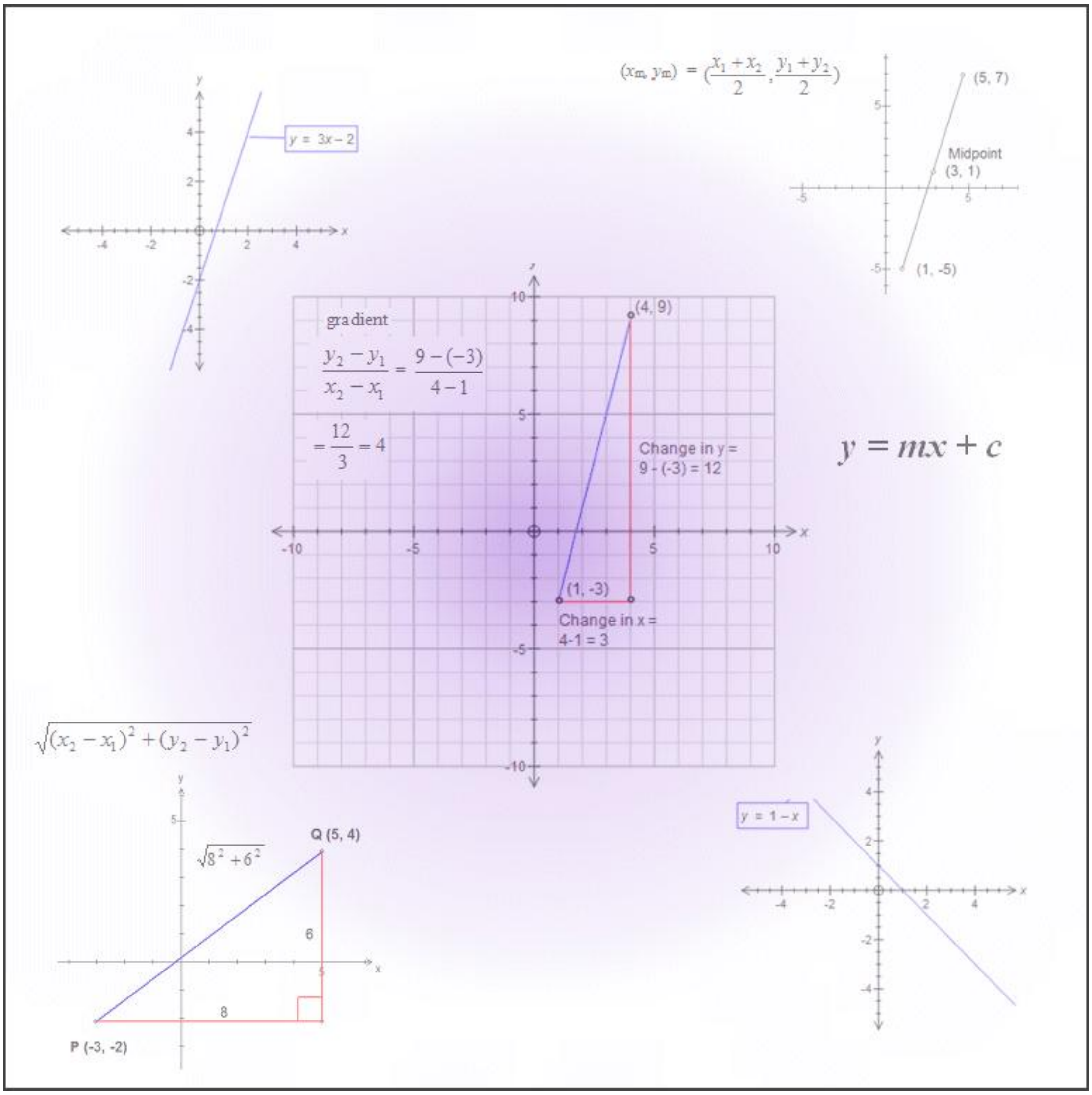
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

STRAIGHT LINE GRAPHS



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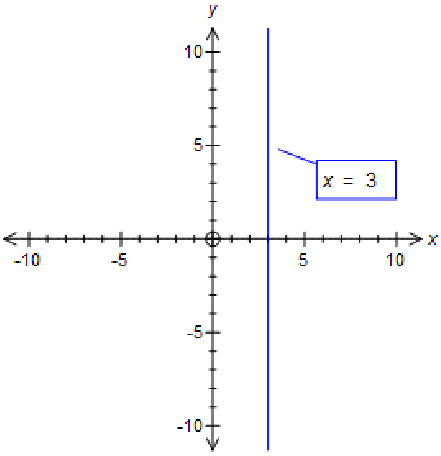
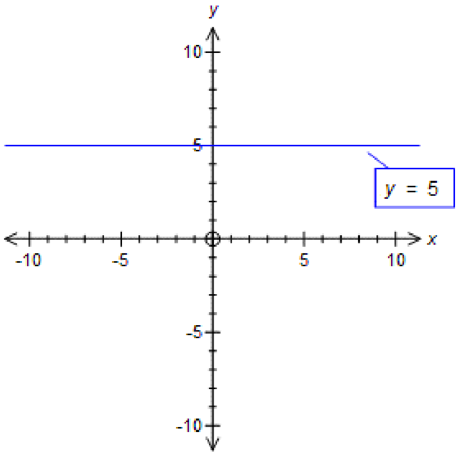
STRAIGHT LINE GRAPHS

A function's graph is a straight line if x and y are related linearly; in other words there are no terms in x^2, $\frac{1}{x}$, x^3 or any other powers or functions of x.

Therefore the graphs of $y = 2x - 3$, $y = 1 - 4x$, $x + 3y = 5$ and $3y = 5x$ are all linear, even though they appear different in form.

The simplest graphs are shown below.





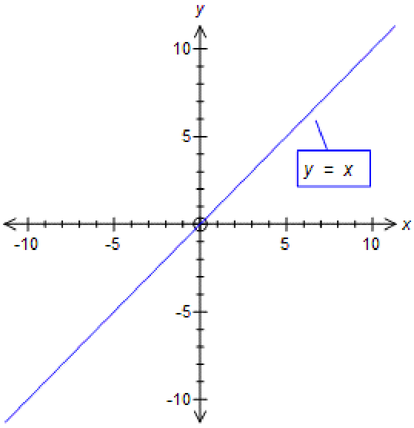
The graphs above are of *constant* functions; in the first one, x is always equal to 3 regardless of the value of y, but in the second one, y always equals 5, no matter what value x takes.

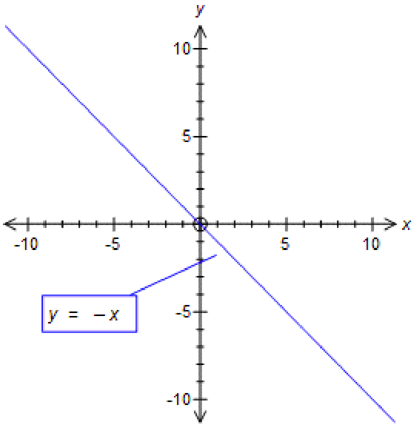
In general, any constant graph of $x = k$ is parallel to the y-axis and k units away from it.
Similarly, any constant graph of $y = k$ is parallel to the x-axis and k units away from it.

The graph of $x = 0$ coincides with the y-axis; that of $y = 0$ coincides with the x-axis.

The next pair of graphs are the main diagonals $y = x$ and $y = -x$.
Notice how each passes through the origin; also notice the slope or gradient of each.

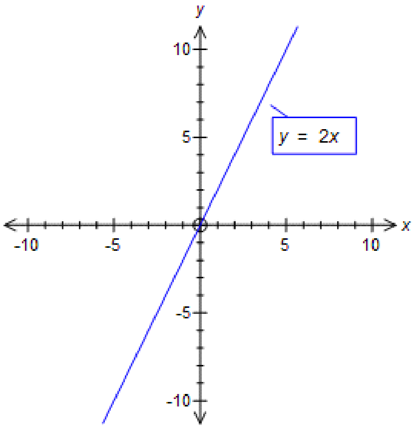
When $y = x$, the graph shows a **positive gradient**, i.e. ***y* increases as *x* increases**.
When $y = -x$, the graph shows a **negative gradient**, i.e. ***y* decreases as *x* increases**.

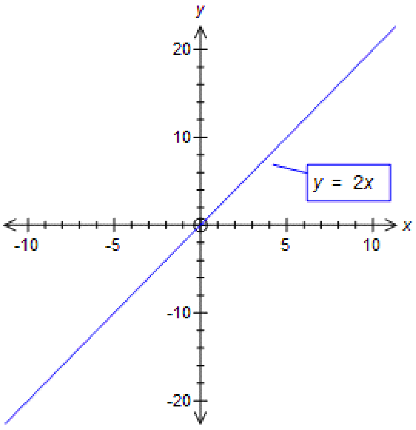




The next sets of graphs also pass through the origin, but they are of the form $y = kx$ where k is a non-zero constant.

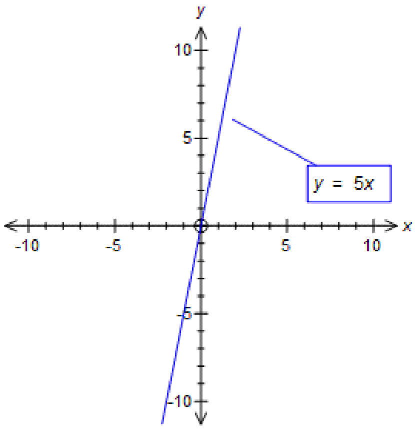
Two graphs of $y = 2x$ are shown below.

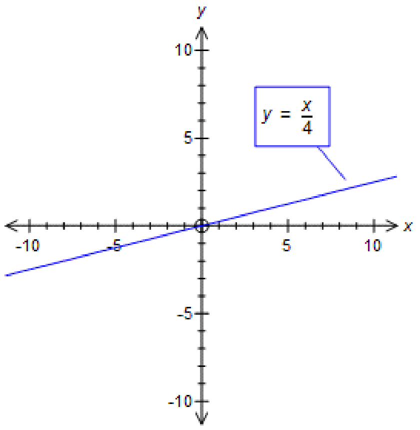




The graph on the left is steeper than that of $y = x$, but the one on the right appears different. This is because the scales of the axes are not uniform. It is important to bear this in mind in later sections, particularly when finding gradients.

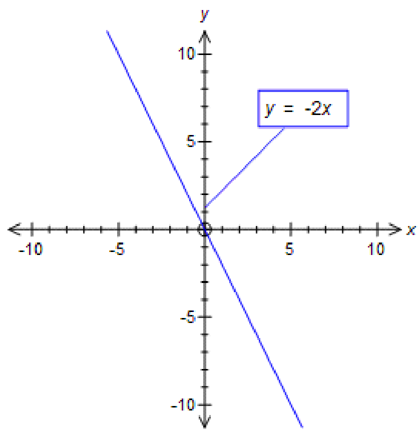
For any graph of $y = kx$ on a uniform scale, where k is positive, the larger k is, the steeper the graph. Conversely, when k is positive and less than 1, the shallower the resulting graph.

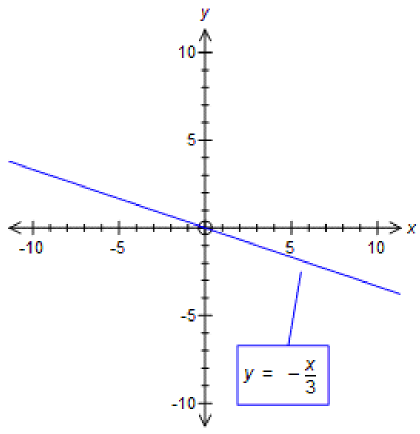




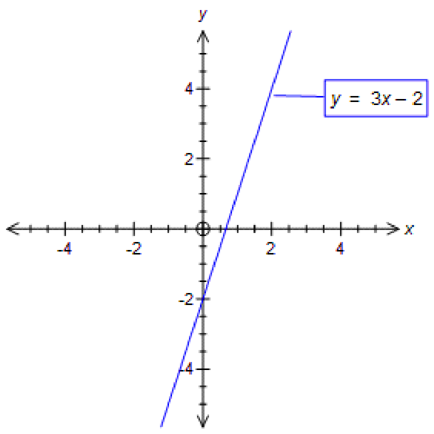
Notice how the gradients of the graphs are still positive; as x increases, so does y.

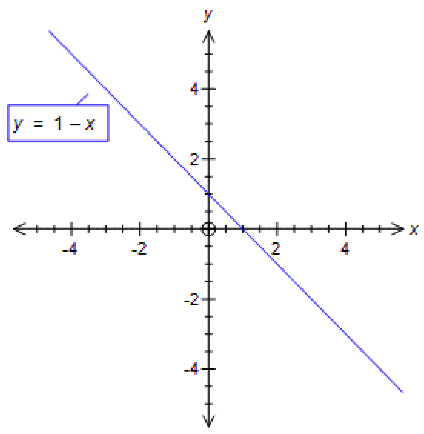
When the value of k for a graph of $y = kx$ happens to be negative, the overall results are similar to the previous examples, but this time the gradients of the graphs are negative.





All of the straight line graphs so far have passed through the origin. However, the vast majority do not !





The graph of $y = 3x – 2$ (above left) does not pass through the origin, but appears to cross the y-axis at the point (0, -2). It also has a positive slope as might be expected of the positive multiple of x, here 3.

The graph of $y = 1 - x$ (above right) appears to cross the y-axis at the point (0, 1). It also has a negative slope, given the negative multiple of x, here -1.

The point where a linear graph cuts the y-axis is also known as the y-intercept, or simply the intercept.

The point where the graph cuts the x-axis is sometimes called the x-intercept, but is more often called the root (as in the solution of an equation).

Plotting a Linear Graph.

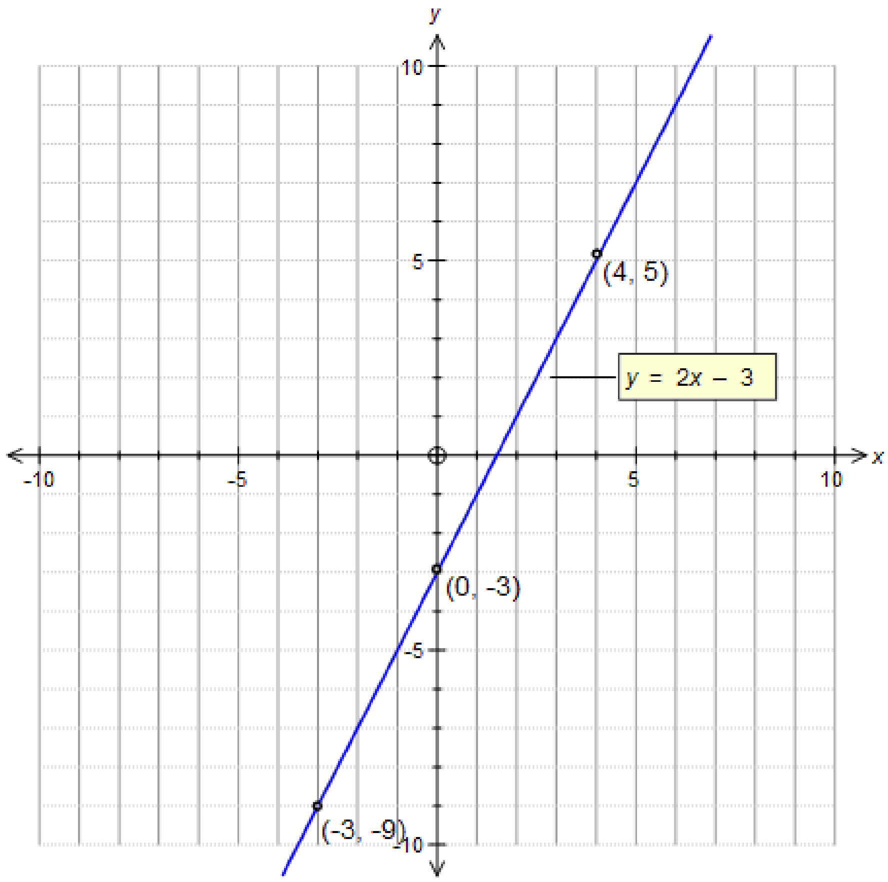
There are two main ways of plotting a graph of a linear function.
We can produce a table of three values by substituting certain values for x, as in the examples below.

Example (1): Plot the graph of $y = 2x - 3$ for values of x between –3 and 5.

We first substitute three values of x into the equation and tabulate the result:

x	-3	0	4
y	-9	-3	5

Then, we plot the three points and draw the straight line passing through them, as in the diagram below.

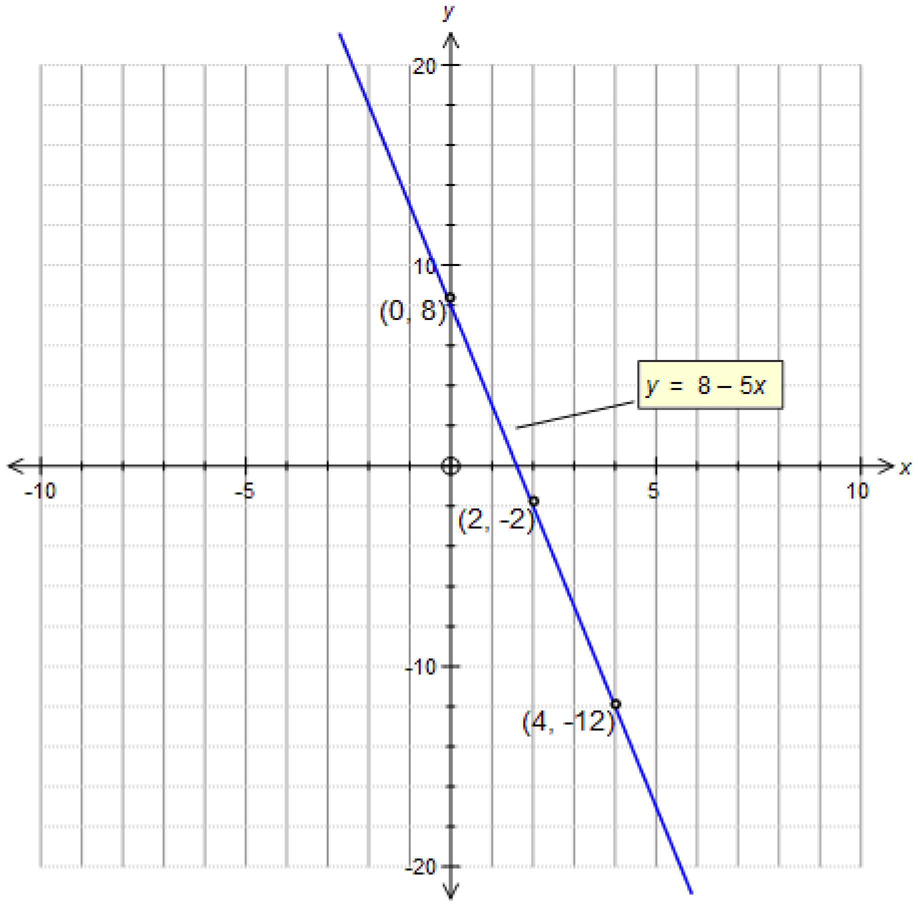


Although two points are sufficient to define a linear graph, it is good practice to plot three, because it would show up any errors in calculating the values of y.

If the three points are not on a straight line, then it is time to check the calculated values of y !

Example (2): Plot the graph of $y = 8 - 5x$ for values of x between –1 and 4.

x	0	2	4
y	8	-2	-12

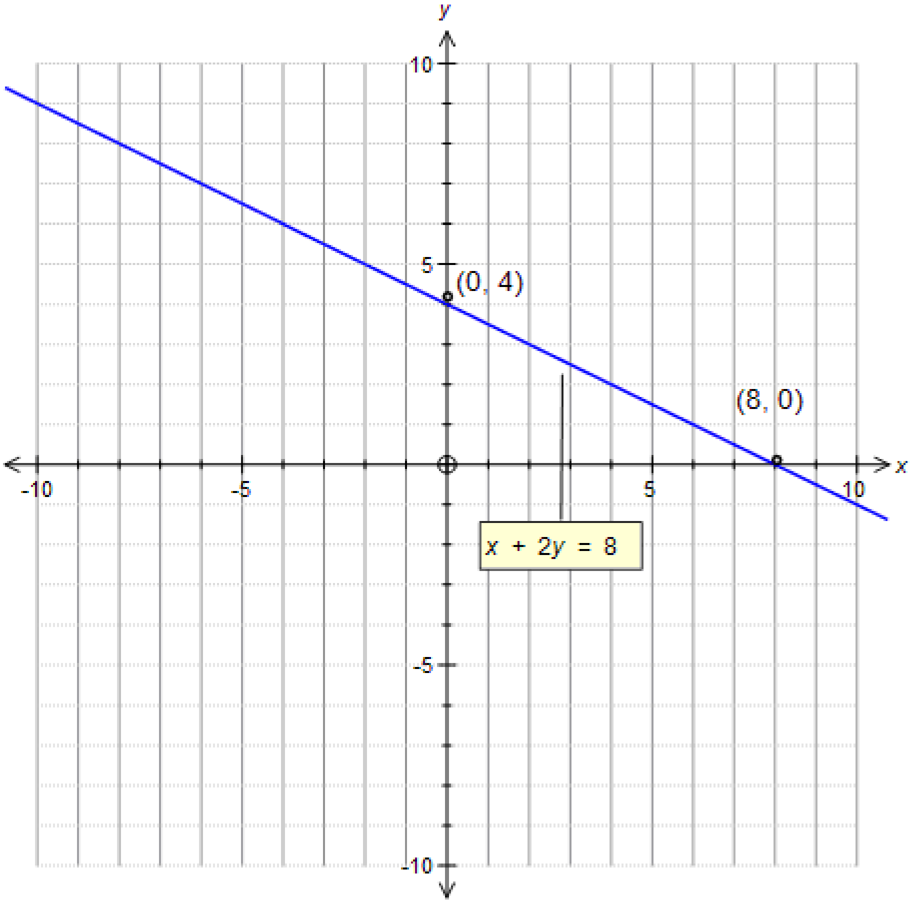


Sometimes the equation might be given in a different form, such as the next example:

Example (3): Plot the graph of $x + 2y = 8$ for values of x between –5 and 15.

Here the easier method of plotting the graph is to substitute $x = 0$ and $y = 0$ into the equation.
When $x = 0$, $2y = 8$, and $y = 4$, so we plot the point (0, 4) – the y-intercept.
When $y = 0$, $x = 8$ and therefore we plot the point (8, 0) – the x-intercept.

Joining the two points gives the graph below.



The Gradient of a Line.

To find the gradient of a linear graph, all we need is to choose two points on it.

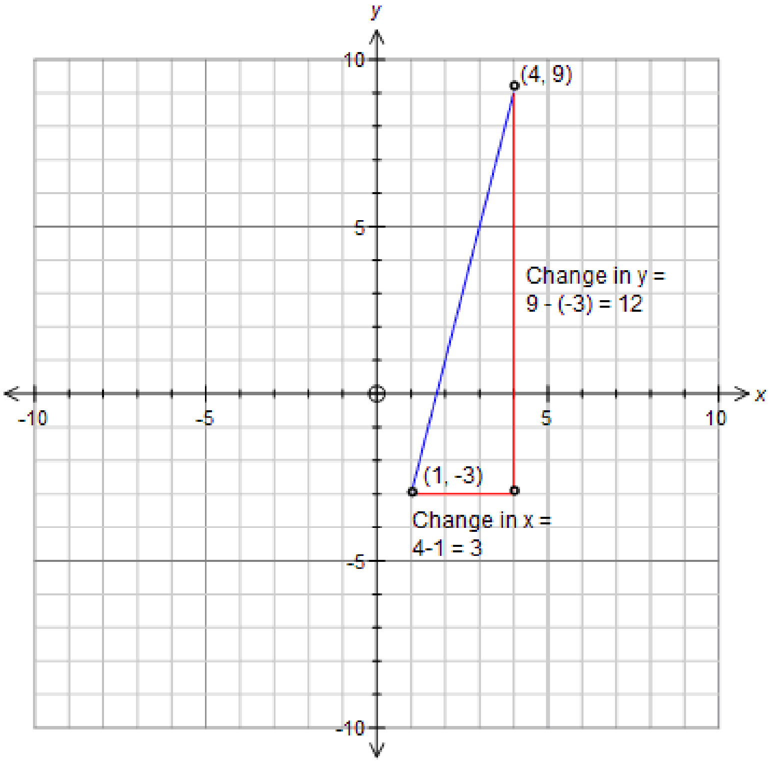
The gradient of a line connecting two points (x_1, y_1) and (x_2, y_2) is given by the formula $\dfrac{y_2 - y_1}{x_2 - x_1}$ - it is the change in the value of y divided by the change in the value of x.

Example (4): Find the gradient of the line passing through the points (1, -3) and (4, 9).

Taking (1, -3) as (x_1, y_1) and (4, 9) as (x_2, y_2), the gradient of the line above is therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-(-3)}{4-1} = \frac{12}{3} = 4.$$

It does not matter which point is taken as (x_1, y_1) – the calculated gradients will be the same.



Example (5): Find the gradient of the line $y = 2x - 3$, using two points in the table from Example 1.

Choosing (0, -3) as (x_1, y_1) and (4, 5) as (x_2, y_2), the gradient works out at $\dfrac{y_2 - y_1}{x_2 - x_1}$ =

$$\frac{5-(-3)}{4-0} = \frac{8}{4} = 2.$$

Example (6): Find the gradient of the line $y = 8 - 5x$, using two points in the table from Example 2.

Choosing (0, 8) as (x_1, y_1) and (2, -2) as (x_2, y_2), the gradient works out at $\dfrac{-2-8}{2} = -\dfrac{10}{2} = -5.$

Example (7): Find the gradient of the line $x + 2y = 8$, from Example 3.

Choosing (0, 4) as (x_1, y_1) and (8, 0) as (x_2, y_2), the gradient works out at $\dfrac{0-4}{8} = -\dfrac{1}{2}.$

Example (8): Find the gradient of the line passing through the points (1, 3) and (1, 7).

Here we run into trouble, since $x_1 = x_2 = 1$, and substituting into the formula would lead to division by zero, which is undefined. (The line is in fact parallel to the y-axis, and its equation is $x = 1$).

In general, if a line is parallel to the y-axis, its gradient is undefined.

Forms of the straight-line equation.

The equation in Example (3) was $x + 2y = 8$, which could be written as $x + 2y - 8 = 0$.

This is of the general form $\boldsymbol{ax + by + c = 0}$, and all straight-line equations can be arranged in this way.

The more common form encountered at GCSE is $\boldsymbol{y = mx + c}$, known as the **gradient-intercept** form.

The gradient-intercept form cannot be used for equations of the "$x = c$" type as attempting to find the gradient of such a line would mean dividing by zero, which is undefined.

Using the results from previous examples, we found that the graph of $y = 2x – 3$ passed through the point (0, -3) - the y-intercept - and that it had a gradient of 2.

Similarly the graph of $y = 8 - 5x$ had a gradient of –5 and its y-intercept was the point (0, 8).
If we rewrite the last example as $y = -5x + 8$, we can see that the gradient is evidently the multiple of x, and that the graph crosses the y-axis where y takes the value of the constant.

- Any graph of the form $\boldsymbol{y = mx + c}$ has a gradient of m and a y-intercept at (0, c).

Sometimes a little algebraic manipulation is needed to put an equation into gradient-intercept form.

Example (6): Rewrite the following equations in "$\boldsymbol{y = mx + c}$" form:
i) $x + y = 5$; ii) $5y = 4x – 3$; iii) $2y – x + 3 = 0$

i) $x + y = 5$ can be rewritten as $y = 5 – x$ (or $y = -x + 5$).
ii) $5y = 4x – 3$ needs to be divided by 5 on each side to give $y = \frac{4}{5}x - \frac{3}{5}$.
iii) $2y – x + 3 = 0$ must be rearranged as $2y = x – 3 \rightarrow y = \frac{1}{2}x - \frac{3}{2}$.

Finding the equation of a straight-line graph.

The equation of a straight-line graph can be determined by just one point and the gradient. If the gradient is not known, two points will also suffice. If one of the stated points is also the *y*-intercept, the equation can be found very easily.

Example (7): Find the equation of the line with gradient 4 passing through the point (0, -7).

We are given both the gradient and the *y*-intercept, so we can immediately say that the equation of the line is $y = 4x - 7$.

Example (8): Find the equation of the line with gradient 2 passing through the point (3, 1).

This time we do not have the *y*-intercept, so we can only say that the equation of the line is $y = 2x + c$ where c is a constant to be determined.

Substituting $x = 3$ and $y = 1$ into the equation gives $1 = 6 + c \rightarrow c = -5$.
The equation of the graph is therefore $y = 2x - 5$.

Example (9): Find the equation of the straight line passing through the points (-1, 2) and (4, 17).

Here we are not given the gradient, m, so we have to work it out first.

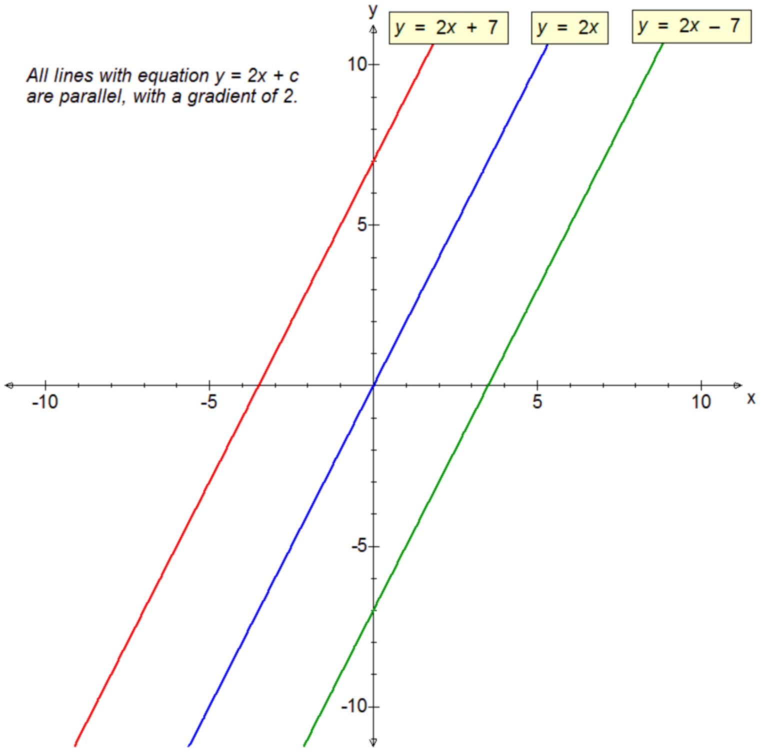
Taking (-1, 2) as (x_1, y_1) and (4, 17) as (x, y) we obtain

$$m = \frac{y - y_1}{x - x_1} \rightarrow m = \frac{17 - 2}{4 - (-1)} \rightarrow m = 3.$$

The gradient of the line is 3, and its equation is $y = 3x + c$.

Substituting $x = 4$ and $y = 17$ into the equation gives $17 = 12 + c \rightarrow c = 5$.
The equation of the graph is therefore $\boldsymbol{y = 3x + 5}$.

Finding the equation of a line parallel to a given line, passing through a specified point.



Parallel lines all have the same gradient, as the graphs above show.

The graphs of $y = 2x + 7$, $y = 2x - 7$ and $y = 2x$ are all parallel, as are the graphs of $y = 2x + c$ where c is any constant.

Example (10): Find the equation of the straight line parallel to $y = 10 - 3x$, and passing through the point (-2, 7).

The required line must have a gradient of -3, so its equation must be $y = -3x + c$ or $c = y + 3x$
.
Substituting $y = 7$ and $x = -2$ gives $c = 7 - 6$, $\rightarrow c = 1$.

$\therefore$ the equation of the required line is $\boldsymbol{y = -3x + 1}$ (or $\boldsymbol{y = 1 - 3x}$).

If the equation of the original line is given in the form $ax + by = c$ rather than in gradient- intercept form, then finding the equation of the parallel line is particularly simple – you just need to substitute x and y to find the new value for c.

Example (10a) : Find the equation of a line parallel to $4x + 3y = 11$, but passing through the point (5, 2).

Substituting $x = 5$, $y = 2$ and recalculating c gives the equation of the parallel line as $\boldsymbol{4x + 3y = 26}$.

Perpendicular straight lines.

When two straight lines are perpendicular, the product of their gradients is –1.

Example (11): Find the equation of the straight line perpendicular to the line $3x - 2y = 8$ and passing through the point (2, -1). Give the result in the form $ax + by = c$.

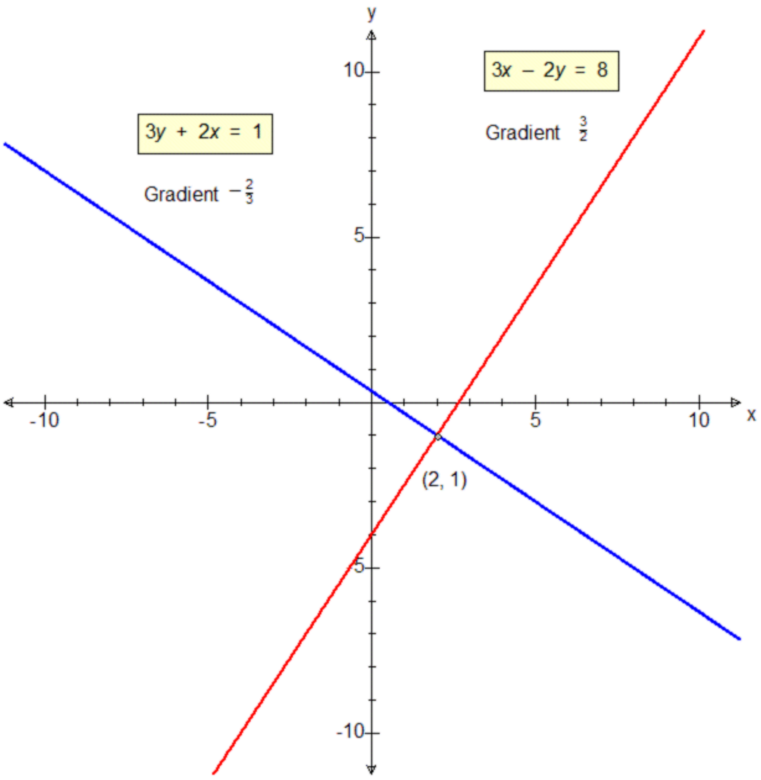
We can find the gradient of the original line by rearranging it into gradient-intercept form:
$3x - 2y = 8 \rightarrow 3x = 2y + 8 \rightarrow \frac{3}{2}x = y + 4 \rightarrow y = \frac{3}{2}x - 4$.
(We are actually only interested in the gradient here).

The gradient of the original line is $\frac{3}{2}$, so the gradient of the perpendicular line must be –1 divided by it, or $-\frac{2}{3}$.

The equation of the perpendicular line is therefore $y = -\frac{2}{3}x + c$.
Multiplying by 3 to get rid of the fractions gives $3y = -2x + c$ and $3y + 2x = c$.

Substituting $x = 2$ and $y = -1$, we have $c = 1$.

The equation of the perpendicular line is $3y + 2x = 1$.



One word of warning here: the perpendicular relationship can only be visualised if the scales of the axes on the graph are **uniform** !

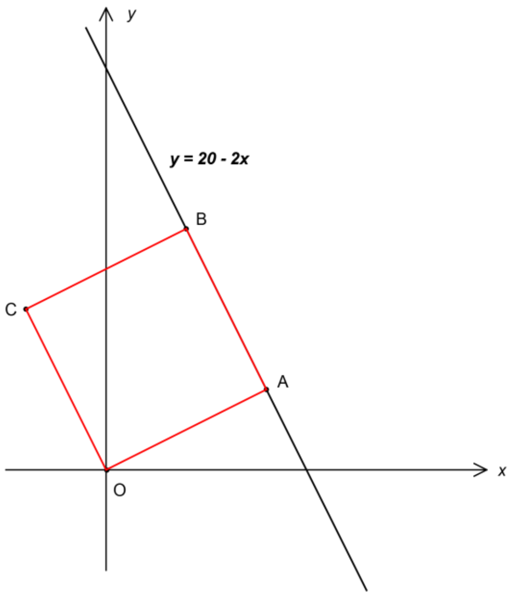
We could have saved ourselves the trouble of converting into gradient-intercept form as follows:

Any line perpendicular to $ax + by = c$ has the equation $bx - ay = c$.
(The constants c need not be equal and can take any numerical values.)

Hence the equation of the straight line perpendicular to the line $3x - 2y = 8$, passing through (2, -1). is $2x + 3y = c$, and to find c we substitute $x = 2$, $y = -1 \rightarrow c = 1$.

The equation of the perpendicular line is $2x + 3y = 1$.

Example (12): *OABC* is a square, with point *O* at the origin. Additionally, the points *A* and *B* lie on the straight line with equation $y = 20 - 2x$.



i) Find the coordinates of point *A*.

ii) Hence show, using congruent triangles, that the coordinates of point *C* are (-4, 8).

iii) Find the area of the square *OABC*.

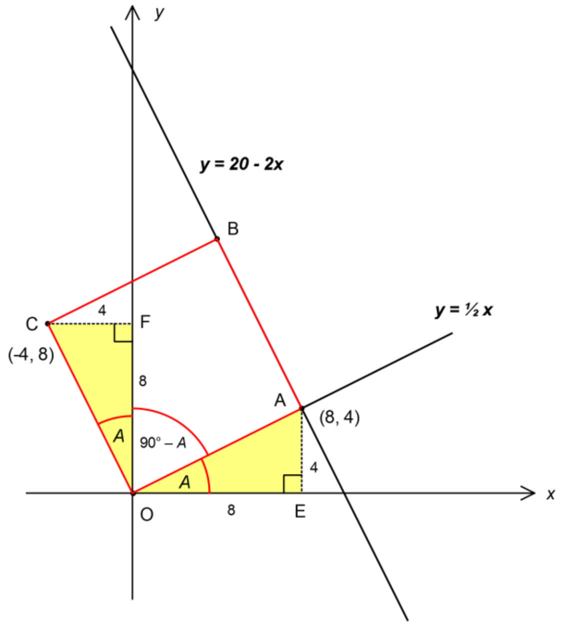
i) The gradient of the line *AB* is -2, and because *OAB* is a right angle, the gradient of *OA* = $\frac{1}{2}$.

Since *OA* also passes through the origin, its equation is $y = \frac{1}{2}x$.

Point *A* lies on the intersection of *OA* and *AB*, so we solve the equations $y = 20 - 2x$ and $y = ½\, x$ simultaneously.

$20 - 2x = ½\, x \quad \rightarrow 40 - 4x = x \quad \rightarrow 5x = 40 \rightarrow x = 8$.

Substituting for $y = ½\, x$, we have $y = 4$, so point *A* has coordinates of (8, 4).



ii) Draw perpendicular lines from *A* to the *x*-axis at *E*, and from *C* to the *y*-axis at *F*, so that $\angle OEA = \angle OFC = 90°$.

Let *OE* = 8 units and *EA* = 4 units (from part (i))
Let $\angle EOA = A$.

Now $\angle EOF = 90°$ (angle between the *x*- and *y*– axes) .
Also $\angle AOC = 90°$ (angle of the square)

Thus $\angle AOF = 90° - A$ by subtraction, and
$\angle FOC = 90° - (90° - A) = A$.

Also, *OA* = *OC* (sides of a square).

Hence $\angle OEA = \angle OFC$, $\angle EOA = \angle FOC$ and $OA = OC$.

Triangles *OEA* and *OFC* are congruent (RHS = RHS).

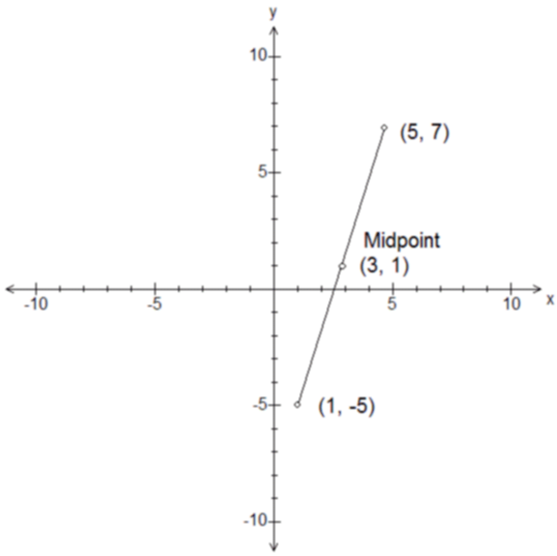
So $OF = OE = 8$ units, and $CF = AE = 4$ units.
Therefore the coordinates of *C* are (-4, 8). (The *y*-coordinate is negative as *C* is to the left of the *y*-axis).

iii) The side *OA* of the square is also the hypotenuse of the triangle *OEA*, and its area is $(OA)^2$.
By Pythagoras, $(OA)^2 = (OE)^2 + (EA)^2 = 8^2 + 4^2 =$ **80 units^2**.

Midpoint of a line.

If a point P has coordinates (x_1, y_1) and a point Q has coordinates (x_2, y_2) , then the midpoint of the line PQ has the coordinates

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



In the above example, the midpoint of the line joining the points (1, -5) and (5, 7) is the point (3, 1).

Note : 3 is halfway between 1 and 5, and 1 is halfway between –5 and 7.

Example (13): Find the midpoint of the line joining the points (-4, 5) and (6, 1).

The coordinates of the midpoint of the line are given as $\left(\frac{(-4)+6}{2}, \frac{5+1}{2}\right)$, simplifying to **(1,3)**.

Sometimes we might be given ‘one end’ and the midpoint, and be asked to find the ‘other end’.

We rearrange the formula as $2(x_m, y_m) = (x_1 + x_2,\ y_1 + y_2)$
$\Rightarrow\ (x_2, y_2) = 2(x_m, y_m) - (x_1,\ y_1)$.

The coordinates of the unknown end can therefore be found by subtracting those of the known end from double those of the midpoint.

Example (14): The midpoint *M* of the line *AB* has coordinates (2, 1). If point *A* is at (7, 5), find the coordinates of point *B*.

Here we are given $A = (x_1, y_1) = (7, 5)$ and the midpoint $M = (x_m, y_m) = (2, 1)$.

$\therefore\ (x_2, y_2) = 2(x_m, y_m) - (x_1,\ y_1)$
$\Rightarrow\ (x_2, y_2) = (4, 2) - (7, 5)$

$\Rightarrow\ (x_2, y_2) = (-3, -3)$.

This answer can also be visualised as follows: point *M* is a translation of *A* by (2 - 7), or -5 units, in *x* and by (1 - 5), or – 4 units, in *y*.

Point *B* is therefore an equivalent translation of *M* by the same amount, so its coordinates are (2 - 5, 1 - 4) or (-3, -3).

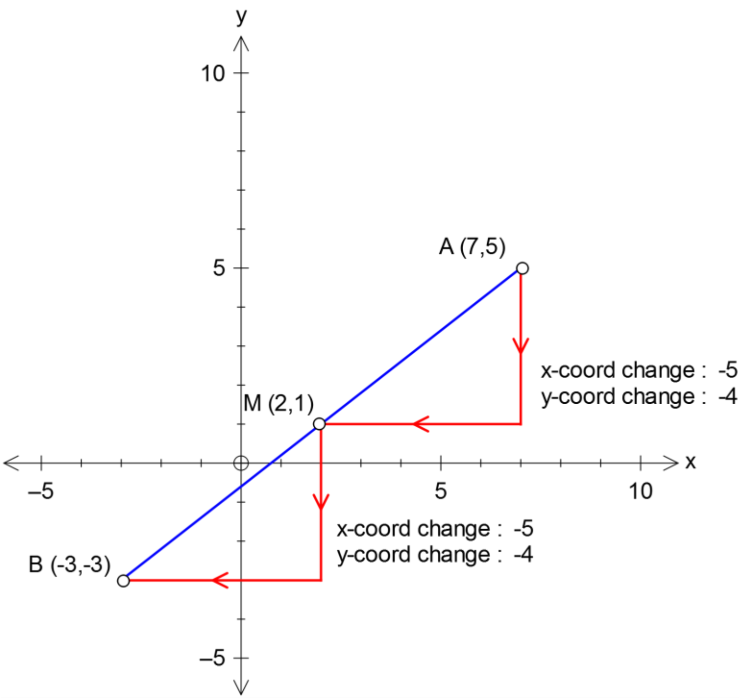
An alternative approach to solving the last problem is to use a “step” method as shown below.

When we get from *A* to *M*, the *x*-coordinate changes from 7 to 2 – a decrease of 5.

The *y*-coordinate changes from 5 to 1 – a decrease of 4.

To get from *M* to *B*, we repeat the changes.

We therefore decrease the *x*- and *y*-coordinates of *M* by 5 and 4 respectively to give those of *B*, namely (-3, -3).



Lines divided in a given ratio.

This is an extension of the method used to find the midpoint of a line, where the division of the line was in the ratio of 1:1.

If the point R divides the line AB in the ratio p:q , then we have to use proportionate division methods.

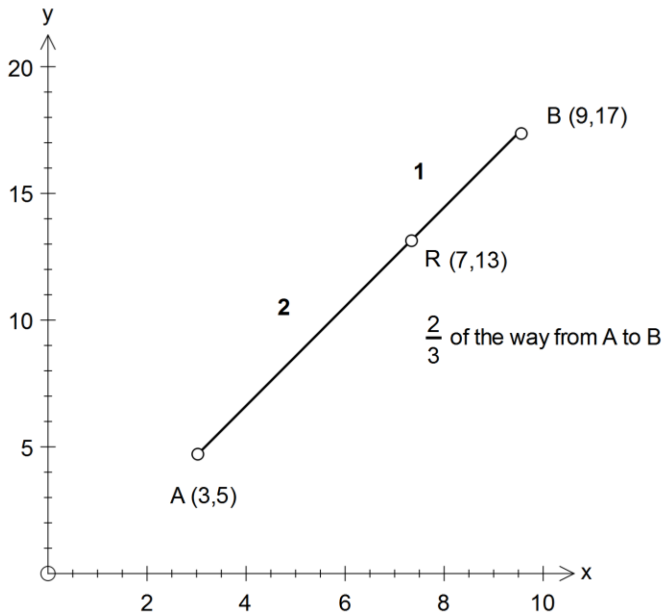
Example (15): The coordinates of points A and B are (3, 5) and (9, 17).
The point R divides AB in the ratio 2:1. Find the coordinates of R.

Adding the proportional parts together we have 2 + 1 = 3, so R is two-thirds of the way along the line joining A to B.

The difference between the x-coordinates of A and B is 9 - 3 or 6, and two-thirds of that difference is 4. The x-coordinate of R is therefore 3 + 4 or 7. (7 is two-thirds of the way from 3 to 9).

Similarly, the difference between the y-coordinates of A and B is 12, and two-thirds of 12 is 8. Hence the y-coordinate of R is 5 + 8 or 13.

$\therefore$ The coordinates of R are (7, 13)



Let point R divide line AB in the ratio p : q.
If A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2) , then the coordinates of R are given by the formula

$$(x_r, y_r) = \left(x_1 + \frac{p}{p+q}(x_2 - x_1),\ y_1 + \frac{p}{p+q}(y_2 - y_1) \right)$$

Again, we might have the case of being given 'one end' and the point of division, and being asked to find the 'other end'.

Example (16): The point *R* divides the line *AB* in the ratio 2 : 3, and the coordinates of points *A* and *R* are (-4, -5) and (6, 3). Find the coordinates of *B.*

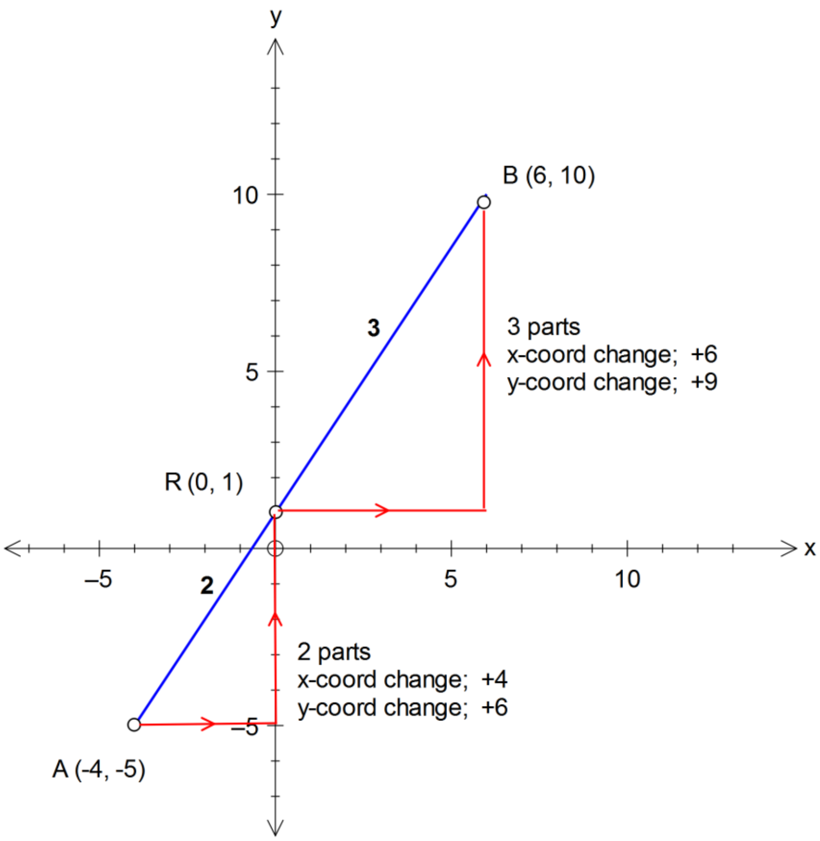
The "step" method is the easier one to use here. The distance from *A* to *R* is 2 proportional parts, and that from *R* to *B* is 3 parts. The difference between the *x*-coordinates of *A* and *R* is 4, and the difference between the *y*-coordinates is 6.

The coordinate differences corresponding to 2 proportional parts are 4 (for *x*) and 6 (for *y*).
From the above, 1 part corresponds to an *x*-difference of 2 and a *y*-difference of 3.

The distance from *R* to *B* amounts to 3 proportional parts, i.e. an *x*-difference of 3×2 or 6, and a *y*-difference of 3×3, or 9.

We need to add 6 and 9 to the *x*- and *y*-coordinates of *R* respectively, to obtain the corresponding coordinates for *B*, which work out as (0 + 6, 1 + 9), or (6, 10).

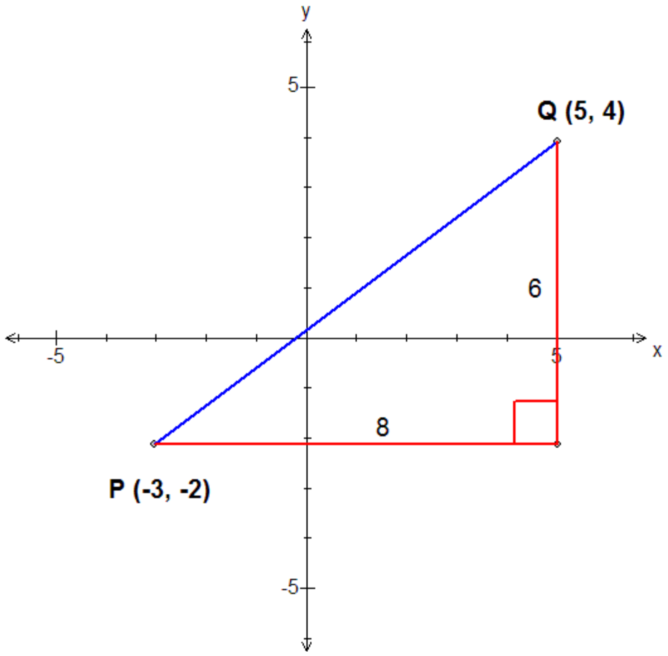
$\therefore$ **The coordinates of *B* are (6, 10).**



The distance between two points.

The distance between two points can be found by applying Pythagoras' theorem.

Example (17): Find the length of the line joining the points *P* (-3, -2) and *Q* (5, 4).



The line joining the two points can be visualised as the hypotenuse of a right-angled triangle whose other two sides run parallel with the axes and whose right angle is at the point (5,-2).

The lengths of the two sides are therefore:

8 units for the one parallel to the *x*-axis, obtained by subtracting the *x*-coordinate of the point (-3,-2) from that of (5,4) - i.e. 5 - (-3) = 8.

6 units for the one parallel to the *y*-axis, obtained by subtracting the *y*-coordinate of the point (-3,-2) from that of (5,4) - i.e. 4 – (-2) = 6.

The length of the hypotenuse, and therefore the distance *PQ*, is $\sqrt{8^2+6^2} = \sqrt{100}$ units, or **10 units.**

In general, the length of a line joining two points (x_1, y_1) and (x_2, y_2) on the plane is expressed as

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

In summary:

We find the difference between the two *x*-coordinates and square it.
Next, we find the difference between the two *y*-coordinates and square it.
Finally, we add the two squares and find the square root of the result.

Example (18): Find the distance between the points (-3, -7) and (5, 8).

Taking (x_1, y_1) as (-3, -7) and (x_2, y_2) as (5, 8), the length of the line joining the two points is

$$\sqrt{(5-(-3))^2+(8-(-7))^2}$$

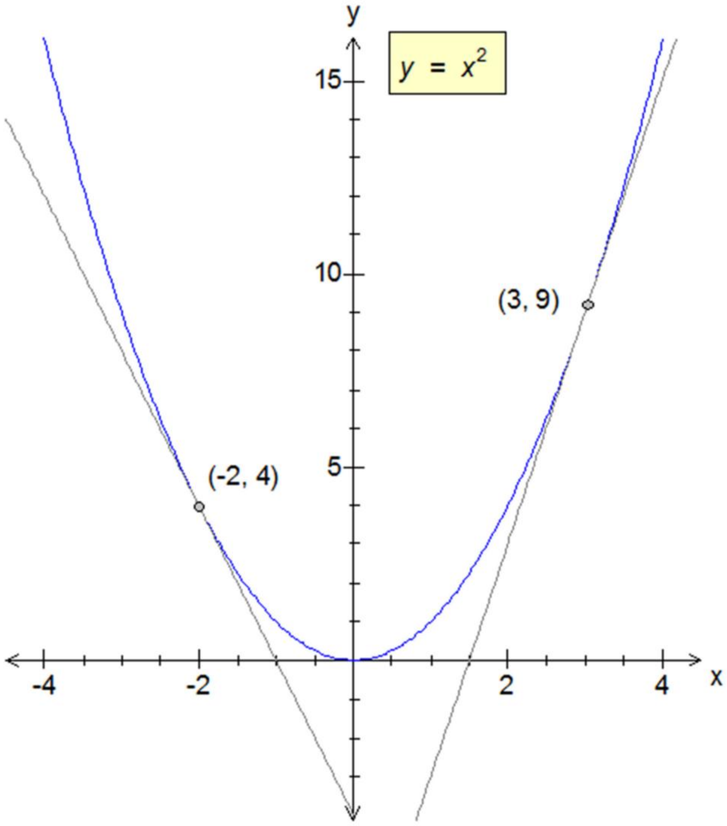
or $\sqrt{8^2+15^2} = \sqrt{289} = 17$ units.

The Gradient of a Curve. (More on "Real Life Graphs").

The full study of this topic is beyond the scope of GCSE, but some exam questions might require estimating the gradient of a curve at a given point.

Example (19). Find the gradient of the curve $y = x^2$ at the points (3, 9) and (-2, 4).

For each of the points in question, we draw a line that just **touches** the curve at the given points without crossing – i.e. the **tangent** to the curve.



As a hint, the small angles on each side of the tangent should be made as equal as possible as in the diagram on the right.

Having drawn the tangents as accurately as possible, we can then work out their gradients by selecting two points on each tangent.

To find the gradient of the tangent at (3, 9) we can choose the points (3, 9) and the x-intercept , which seems to be at (1.5, 0).

For the tangent at (-2, 4) we can choose the points (-2, 4) and the x-intercept , which this time seems to be at (-1,0).

Choosing (1.5, 0) as (x_1, y_1) and (3, 9) as (x_2, y_2), the gradient at (3, 9) works out as

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-0}{3-1.5} = \frac{9}{1.5} = 6.$$

Choosing (-2, 4) as (x_1, y_1) and (-1,0) as (x_2, y_2), the gradient at (-2, 4) works out as

$$\frac{0-4}{(-1)-(-2)} = \frac{-4}{1} = -4.$$

When x = 3, the gradient is equal to 6.

When x = -2, the gradient is equal to –4.

The gradient in each case seems to be twice the x-value.