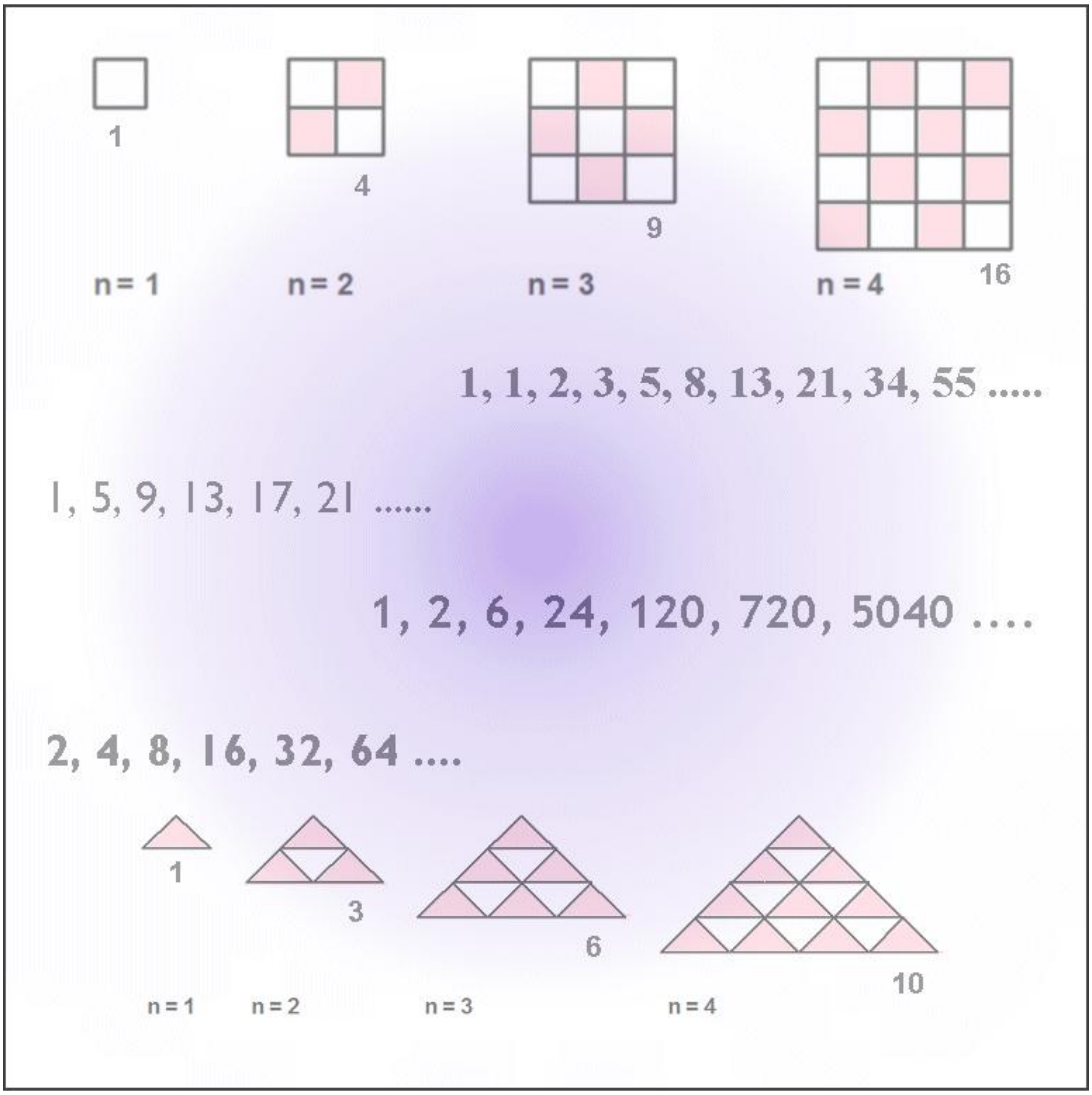
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SEQUENCES



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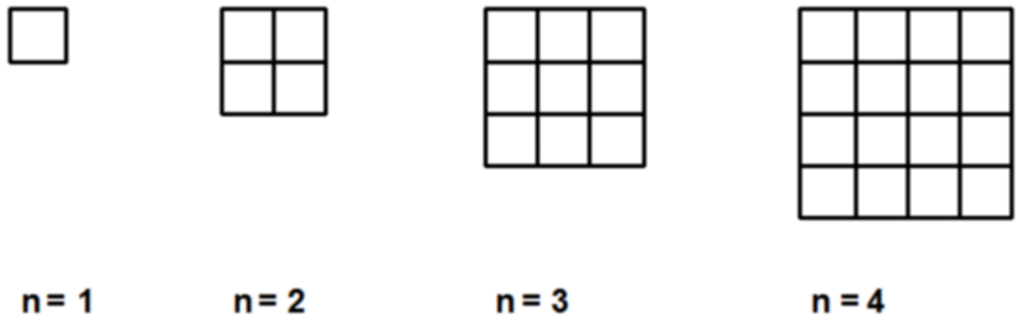
SEQUENCES

A **sequence** is a list of numbers following some rule for finding succeeding values.

Each number in a sequence is called a **term**.

A sequence can be defined by a formula for the n^{th} term, where n is the position of the term in the sequence.

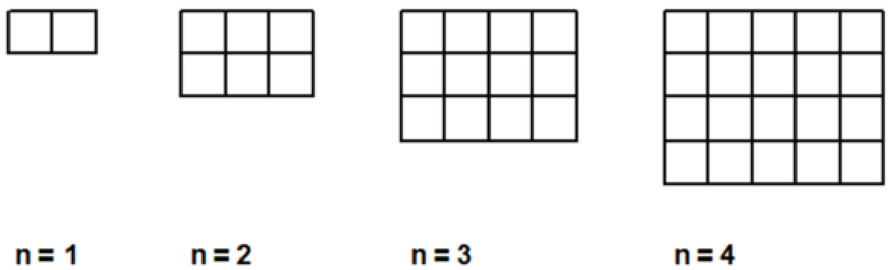
Example (1): One example of a sequence is that of the square numbers, illustrated below.



The first term of the sequence is 1, the second 4, the third 9 and the fourth, 16.
The number of squares in each of the diagrams is the same as the square of its position in the sequence.

The 5^{th} term in the sequence will therefore be 5^2 or 25, and we can also generalise by saying that the n^{th} term is equal to n^2.

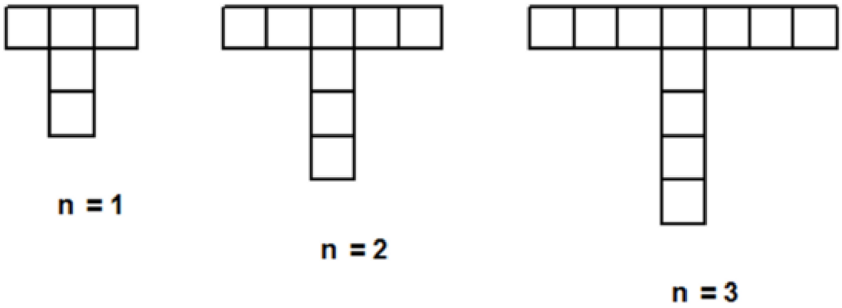
Example (2): Investigate the sequence of rectangles below to find a general formula for the width, the height, and thus the area, in unit squares, of the n^{th} rectangle.



The first rectangle is 2 squares wide × 1 square high.
The second rectangle is 3 squares wide × 2 squares high.
The third rectangle is 4 squares wide × 3 squares high.
The fourth rectangle is 5 squares wide × 4 squares high.

We can see an obvious pattern here.
The height of each rectangle equals the position in the sequence – it can be represented as $\boldsymbol{n}$.
The width of each rectangle exceeds the position in the sequence by 1– it can be represented as $\boldsymbol{n + 1}$.
Since the area of a rectangle is given by height × width, the area of the n^{th} one is $\boldsymbol{n(n + 1)}$.

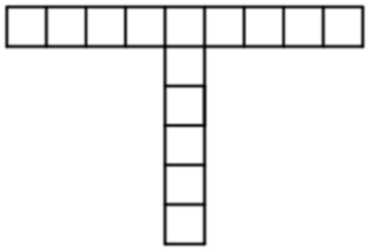
Example (3): Investigate the sequence of T-shapes below to draw the next T-shape in the sequence, and also to find a general formula for the total number of unit squares in the n^{th} T-shape.



Spotting the pattern here is not quite as easy as in the last example, but we can see that:

An extra unit square is added to the vertical bar of the T as n goes up by 1.
Two extra unit squares (one on each side) are added to the horizontal bar of the T as n goes up by 1.

The fourth T-shape in the sequence therefore looks like the diagram on the right.



The first T-shape is made up of 5 unit squares.
The second one is made up of $5 + (3 \times 1)$ or 8 unit squares.
The third one is made up of $5 + (3 \times 2)$ or 11 unit squares.
The fourth one is made up of $5 + (3 \times 3)$ or 14 unit squares.

We can spot the pattern here: the n^{th} T-shape is made up of $\mathbf{5 + 3(}\boldsymbol{n}\mathbf{-1)}$ unit squares.

(An equally valid expression is $\mathbf{2 + 3}\boldsymbol{n}$).

The first three examples were shape-based and intuitive, but it is also possible to determine the general term of a sequence by listing enough of its elements to spot a pattern.

Sequences with a common difference.

These sequences are generated by **adding a constant value** to each term as we go along.

We can see that the next two terms in the sequence 7, 14, 21, 28 .... are 35 and 42, the terms being the first multiples of 7. We can also see that the n^{th} term is $7n$.

Since each term can be obtained from the previous one by adding 7, the common difference is 7.

The common difference can be found by taking a term after the first, and subtracting the one before it.

The general formula for the n^{th} term is $u_n = an + b$ where a is the common difference and b is a constant to be added. This constant is a hypothetical "0^{th} term" derived by subtracting the common difference from the first term.

Note the use of the subscripted variable u_n to denote the n^{th} term of a sequence. By this same notation, the first three terms of a sequence are u_1, u_2 and u_3.

(This type of sequence is also called an **arithmetic progression**.)

Examples (4a): Find the next two terms of the following sequences, together with a general formula for the n^{th} term:

i) 1, 5, 9, 13, 17, ...
ii) 44, 39, 34, 29, 24, ...

In i) we see that the first term is equal to 1 and the common difference is 4.

The next two terms are 21 and 25.

The formula for the general term is therefore $u_n = 4n + b$.
The first term is 1, so the "0^{th} term", u_0, is 1 – 4 or -3, giving a general formula of $\boldsymbol{u_n = 4n - 3}$.
The 100^{th} term, u_{100}, is therefore $(4\times100) - 3$ or 397.

In ii) we can see that successive terms decrease in value, so the common difference will be negative.
The first term $u_1 = 44$ and the common difference is –5.
The next two terms are 19 and 14.

This time the expression for the general term is $u_n = -5n + b$.
The first term is 44, so $u_0 = 44 - (-5)$ or 49, giving a general formula of $\boldsymbol{u_n = 49 - 5n}$.
Hence $u_{100} = 49 - (5\times100)$ or -451.

Example (4b): For the sequence 1, 5, 9, 13, 17, ...

i) Find the position of the number 61.
ii) Show that the number 75 is not a term of the sequence.

i) Because the formula $u_n = 4n - 3$ can be used to find the n^{th} term in the sequence, we solve the equation $4n - 3 = 61$, and the solution is $n = 16$.
The number 61 is therefore the 16^{th} term in the sequence.

ii) When we solve $4n - 3 = 75$ as in i), we have the result $n = 19.5$, which is not a positive whole number, so 75 is not a member of the sequence.

Sequences with a common ratio.

These sequences are generated by **multiplying** each term by **a constant value** as we go along.

Take the sequence 3, 9, 27, 81 .... as an example.

The common ratio can be found by taking a term after the first, and dividing it by the one before it.

The general expression for the n^{th} term is $b(a^n)$ where a is the common ratio and b is a constant multiplier. This constant is a hypothetical "0^{th} term" obtained by dividing the first term by the common ratio.

Since each term can be obtained from the previous one by multiplying by 3, the common ratio is 3.

We can see that the next two terms in the sequence 3, 9, 27, 81 .... are 243 and 729, the terms being the first powers of 3. We can also see that the n^{th} term, u_n , is 3^n.

(This type of sequence is also called a **geometric progression**.)

Examples (5): Find the next two terms of the following sequences, together with a general expression for the n^{th} term:

i) 3, 6, 12, 24, 48, ...
ii) 800, 80, 8, 0.8, 0.08,...........

In i) we see that the first term, u_1 , is equal to 3 and the common ratio r (e.g.6 ÷ 3) is 2.

The next two terms are 96 and 192.
The first term, u_1 is 3, so $u_0 = 3 \div 2$ or 1.5, with the resulting general formula $\boldsymbol{u_n = 1.5(2^n)}$.

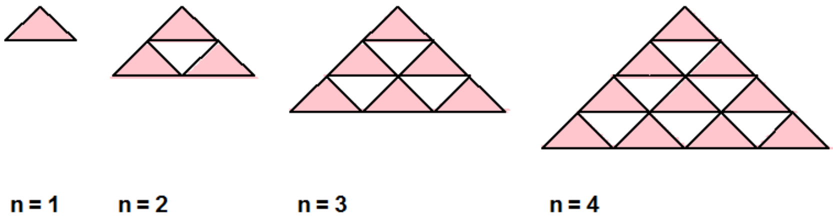
In ii) the successive terms decrease in value, and so the common ratio will be less than 1.
The first term is 800 and the common ratio is 0.1.
The next two terms are 0.008 and 0.0008.

Since the first term, u_1, is 800, the "0^{th} term", u_0 is 800 ÷ 0.1 or 8000, with the resulting formula of $\boldsymbol{u_n = 8000(0.1^n)}$ for the general n^{th} term.

Quadratic Sequences.

These are trickier than the last two, because we have two sets of differences to consider.

Example (6): Take the sequence of triangular numbers.



The triangular numbers are represented here by the shaded triangles.

The first triangular number is 1.
The second triangular number is 1 + 2 or 3.
The third triangular number is 1 + 2 + 3 or 6.
The fourth triangular number is 1 + 2 + 3 + 4 or 10.

The sequence of triangular numbers therefore goes 1, 3, 6, 10, ...

The difference between the first term and the second is 1.
The difference between the second term and the third is 2.
The difference between the third term and the fourth is 3.

The first differences form an arithmetic sequence, but the second differences are constant at 1.
All quadratic sequences have a constant second difference.

We can set up a list of differences as follows :

Sequence terms	1		3		6		10
1st differences		*2*		*3*		*4*	
2nd differences			*1*		*1*		

To find the next two terms of the series, we put two more instances of 1 in the “2nd difference” row.
Then we can put 4 + 1 = 5 and 5 + 1 = 6 into the “1st differences” row above.
Finally, we can put 10 + 15 = 15 and 15 + 6 = 21 into the top row to get the next triangular numbers.

Sequence terms	1		3		6		10		**15**		**21**
1st differences		*2*		*3*		*4*		***5***		***6***	
2nd differences			*1*		*1*		***1***		***1***		***1***

Sequence terms	1		3		6		10		15
1st differences		*2*		*3*		*4*		*5*	
2nd differences			*1*		*1*		*1*		

This method is not suitable if we wanted to find, say, the 60th triangular number, since filling in such a long list would be tedious. We need to find the expression for a general term for the sequence, u_n.

A general term of this sequence is $u_n = an^2 + bn + c$ where a, b and c are constants.

We can find a by halving the second difference.

In the case of the triangular numbers, the second differences are all equal to 1, so $a = ½$.

Hence $u_n = ½n^2 + bn + c$.

Position n	1	2	3	4	5
Sequence term u_n	1	3	6	10	15
Value of $½n^2$	½	2	4½	8	12½
$bn + c$ (after subtracting)	**½**	**1**	**1½**	**2**	**2½**

After subtracting $½n^2$ from each term, we are left with a linear sequence $u_n = bn + c$ with a common difference of ½ and a first term of ½.

The general term of this linear sequence is $u_n = ½n$, and so the general term of the original is

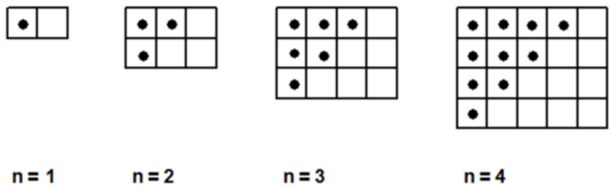
$u_n = ½n^2 + ½n$ or $\mathbf{½\, n\, (n + 1)}$.

Hence the 60th triangular number is ½ × 60 × 61 = 1830.

The result resembles that of the rectangle example (2).

Each rectangle can be split into two triangles as shown to the right.

The illustrated example with the triangular numbers could have been solved by other intuitive methods, but sometimes the above method is the only option.



Example (7):

i) Find the next two terms of the sequence 4, 7, 16, 31, 52......
ii) Find the general term for the nth term u_n.
iii) Hence find the 10th term.

i) Calculate the differences first :

Sequence terms	4		7		16		31		52		**79**		**112**
1st differences		*3*		*9*		*15*		*21*		***27***		***33***	
2nd differences			*6*		*6*		*6*		***6***		***6***		

The first differences for the sequence 3, 9, 15, 21 .... and the second differences are constant at 6.

We can therefore put two more instances of 6 in the "2nd difference" row.
Then we can put 21 + 6 = 27 and 27 + 6 = 33 into the "1st differences" row above.
Finally, we put 52 + 27 = 79 and 79 + 33 = 112 into the uppermost row.

The next two terms of the sequence are therefore 79 and 112.

ii) The general term is $u_n = an^2 + bn + c$.

The second differences are all equal to 6, and half of that is 3, so a = 3 and $u_n = 3n^2 + bn + c$.

Position n	1	2	3	4	5
Sequence term u_n	4	7	16	31	52
Value of $3n^2$	3	12	27	48	75
$bn + c$ (after subtracting)	**1**	**-5**	**-11**	**-17**	**-23**

After subtracting $3n^2$ from each term, we are left with a linear sequence $u_n = bn + c$ with a common difference of -6 and a first term of 1.

The general formula for the n^{th} term of this linear sequence is $u_n = -6n + 7$, hence the general n^{th} term of the original sequence is $\mathbf{u_n = 3n^2 - 6n + 7}$.

iii) We find the 10th term of the sequence by substituting $n = 10$ into the formula $u_n = 3n^2 - 6n + 7$.

Thus u_{10} = 300 – 60 + 7 = **247.**

Finding the general term of a quadratic sequence – another approach.

If the general term of a quadratic sequence is $u_n = an^2 + bn + c$ **,** we can substitute for n to obtain expressions for the first few terms.

For the first term, $n = 1$, so $u_1 = a(1^2) + b(1) + c$, or simply $a + b + c$.
The second term is $u_2 = a(2^2) + b(2) + c$, or $4a + 2b + c$.
The third term is $u_3 = a(3^2) + b(3) + c$, or $9a + 3b + c$.
The fourth term is $u_4 = a(4^2) + b(4) + c$, or $16a + 4b + c$.

We now have enough data to set up a difference table.

Position n	1		2		3		4
Sequence term u_n	$a + b + c$		$4a + 2b + c$		$9a + 3b + c$		$16a + 4b + c$
1st differences		$3a + b$		$5a + b$		$7a + b$	
2nd differences			$2a$		$2a$		

The highlighted values are the key to finding the general formula.
We can see that the second differences are all twice the quadratic coefficient a.
Once a has been found, we can find the linear coefficient b by checking the difference between the first and second terms. Having obtained a and b, we can find the constant term c.

Example 7(a):
Use the method above to find the expression for the nth term u_n of the sequence 4, 7, 16, 31......

Position n	1		2		3		4
Sequence term u_n	4		7		16		31
1st differences		3		9		15	
2nd differences			6		6		

The second difference is 6 throughout, so $2a = 6$ and thus $a = 3$, and the quadratic term is $\mathbf{3n^2}$.

The difference between the first and second terms is 3, so $3a + b = 3$, and substituting $a = 3$, we have $9 + b = 3$, and so $b = -6$, with the linear term of **$-6n$**.

The first term is 4, so $a + b + c = 4$. Substituting $a = 3$ and $b = -6$, we have $3 – 6 + c = 4$, so $c = 7$, giving a constant term of **7**.

The generalised term of the sequence 4, 7, 16, 31... is therefore $\mathbf{u_n = 3n^2 - 6n + 7}$**.**

Cubic Sequences.

These are an extension of quadratic sequences, with three sets of differences to consider, but are only concerned with recognising them at GCSE. There will be no need to find the general formula for such a sequence !

The sequence of cube numbers, where the general term is $u_n = n^3$, proceeds 1, 8, 27, 64, 125, 216 .....

The difference tables look as follows :

Position n	1		2		3		4		5		6
Sequence terms	1		8		27		64		125		216
1st differences		*7*		*19*		*37*		*61*		*91*	
2nd differences			*12*		*18*		*24*		*30*		
3rd differences				***6***		***6***		***6***			

Notice how the second differences still form an arithmetic sequence, but that the third differences are now constant.

Example (8): Show that the sequence 1, 13, 47, 115, 229, 401 .... is a cubic sequence.

We can calculate sets of differences and tabulate as follows:

Sequence terms	1		13		47		115		229		401
1st differences		*12*		*34*		*68*		*114*		*172*	
2nd differences			*22*		*34*		*46*		*58*		
3rd differences				***12***		***12***		***12***			

The third differences are constant, and so the sequence is a cubic sequence.

(In actual fact the general term of the sequence is $u_n = 2n^3 - n^2 + n - 1$).

Generating sequences by rules.

Another way of generating a sequence is to follow some rule relating the terms.
Thus the sequence 1, 5, 9 ,13, 17... can be generated by the rules:

- Start with 1.
- Generate the next term by adding 4 to the term before it.

Another way of expressing these rules is to use an **inductive definition** (sometimes called a **recurrence relation**) using subscripted variables :

$u_1 = 1$; $u_{k+1} = u_k + 4$

Thus u_1 is the first term of the sequence (here 1), u_2 the second, and u_k the general kth term.

The statement $u_{k+1} = u_k + 4$ means "To find the next term, take the current one and add 4."

Similarly the sequence 3, 6, 12, 24, 48, ... can be generated by the rules:

- Start with 3.
- Generate the next term by doubling the term before it

In inductive definition form:

$u_1 = 3$; $u_{k+1} = 2u_k$

Thus u_1 , the first term of the sequence, is 3.
Also, the statement $u_{k+1} = 2u_k$ means "To find the next term, double the current one."

One disadvantage of this definition is that it is not always easy to deduce a formula for the n^{th} term.

Example (9):

Generate the first six terms of the sequence generated by the following rule:

- Set the first term to 5.
- Generate each further term by doubling and previous term and subtracting 3 from the result.

Also, write out an inductive definition for the sequence.

The first term is (2 × 5) -3 or 7; the second is (2 × 7) -3 or 11 ; the third is (2 × 11) -3 or 19.
Similar calculations lead to the next three terms of 35, 67 and 131.

When expressed in inductive definition form, we have $u_1 = 5$; $u_{k+1} = 2u_k -3$.

"Fibonacci" Sequences.

These sequences are named after a 13^{th} – Century mathematician, Leonardo of Pisa, 'Fibonacci – son of Bonaccio' who first investigated them. The next example is the original Fibonacci sequence.

Example (10):

Generate the first ten terms of the sequence generated by the following rule:

- Set the first two terms to 1 and 1.
- Generate each further term by adding together the two previous ones.

Also, write out an inductive definition for the sequence.
(Do not attempt to find a general formula for the n^{th} term !)

The third term is 1 + 1 or 2, the fourth one is 1 + 2 or 3 and the fifth one is 2 + 3 or 5.

The sequence thus goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ....

The inductive definition is $u_1 = 1$; $u_2 = 1$; $u_{k+1} = u_{k-1} + u_k$.

This sequence can be seen in many examples in nature, especially in plants.

The scales on a pine-cone form clockwise and anticlockwise spirals in the Fibonacci ratio of 5 : 8, as do the 'bumps' on pineapples (ratio 8:13). The most striking example is in the clockwise and anticlockwise arrangements of the central flower / seed heads of sunflowers, where the most usual ratios are 21 : 34 and 34 : 55.

Example (10a):

Generate the first six terms of the sequence generated by the following rule:

- Set the first two terms to 2 and 5.
- Generate each further term by adding together the two previous ones.

This is a generalised version of the Fibonacci sequence with the same term-to-term rules as the original, but with different starting values.

The first six terms are therefore 2, 5, 7, 12, 19 and 31.
The inductive definition is $u_1 = 2$; $u_2 = 5$; $u_{k+1} = u_{k-1} + u_k$.