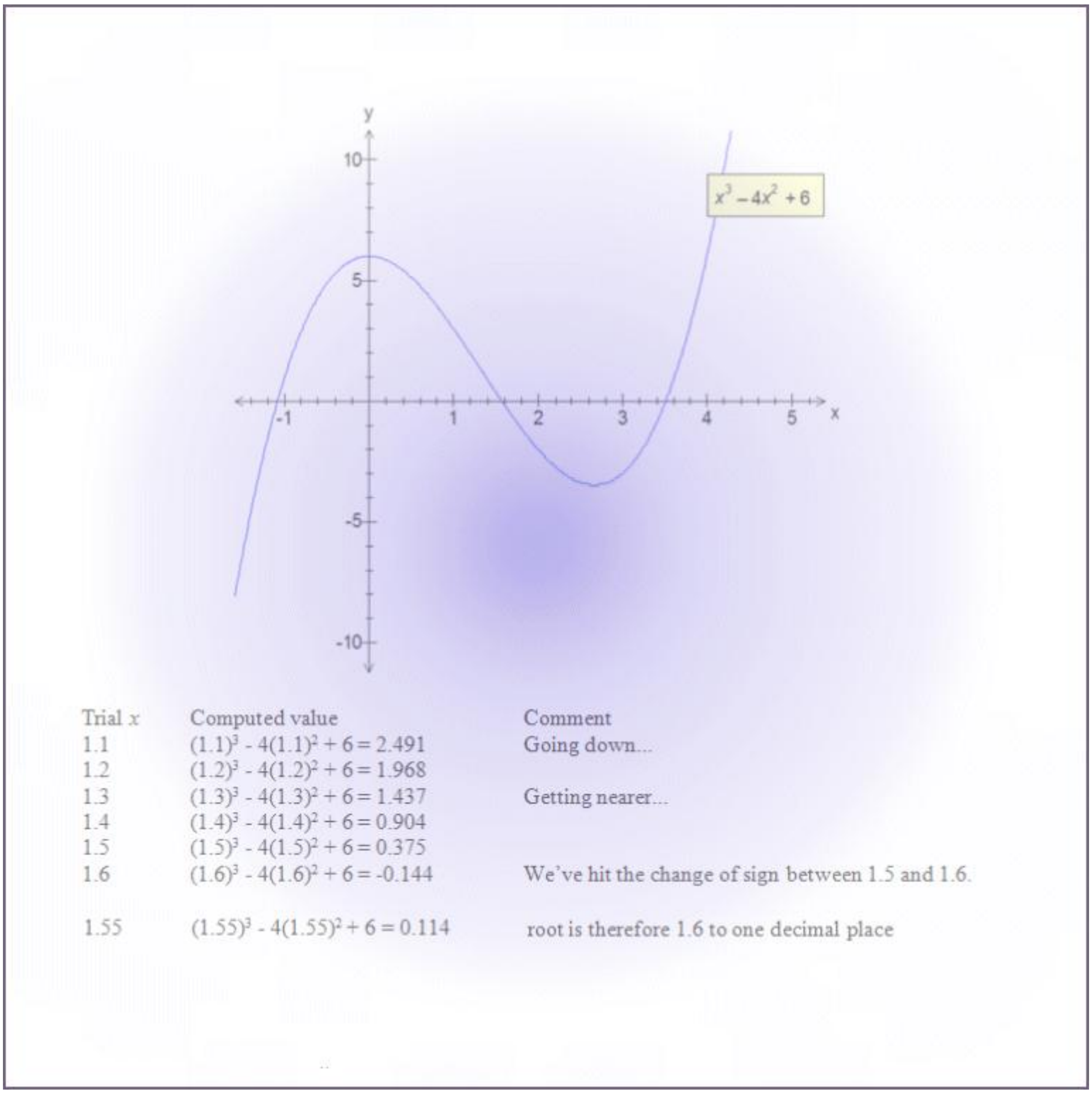
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

NUMERICAL METHODS FOR SOLVING EQUATIONS



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NUMERICAL METHODS FOR SOLVING EQUATIONS.

Many algebraic equations cannot be solved using the standard methods of factorisation and substitution into formulae, especially when they are more complicated than quadratics.

Take the cubic equation $x^3 - 4x^2 + 6 = 0$

This equation cannot be factorised, and there is also no general formula for solving cubic equations as there is for quadratic equations, so other methods must be used.

One such method, studied in earlier years, is 'trial and improvement' using a decimal search. This involves finding a value of x which makes $x^3 - 4x^2 + 6$ as close to 0 as the accuracy allows.

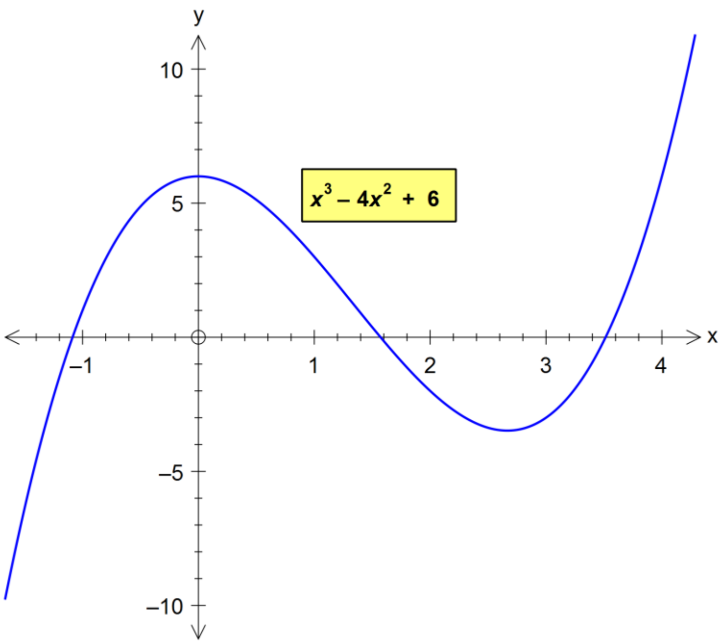
This method is no longer part of the GCSE syllabus, for reasons which will become clear.

Example (1): ("Legacy" example)

The graph of $y = x^3 - 4x^2 + 6$ is shown below, and has three solutions of $x^3 - 4x^2 + 6 = 0$,
given that its graph cuts the x-axis at three points, with those solutions being the x-coordinates of those points. .

One of the solutions is close to -1, another is between 1 and 2, and a third one is between 3 and 4.

Use trial and improvement to find the solution between 1 and 2 correct to one decimal place.



We know that the solution lies between 1 and 2, so we begin with those trial values of x .

Trial x	**Computed value of $x^3 - 4x^2 + 6$**	**Comment**
1	$(1)^3 - 4(1)^2 + 6 = 3$	Too high (> 0)
2	$(2)^3 - 4(2)^2 + 6 = -2$	Too low (< 0)

The solution seems to be about half-way between 1 and 2, so we can substitute $x = 1.3, 1.4, 1.5, 1.6$ .... into the equation . It must also be noted that $x^3 - 4x^2 + 6$ *decreases* as x increases within this interval.

Trial x	**Computed value of $x^3 - 4x^2 + 6$**	**Comment**
1.3	$(1.3)^3 - 4(1.3)^2 + 6 = 1.437$	Too high (> 0)
1.4	$(1.4)^3 - 4(1.4)^2 + 6 = 0.904$	Too high
1.5	$(1.5)^3 - 4(1.5)^2 + 6 = 0.375$	Still too high, but getting nearer
1.6	$(1.6)^3 - 4(1.6)^2 + 6 = -0.144$	Too low (< 0)

We now know that the solution lies between 1.5 and 1.6, so we substitute $x = 1.55$ (the mean of 1.5 and 1.6) to obtain the correct first decimal place.

When $x = 1.55$, $x^3 - 4x^2 + 6 = 0.114$, which is greater than 0.

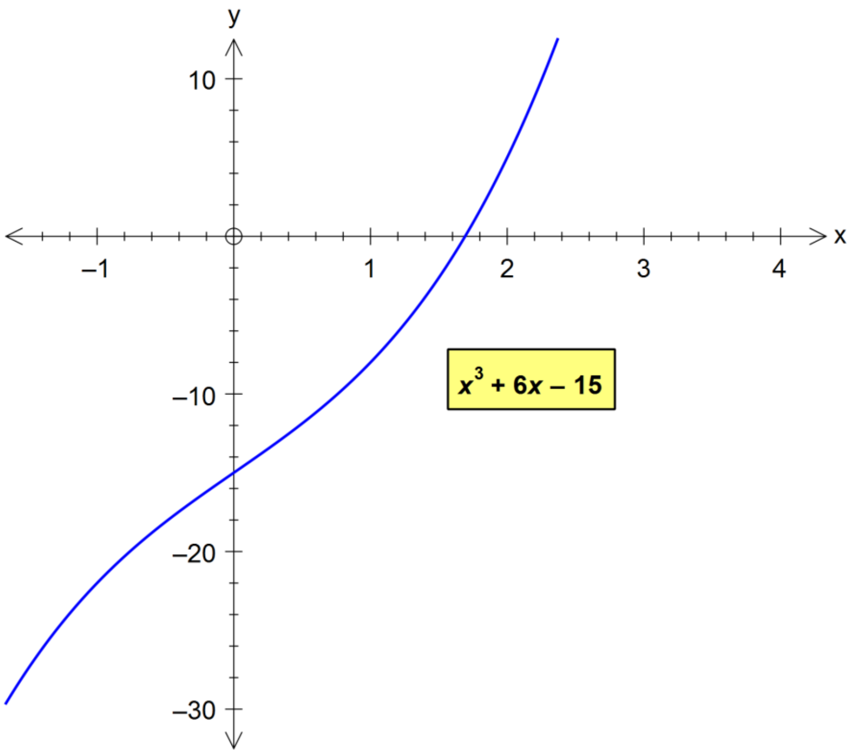
The change between the value of $x^3 - 4x^2 + 6$ from being 'too high' to being 'too low' occurs in the interval from 1.55 to 1.6.

Since all values in this particular interval round to 1.6 to 1 decimal place, the required solution of the equation $x^3 - 4x^2 + 6 = 0$ is $x = 1.6$ to 1 decimal place.

Iterative methods.

The decimal searching technique used the first two examples was simple, but there was a drawback. The process was *slow*, requiring a large number of trials for even a moderate level of accuracy.

Example (2): Find the solution of $x^3 + 6x = 15$ to four decimal places.



The equation $x^3 + 6x = 15$ can be rewritten as $x^3 + 6x - 15 = 0$, and it has one solution between 1 and 2, where the graph intersects the *x*-axis.

Trial and improvement would be unsuitable here, because of the large number of trials needed. If it takes on average five trials to achieve accuracy to one decimal place, it would take four times as long, or about 20 attempts, to achieve four decimal-place accuracy.

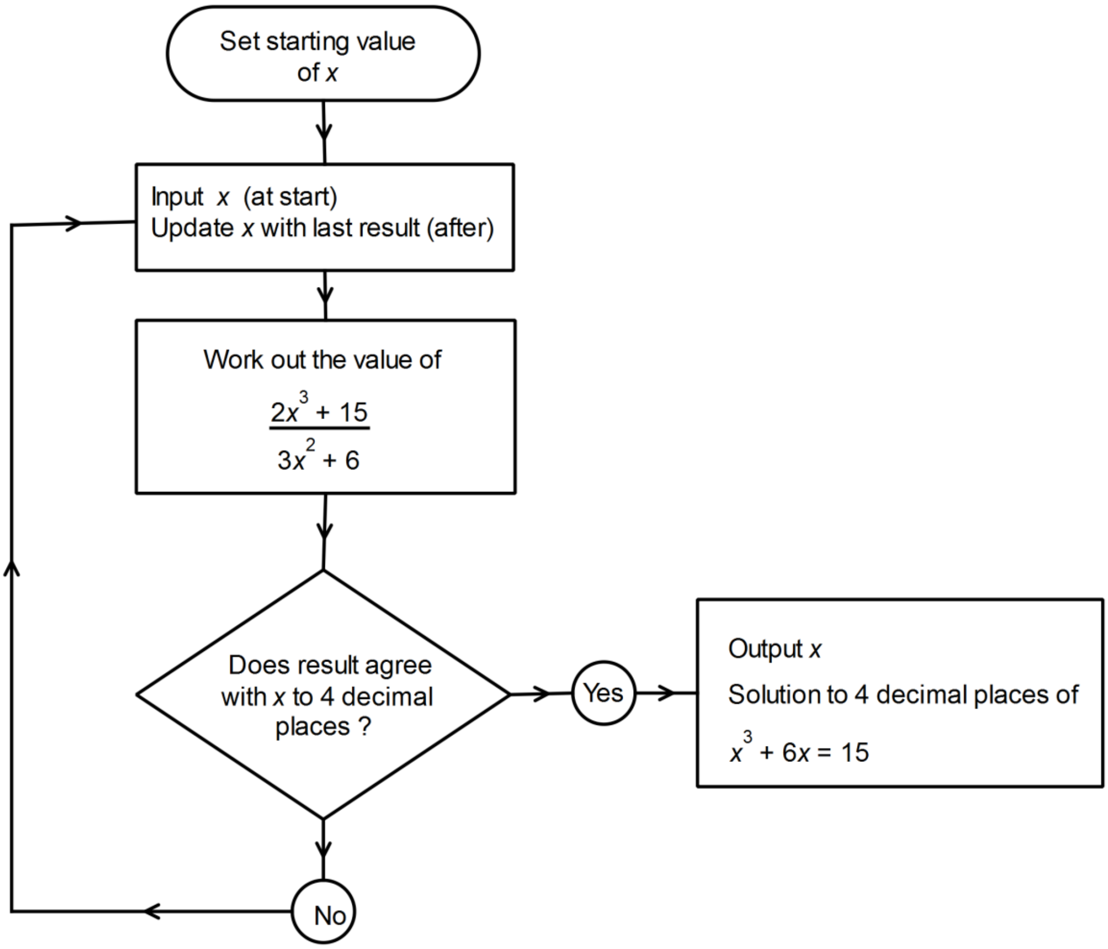
This calls for a different method known as **iteration**.

In iteration, we use an initial estimated solution as the input to a formula, and the resulting output provides an improved estimate. This improved estimate can then be input into the formula again, until the desired level of accuracy is reached .

The equation $x^3 + 6x = 15$ has a single solution between 1 and 2.

Use the following flowchart to calculate this solution to four decimal places.

Choose a starting value of $x = 2$, and keep four decimal places in the working.



You may notice that the expression $\dfrac{2x^3+15}{3x^2+6}$ does not have any obvious relationship with the equation $x^3 + 6x = 15$, but finding such iterative formulae is outside the scope of GCSE.

All questions will have a suitable formula provided for you !

By choosing the starting value of x = 2 as input, we calculate $\frac{2(2^3)+15}{3(2^2)+6} = \frac{31}{18} = 1.7222$.

The new value of x does not agree sufficiently with the input value of 2,

so we take the "No" branch and update x with the new value of 1.7222 into the formula to obtain

$\frac{2(1.7222^3)+15}{3(1.7222^2)+6} = 1.6926$. This result is still not close enough to 1.7222, so we take the "No" branch again.

Substituting x = 1.6926 gives us the next value of x, namely $\frac{2(1.6926^3)+15}{3(1.6926^2)+6} = 1.6923$.

The iterations appear to be closing in on the required solution, with only the fourth decimal place different after this last calculation.

We still take the "No" branch, substitute x = 1.6923, giving our next value of x which is
$\frac{2(1.6923^3)+15}{3(1.6923^2)+6} = 1.6923$ to 4 decimal places.

Since this output value of x now agrees to four decimal places with the input, we can now end the iterative loop and take the "Yes" branch.

The output value of x = 1.6923 is the solution of the equation $x^3 + 6x = 15$ to the guaranteed accuracy of four decimal places after four iterations.

This method is more accurate than the decimal search, and is also a great deal quicker.

We can check by substituting x = 1.6923 into $x^3 + 6x$; the result is 15.0003, very close to 15.

Another way of expressing an iterative formula without a flowchart is to use subscripts where x_1 is the starting trial value of the solution and x_2, x_3 .... are the subsequent iterations.

This is similar to generating sequences by inductive definition – see "Sequences" document.

The formula used in the last example could be defined as $x_{n+1} = \frac{2x_n^{\ 3} + 15}{3x_n^{\ 2} + 6}$

where x_n is the current estimate for x and x_{n+1} is the next one.

Thus in the last example, we had $x_1 = 2$, $x_2 = 1.7222$, $x_3 = 1.6926$ and $x_4 = 1.6923$.

A very useful hint.

In the last example, we had been calculating iterated values of x, rounding them to four decimal places, and re-inputting the new values into the iterative formula.

The work was rather tedious, but fortunately, most calculators have an **Ans** key to perform such calculations very rapidly.

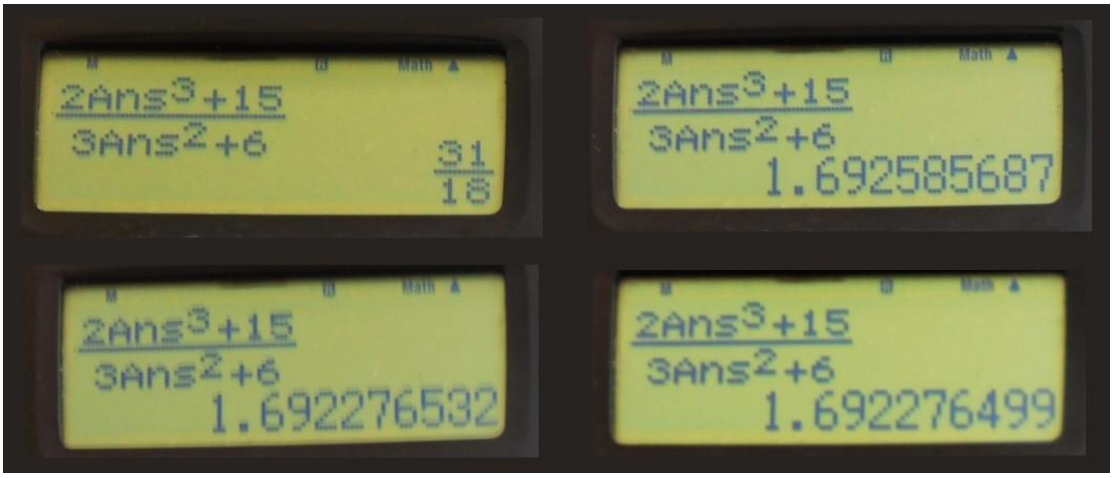
This example has the **Ans** key on the bottom row.

For the last example , we can set the starting value by pressing the keys **2** = = .

(On more modern calculators, the = key can be pressed once only.)

Next, we enter the iterative formula using the **Ans** button, among others. (We must enter **Ans** followed by x^3 when entering the top line, similarly **Ans** followed by x^2 when entering the bottom line).

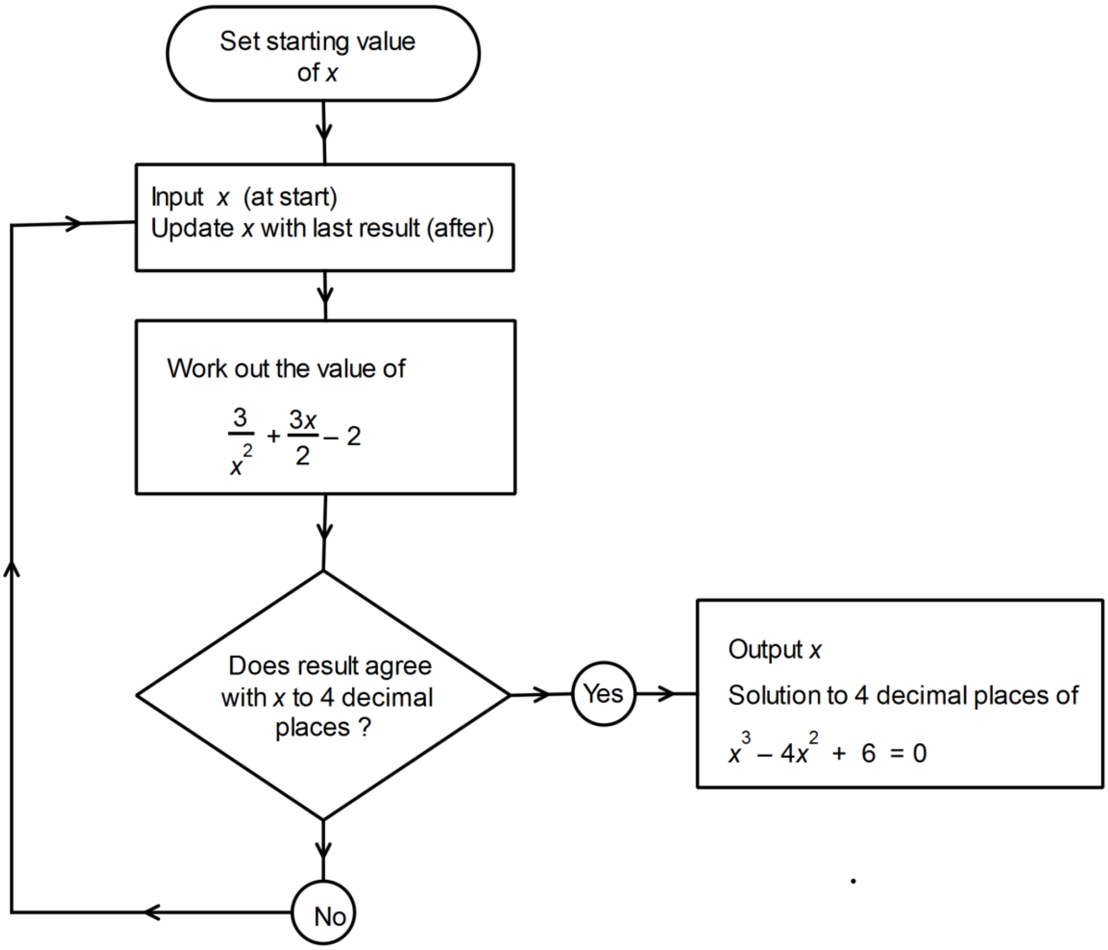
Having started with the value of $x_1 = 2$, it is just a matter of repeatedly pressing the = key to obtain the values of x_2 , x_3 and so forth. Notice how the value of x_2 appears as a fraction, and how rapidly the values converge to 4-decimal place accuracy.



Example (3):

The equation $x^3 - 4x^2 + 6 = 0$ from Example (1) had a solution of 1.6 to one decimal place.
Use the following flowchart to calculate this same solution to four decimal places, and practise by using the **Ans** key on the calculator .

Choose a starting value of $x = 2$.



By choosing the starting value of x_1 = 2 as input, we calculate $\frac{3}{x^2}+\frac{3x}{2}-2$.

On the calculator, it is a matter of pressing the 2, =, and = keys, followed by entering

$\frac{3}{Ans^2}+\frac{3(Ans)}{2}-2$, and then pressing the = key throughout.

This takes care of all the calculations, inasmuch as we just press the = key until sufficient accuracy has been reached .

In other words, a “No” result means “press the = key again”, and a “Yes” means “End of sum”.

Pressing the = key first time gives us the next trial value of *x*, namely x_2, or 1.75, which does not agree sufficiently with x_1, so we press the = key again to give us x_3, or 1.6046.

Pressing the = key again gives us x_4 = 1.5721, x_5 = 1.5720 and x_6 = 1.5720

Since the values of x_5 and x_6 agree to four decimal places, we now take the “Yes” branch and end the iterative process..

One solution of the equation $x^3 - 4x^2 + 6 = 0$ is 1.5720 to 4 decimal places.

This example could have been rephrased as follows, without the need of a flowchart :

Example (3) Version : The equation $x^3 - 4x^2 + 6 = 0$ has a solution between 1 and 2.

Use the iterative formula $x_{n+1} = \frac{3}{x_n^{\ 2}} + \frac{3x_n}{2} - 2$ and $x_1 = 2$ to find the values of x_2, x_3, x_4 and x_5 to four decimal places.

In tabular form :

n	x_n
1	2
2	1.75
3	1.6046
4	1.5721
5	1.5720

Example 3(b) : Repeat the last example, using the iterative formula $x_{n+1} = \frac{x_n^{\ 3} + 12}{x_n(8 - x_n)}$.

On calculator: press the 2, =, and = keys, enter $\frac{Ans^3 + 12}{Ans(8 - Ans)}$, then keep pressing the = key throughout.

Results in tabular form :

n	x_n
1	2
2	1.6667
3	1.5754
4	1.5719
5	1.5720

Example (4): The equation $x^3 - 4x^2 + 6 = 0$ also has a solution between -1 and -2.

Use the iterative formula $x_{n+1} = 0.4 - \frac{0.6}{x_n^{\ 2}} + 0.9x_n$ and $x_1 = -1$ to find the values of x_2, x_3, x_4 and x_5 to four decimal places.

Calculator: press the -1, =, and = keys, enter $0.4 - \frac{0.6}{Ans^2} + 0.9(Ans)$, and press the = key throughout.

In tabular form :

n	x_n
1	-1
2	-1.1
3	-1.0859
4	-1.0861
5	-1.0861

Example (5): The equation $x^3 - 4x^2 + 6 = 0$ also has a solution between 3 and 4.

i) Use the iterative formula $x_{n+1} = \frac{5x_n^{\ 3} - 36}{11x_n^{\ 2} - 24x}$ and $x_1 = 4$ to find the values of x_2, x_3, x_4 and x_5 to four decimal places.

Calculator: press the 4, =, and = keys, enter $\frac{5Ans^3 - 36}{11Ans^2 - 24Ans}$, then keep pressing the = key..

i) In tabular form :

n	x_n
1	4
2	3.55
3	3.5131
4	3.5142
5	3.5141