M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

INTRODUCTION TO ALGEBRAIC PROOF

Term No.	Value
1	p
2	q
3	$p+q$
4	$p+2q$
5	$2p+3q$
6	$3p+5q$
7	$5p+8q$
8	$8p+13q$
9	$13p+21q$
10	$21p+34q$
Sum of 10	$\mathbf{55p+88q}$

$$\begin{aligned}(2k+1)^2 &= 4k^2+4k+1\\ &= 4(k^2+k)+1.\end{aligned}$$

$$(n+5)^2-(n+1)^2$$
$$=(n^2+10n+25)-(n^2+2n+1)$$
$$=8n+24=8(n+3).$$

x	x^2+x+11
1	13
2	17
3	23
4	31
5	41
6	53

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Introduction to Algebraic Proof.

This section is a brief introduction to on how to prove mathematical conjectures using algebra.

The following methods are the ones normally encountered at GCSE :

- Proof by algebraic reasoning
- Proof by exhaustion
- Disproof by counterexample

The important thing about proving a conjecture is that 'every step must be justified'.

Example (1) : Prove that the product of three consecutive positive integers is a multiple of 6.

We cannot just say "$1 \times 2 \times 3 = 6$, $2 \times 3 \times 4 = 24$, $3 \times 4 \times 5 = 60$. It works for those examples, so it works in all cases." We need to be more rigorous than that !

i) We can take sequences of three consecutive positive integers such as 1-2-3, 2-3-4 and 3-4-5 and see that at least one of them must be even and that one is a multiple of 3.

ii) The product must therefore have factors of both 2 and 3. The L.C.M. of 2 and 3 is 6, since 2 and 3 have no common factor.

$\therefore$ The product of three consecutive positive integers is a multiple of 6.

Is this proof ? No, as far as statement i) is concerned. Looking at the number sequences and observing a pattern is not rigorous enough to justify the argument. Statement ii) is rigorous and concise, and thus can be left as it is when rewriting the 'proof'.

<u>True proof:</u>

To prove that the product of three consecutive positive integers is a multiple of 6 we must choose the general case of three consecutive positive integers: k, k+1 and k+2 and use the properties of division of integers.

First we check for the presence of even integers. The integer k can either be even (have no remainder on dividing by 2) or odd (have remainder of 1 on dividing by 2).

If k is even, then the product is even (is a multiple of 2). If k is odd , then $k + 1$ will be even because the sum of two odd numbers is even, and so the product will be even due to the k+1 term.

By similar logic, the integer k can have three possible remainders when it is divided by 3.
It can be a multiple of 3 (no remainder), or will have a remainder of 1 or 2.

If dividing k by 3 leaves a remainder of 2, then $k + 1$ will be a multiple of 3; if dividing k by 3 leaves a remainder of 1, then $k + 2$ will be a multiple of 3.

Therefore there will always be one number of the sequence divisible by 3.

The product must therefore have factors of both 2 and 3. The L.C.M. of 2 and 3 is 6, since 2 and 3 have no common factor.

$\therefore$ The product of three consecutive positive integers is a multiple of 6 (the L.C.M. of 2 and 3).

The proof in Example 1 used a combination of exhaustion and mathematical reasoning. Here are a few more examples with full working:

Proof by algebraic reasoning.

This uses mathematical logic and uses well-established results to prove a conjecture or a theorem.

Example (2): The "Fibonacci" sequence is defined by the following rules:

- The first two terms are 1 and 1.
- Each subsequent term is generated by adding together the two previous ones.

The first ten terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 .

A curious fact is that the seventh term of the sequence is 13, and the sum of the first ten terms is

1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143, which is 11 times the seventh term..

This might seem unremarkable, but if we generate a similar sequence (aka a Lucas sequence) with any starting numbers other than 1 and 1, the sum of the first ten terms is still 11 times the seventh number !

Take the sequence 1, 3, 4, 7, 11, 18, 29, 47, 76, 123. The seventh term is 29, and the sum of the first ten terms is 11 times 29, or 319 !

Prove that this rule holds true for all generalised Fibonacci sequences !

Call the first two terms of such a sequence p and q.

The table shows the first ten terms, along with their total sum :

Term No.	Value
1	p
2	q
3	$p + q$
4	$p + 2q$
5	$2p + 3q$
6	$3p + 5q$
7	$5p + 8q$
8	$8p + 13q$
9	$13p + 21q$
10	$21p + 34q$
Sum of 10	$\mathbf{55p + 88q}$

The seventh term is $5p + 8q$ and the sum of the ten terms is $55p + 88q = 11(5p + 8q)$.

Hence the sum of the first 10 terms of any generalised Fibonacci (Lucas) sequence is always 11 times the seventh term.

Example (3): By looking at the square numbers of 9, 25, 49 and 81, it can be seen that they all leave a remainder of 1 when divided by 4.

Prove that this fact holds true for all odd square numbers greater than 1.

Any odd number greater than 1 can be expressed as $2k + 1$ where k is a positive integer.
Squaring, we have $(2k + 1)^2 = 4k^2 + 4k + 1$.
Subtracting the remainder of 1 gives us $4k^2 + 4k$ which factorises into $4(k^2 + k)$

Hence $(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
This right-hand expression is a multiple of 4 plus a remainder of 1.

Example (4): Prove that $(n + 5)^2 – (n + 1)^2$ is a multiple of 8 for all integers n.

Here we must expand and simplify the expression, and then take out a factor of 8.

Proof:

$(n + 5)^2 – (n + 1)^2 = (n^2 + 10n + 25) – (n^2 + 2n + 1) = 8n + 24 = 8(n + 3)$.
The right-hand expression is evidently a multiple of 8.

Proof by exhaustion.

This uses exhaustive testing when the set of results to be tested is finite. This is often used together with mathematical reasoning.

Example (5): Prove that no square number ends in 2, 3, 7 or 8.

We begin by taking the squares of the integers from 0 to 9; they are 0, 1, 4, 9, 16, 25, 36, 49, 64 and 81.

Next, we can express any integer greater than 10 as $10m + n$ where m and n are integers, $m > 0$ and $0 \le n \le 9$.

Squaring $10m + n$ gives $100m^2 + 20mn + n^2 = 10m(10m + 2n) + n^2$.

The terms involving m are divisible by 10, and so the value of m will have no effect on the last ('units') digit in the square, i.e. $(10m + n)^2$ ends in the same digit as n^2.

(For example, the squares of 17, 27, 37,.... end in 9 because the square of 7 does so).

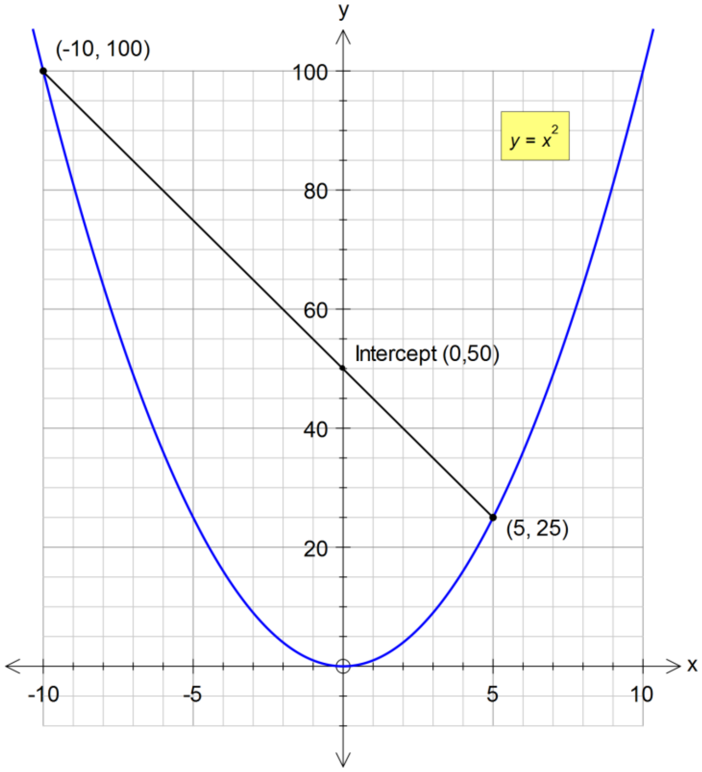
The squares of the integers from 0 to 9 have a 'units' digit of 0, 1, 4, 5, 6 and 9 – there are none ending in 2, 3, 7 or 8.

$\therefore$ no square number ends in 2, 3, 7 or 8.

Example (6): The graph of $y = x^2$ is shown on the right, along with the chord joining the points (-10,100) and (5, 25). The y-intercept of the chord can be seen to be 10 × 5, or 50.

If the chord were to connect (-6,36) to (8,64), then the y-intercept would be 6 × 8, or 48.

Prove algebraically that any general chord connecting $(-a, a^2)$ and (b, b^2) intersects the y-axis at $(0, ab)$.
(Note that the square of $-a$ is equal to the square of a).



We need to find the equation of the chord connecting $(-a, a^2)$ and (b, b^2) .

Its gradient is $\dfrac{b^2 - a^2}{b - (-a)}$

$$= \frac{(b+a)(b-a)}{b+a} = b - a.$$

Its general equation is $y = mx + c$ where m is the gradient and c is the y-intercept.

Substituting (b, b^2) for (x, y) we have

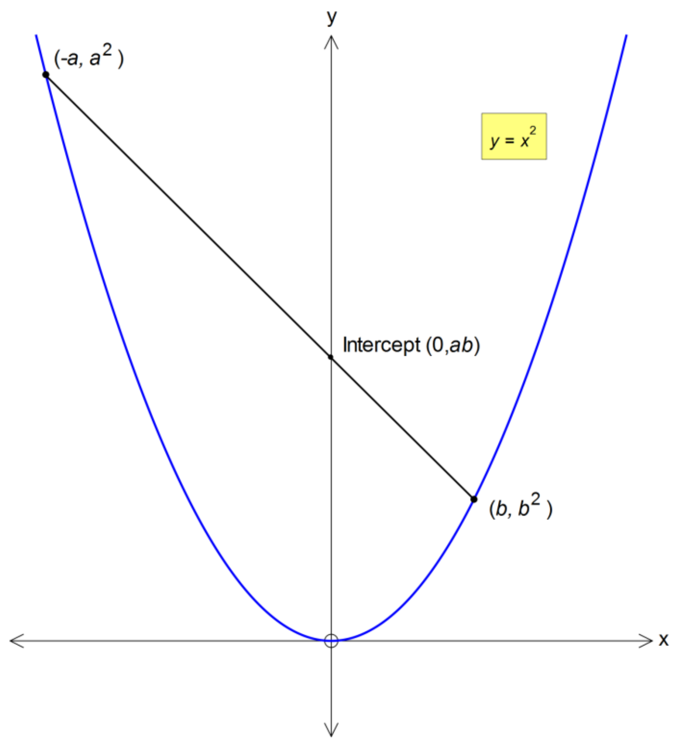
$b^2 = (b-a)b + c$ and thus $b^2 = b^2 - ab + c$.

Subtracting b^2 from both sides and rearranging,

$c - ab = 0$ and thus $c = ab$.

The equation of the chord is therefore
$y = (b-a)x + ab$.

Hence when $x = 0$, $y = ab$.



In other words, the y-intercept of the chord connecting $(-a, a^2)$ and (b, b^2) is always $(0, ab)$.

Disproof by counterexample.

When we had to prove conjectures to be true in the earlier examples, we had to use exhaustive and rigorous methods. Disproving a conjecture is easier - a single counterexample suffices.

Example (7): Carl states :

"The square of any number less than 1 is less than the number itself"

i) Prove that Carl's statement is incorrect as it stands.
ii) How can Carl's statement be made correct by just adding one word ?

i) At first, Carl's reasoning seems sound enough. If we square one half, we obtain a quarter, and when we square 0.1, we have 0.01.

However, we run into difficulties when we consider zero or negative numbers:

The number -2 is less than 1, but its square is 4, which is greater than -2.
The square of 0 is 0.

ii) Carl's statement can be corrected to read

"The square of any **positive** number less than 1 is less than the number itself".

Example (8): Prem has tabulated the values of $x^2 + x + 11$ for the first few positive integers:

x	$x^2 + x + 11$
1	13
2	17
3	23
4	31
5	41
6	53

He notices that all the calculated values are prime, and conjectures :
"The value of $x^2 + x + 11$ is prime for all positive x."

Prove whether Prem's conjecture is true or false

Substituting values x = 7, 8 and 9 gives $x^2 + x + 11$ = 67, 83, 101... , which are all prime.
Unfortunately if x = 10 , then $x^2 + x + 11$ = 121 which is not a prime, it being the square of 11.

Hence Prem's conjecture is false.

Alternatively, we could have substituted x = 11 to obtain $11^2 + 11 + 11$, which can have the factor of 11 taken out to give = 11(11 + 1 + 1) or $11 \times 13 = 143$ – again disproving the conjecture.