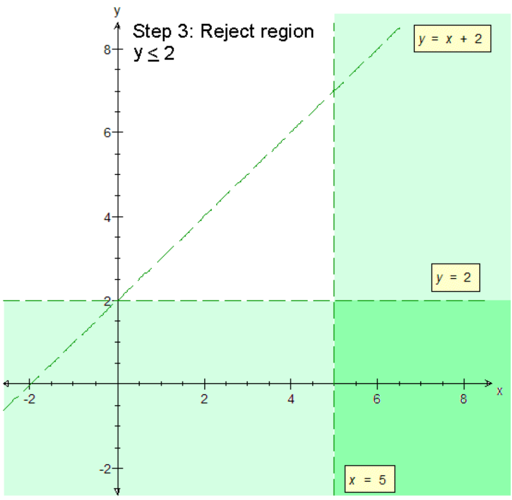
Here the inequalities are not strict, so all the boundary lines are shown solid.

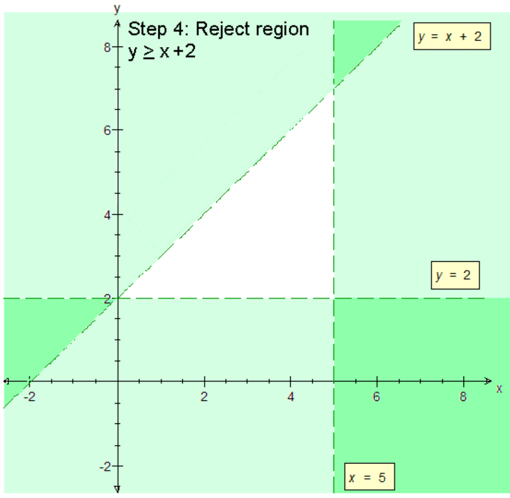
The feasible region satisfying **both** inequalities is now finite – it is the white rectangle (including edges) bounded by the two pairs of parallel lines, and its vertices are at (-1, 2), (2, 2), (-1, 4) and (2, 4).

Example (10): Show graphically the region bounded by the following three inequalities:
$x < 5$; $y > 2$; $y < x + 2$.
In addition, give the coordinates of all points (p, q) inside this feasible region where p and q are integers.









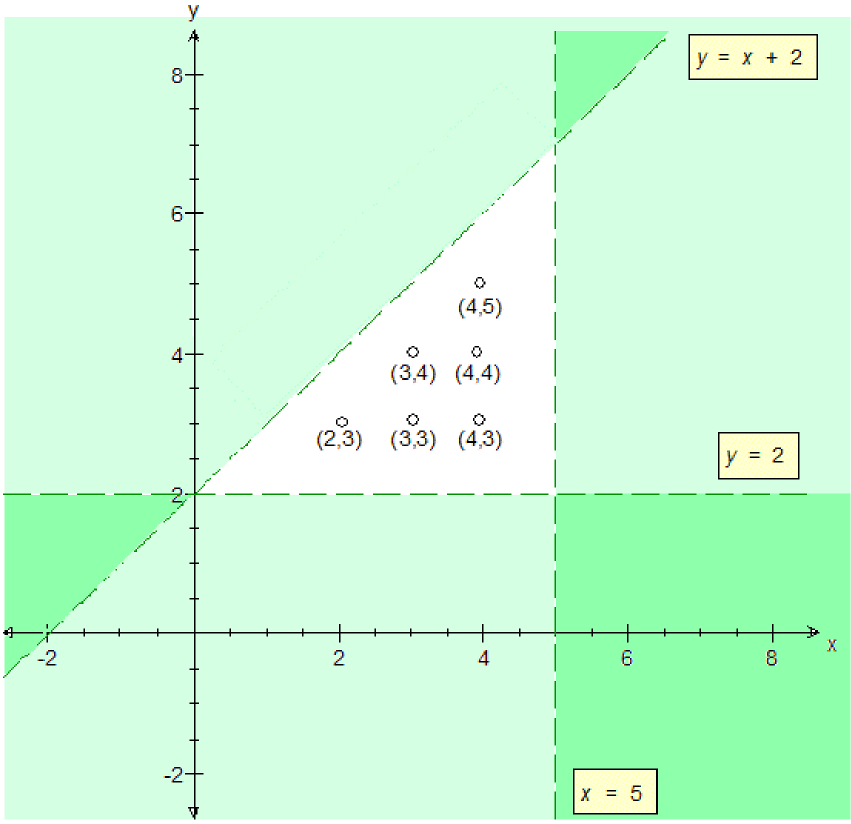
Since we are dealing with strict inequalities, the lines $x = 5$, $y = 2$ and $y = x + 2$ are shown dotted rather than solid.

It is easy enough to determine which regions to reject in Steps 2 and 3.

For Step 4, we can choose the origin as a test point for the inequality $y < x + 2$.

When $(x, y) = (0, 0)$, then we have $0 < 2$, which is a true statement.
We therefore keep the region on that side of the line $y = x + 2$, (i.e. below and right of the line) unshaded, and reject and shade the opposite one (above and left).

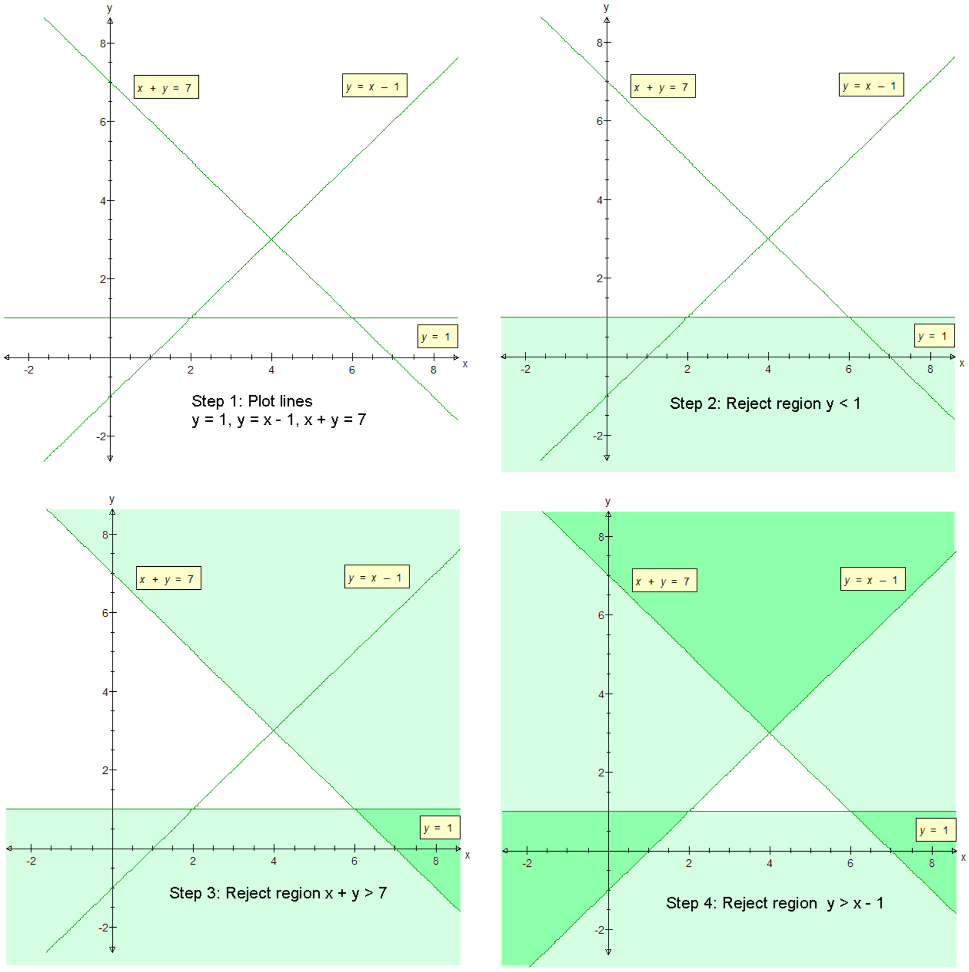
It now only remains to give the coordinates of the points (p, q) inside the triangular region remaining , and where p and q are integers.



The edges of the triangular region do not count (strict inequalities !), and so we have only the six points highlighted in the right-hand diagram.

Example (11): Show graphically the region bounded by the following three inequalities:
$y \geq 1$; $x + y \leq 7$; $y \leq x - 1$.

In addition, give the coordinates of all the vertices of the feasible region.



The inequalities are not strict here; the lines $y = 1$, $x + y = 7$ and $y = x - 1$ are shown solid.

For Steps 3 and 4, we choose the origin as a test point for the inequalities $x + y \leq 7$ and $y \leq x - 1$.

When $(x, y) = (0, 0)$, substituting into $x + y \leq 7$ gives $0 + 0 \leq 7$ which is true. We therefore *keep* the region to the side of the line $x + y = 7$ where the origin lies, and *reject* the opposite side.

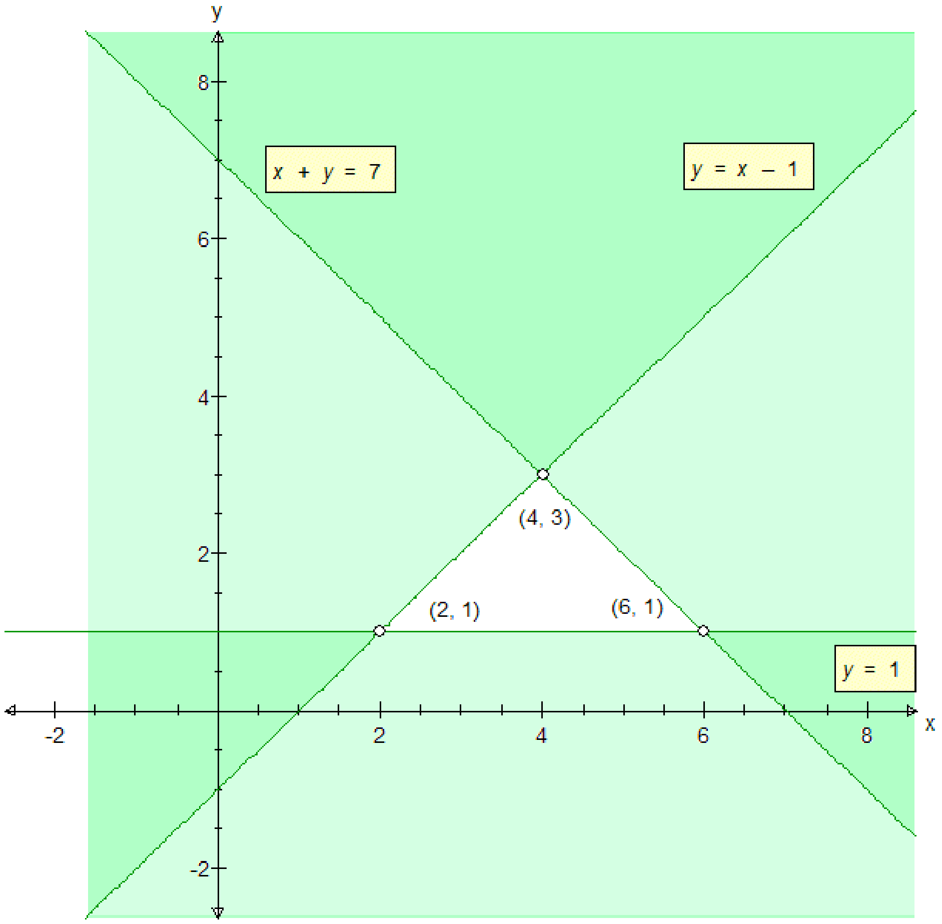
Again, when $(x, y) = (0, 0)$, substituting into $y \leq x - 1$ gives $0 \leq -1$ which is false. Hence we *reject* the region to the side of the line $y = x - 1$ where the origin lies, and *keep* the opposite side.

To find the coordinates of the vertices of the resulting triangular feasible region, we solve three pairs of simultaneous equations;

$x + y = 7$ and $y = 1$:
Substituting $y = 1$ into the first equation gives $x = 6$, so the intersection of those lines is (6, 1) .

$y = x - 1$ and $y = 1$:
Substituting $y = 1$ into first equation gives $x = 2$, so the intersection of those lines is (2, 1) .

$x + y = 7$ and $y = x - 1$:
Substituting $y = x - 1$ into first equation gives $2x - 1 = 7$,
so $x = 4$ and $y = 3$ and the intersection of those lines is (4, 3) .



Example (12): Find the set of inequalities corresponding to the feasible unshaded triangular region.

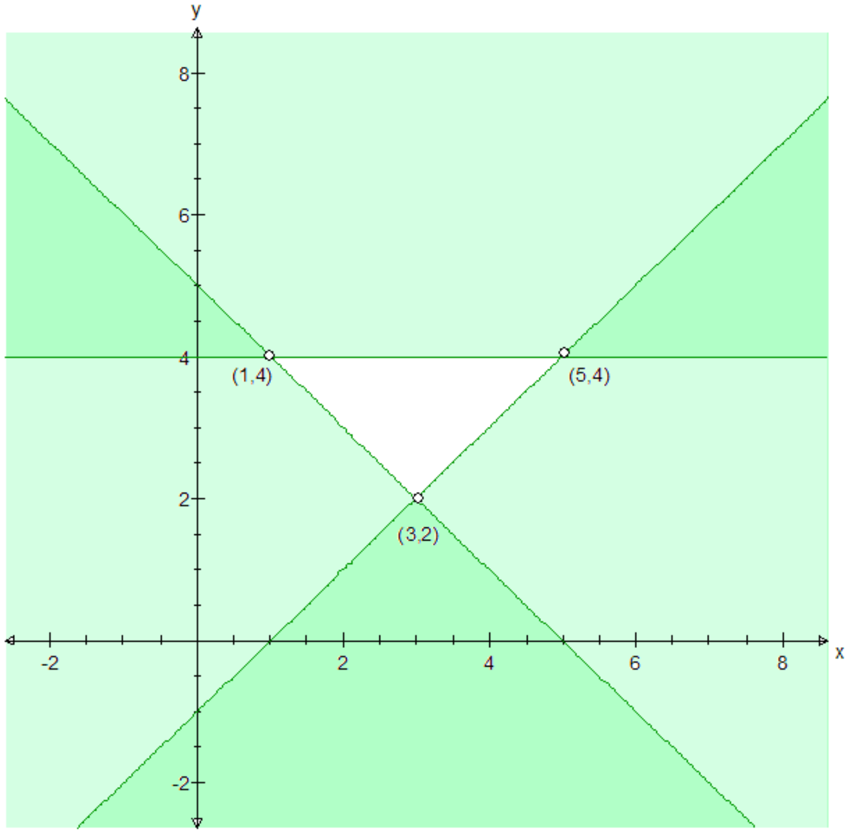
The line through (1,4) and (5,4) is parallel to the x-axis, so its equation is $y = 4$.

The line passing through (1,4) and (3,2) has a y-intercept at (0, 5) and a gradient of
$\frac{2-4}{3-1} = \frac{-2}{2} = -1$.

Its equation is thus $y = -x + 5$ (or $y = 5 - x$).

The line passing through (3,2) and (5,4) has a y-intercept at (0, -1) and a gradient of $\frac{5-3}{4-2} = \frac{2}{2} = 1$.

Its equation is thus $y = x - 1$.



Having found the equations, we then determine the corresponding inequalities (none of them strict, since the lines are all solid).

Line $y = 4$:

Since the area *above* the line is shaded out and rejected, the corresponding inequality is $y \leq 4$.

Line $y = 5 - x$:

The origin (0,0) is on the rejected side here, so we need to find which of $y \leq 5 - x$ and $y \geq 5 - x$ is false for $(x,y) = (0,0)$;

here it is $y \geq 5 - x$, as the statement $0 \geq 5$ is false.

The corresponding inequality is therefore
$y \geq 5 - x$.

Line $y = x - 1$:

The origin (0,0) is on the desired side here, so we need to find which of $y \leq x - 1$ and $y \geq x - 1$ is true for $(x,y) = (0,0)$. Here it is $y \geq x - 1$, as the statement $0 \geq -1$ is true.

The corresponding inequality is therefore $y \geq x - 1$.

$\therefore$ the feasible triangular region is bounded by the inequalities $y \leq 4$, $y \geq 5 - x$ and $y \geq x - 1$.

Quadratic Inequalities. (IGCSE only) (GCSE from 2017)

Quadratic inequalities are not unlike quadratic equations, but with a few traps within.

The quadratic equation $x^2 - 16 = 0$, rewritten as $x^2 = 16$, has two solutions, namely $x = 4$ and $x = -4$. Two corresponding inequalities are $x^2 < 16$ and $x^2 > 16$.

Examples (13) : Solve these quadratic inequalities: i) $x^2 < 16$; ii) $x^2 > 16$

i) We might be tempted to simply take square roots of both sides of the expression $x^2 < 16$ to obtain $x < 4$, but we see that -5 is less than 4 but its square is 25, which is greater than 16.

The actual solution set for x is $-4 < x < 4$.
(The general solution for the inequality $x^2 < a^2$ is in fact $-a < x < a$.)

ii) We might again be tempted to take square roots to get $x > 4$, but we see that -5 is less than 4 but its square is 25, which is greater than 16.

The actual solution set for x consists of two separate sets: $x > 4$ or $x < -4$.
(The general solution for the inequality $x^2 > a^2$ is in fact $x < -a$ or $x > a$.)

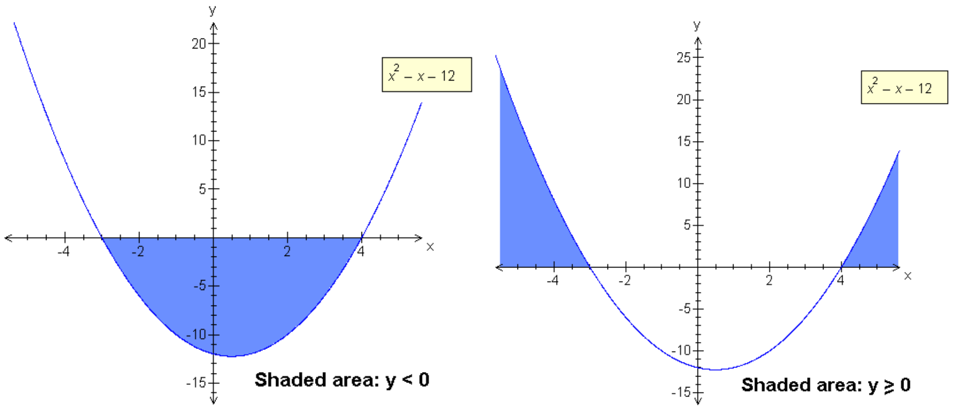
The last result cannot be combined into a single inequality: an expression like $-a > x > a$ is nonsense.

To solve a trickier quadratic inequality, the procedure is to first solve the corresponding equation, followed by sketching the corresponding graph and selecting the required regions.

Examples (14) : Solve the quadratic inequalities a) $x^2 - x - 12 < 0$; b) $x^2 - x - 12 \geq 0$.

The corresponding equation factorises into $(x+3)(x-4) = 0$, giving roots of -3 and 4.

Sketches of the graphs show that there are two different types of solution.



The solution to a) is shown as a single region in the graph on the left, i.e. where the graph passes below the x-axis. All values of x between -3 and 4 satisfy the inequality $x^2 - x - 12 < 0$. The end values of -3 and 4 do not, since this is a strict inequality.

$\therefore$ The solution of $x^2 - x - 12 < 0$ is therefore $-3 < x < 4$.

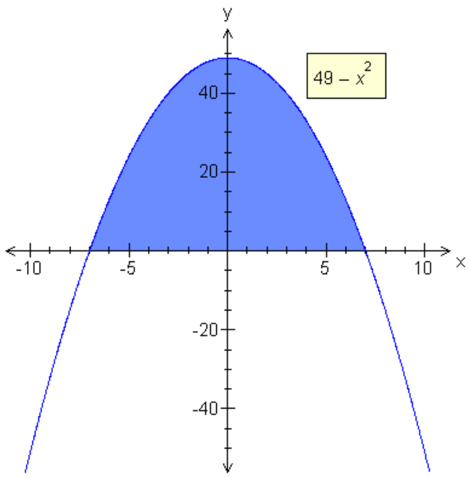
The solution to b) occurs in two regions as shown in the graph on the right, i.e. where the graph lies above the x-axis. This time all values of x less than or equal to -3 satisfy the inequality $x^2 - x - 12 \geq 0$, as do all values of x greater than or equal to 4. (Note that this is not a strict inequality).

$\therefore$ The solution of $x^2 - x - 12 \geq 0$ is therefore $x \leq -3$ or $x \geq 4$.

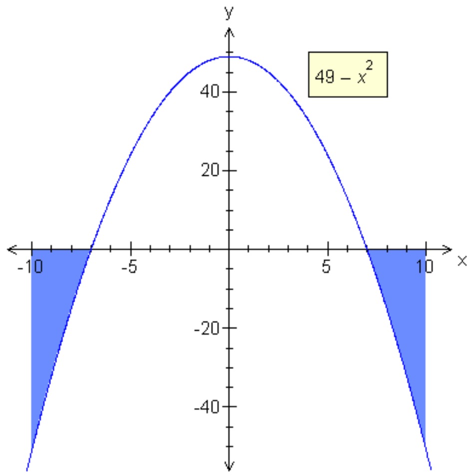
Examples (15) : Solve the quadratic inequalities a) $49 - x^2 > 0$; b) $49 - x^2 \leq 0$.

The corresponding equation is a difference of two squares, factorising into $(7 + x)(7 - x) = 0$, giving roots of –7 and +7.

(This time, the sketch will show an upturned graph since the term in x^2 is negative.)



Shaded area: y > 0



Shaded area: y ≤ 0

The solution to a) is shown on the left. All values of x between –7 and 7 (excluding –7 and 7 themselves) satisfy the strict inequality $49 - x^2 > 0$.

$\therefore$ The solution of $49 - x^2 > 0$ is therefore $-7 < x < 7$.

The solution to b) is shown on the right. This time all values of x less than or equal to -7 satisfy the inequality $49 - x^2 \leq 0$, as do all values of x greater than or equal to 7.

$\therefore$ The solution of $49 - x^2 \leq 0$ is therefore $x \leq -7$ or $x \geq 7$.

We could have rewritten inequality a) as $x^2 < 49$ and b) as $x^2 \geq 49$ and obtained the results using the quicker method in Example (13).