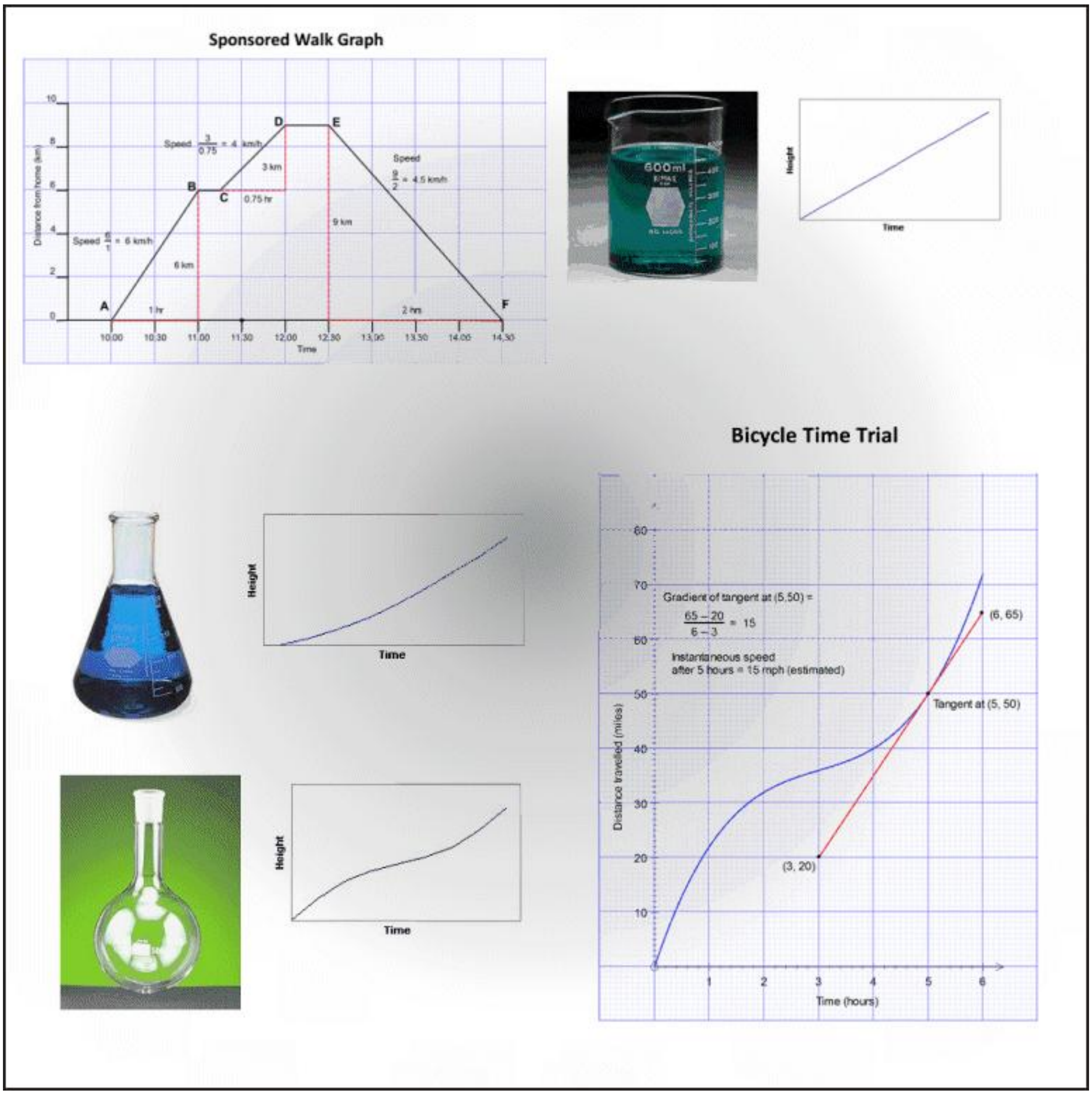
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

REAL LIFE GRAPHS



Version: 2.1 Date: 20-10-2015

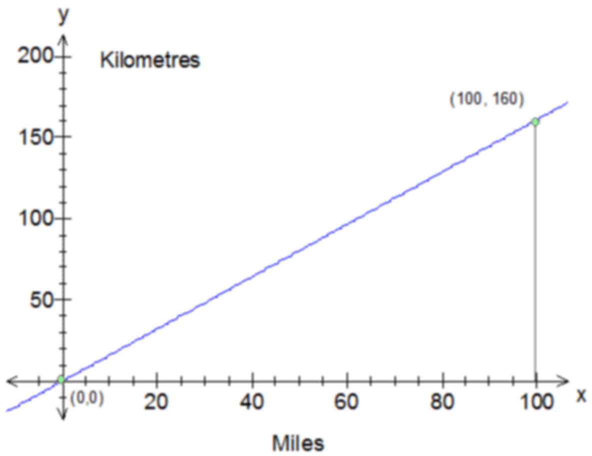
REAL-LIFE GRAPHS

Gradients - revision.

Many practical situations can be approximated or illustrated using straight-line graphs.

The graph on the right is a conversion graph between miles and kilometres.

By selecting two points on the graph, namely (0, 0) and (100,160), we can work out the gradient.



The gradient is therefore the change in *y* (km) divided by the change in *x* (miles) .

$\frac{160-0}{100-0}$ or 1.6 - the number of km in a mile.

Similar graphs could be used for currency conversion (e.g. £ to €).

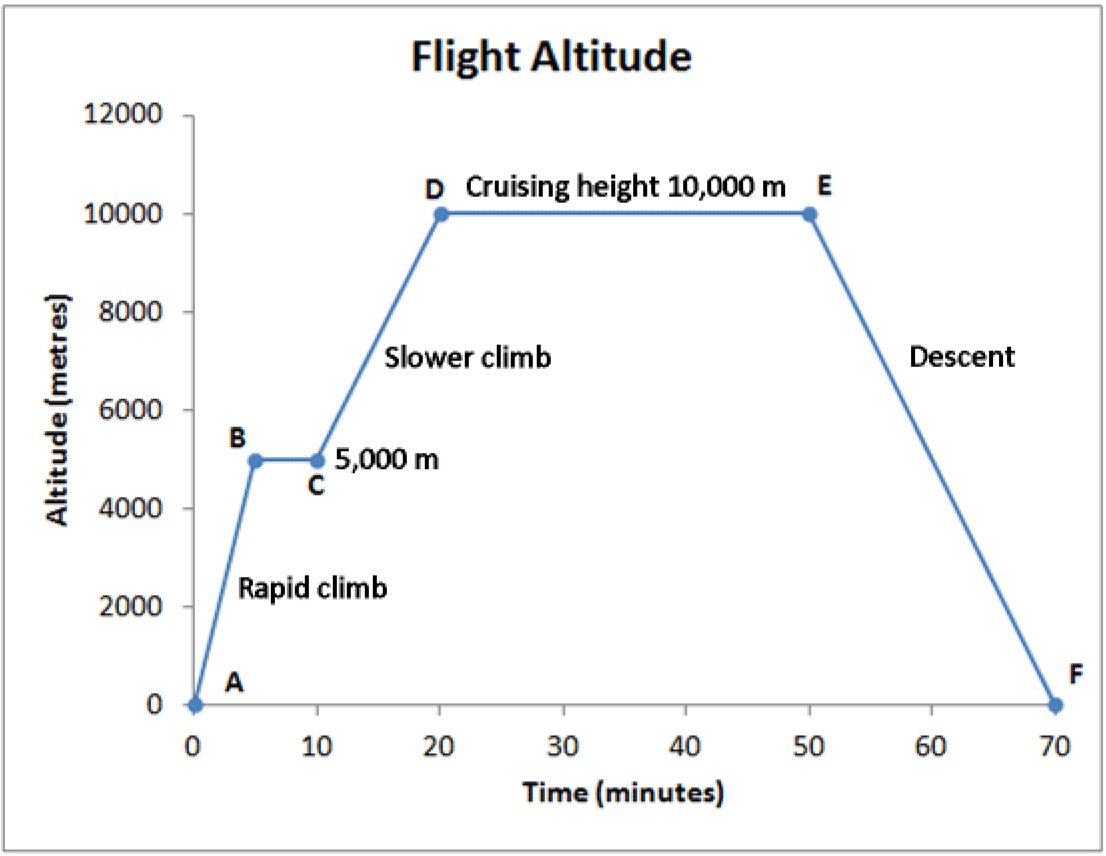
Other examples (note how they all have **time** on their horizontal axis) :

Real-life Situation	**Horizontal axis**	**Vertical Axis**	**Gradient**
Aircraft climbing after take-off	Time (sec)	Altitude (metres)	Climb rate in metres /s
Water pumped into a reservoir	Time (sec)	Volume (litres)	Flow rate in litres / s
Car travelling in a straight line	Time (hrs)	Distance (km)	Speed in km / h
People entering a stadium via turnstiles	Time (min)	Number of people through turnstile	People per minute

Many real-life situations lend themselves to graphical representation. The most important among them are **distance-time** graphs (more correctly **displacement-time** graphs) and **velocity-time** graphs.

Example (1): A plane takes off from Manchester and climbs for five minutes to a height of 5,000 metres, stays at that height for another five minutes, and then takes ten minutes to reach its cruising altitude of 10,000 metres. It remains at this cruising height for half an hour until beginning a twenty-minute descent, landing at Paris after a 70-minute flight.

Draw a distance-time graph to represent the plane's altitude during the flight, assuming steady climbing and descent.

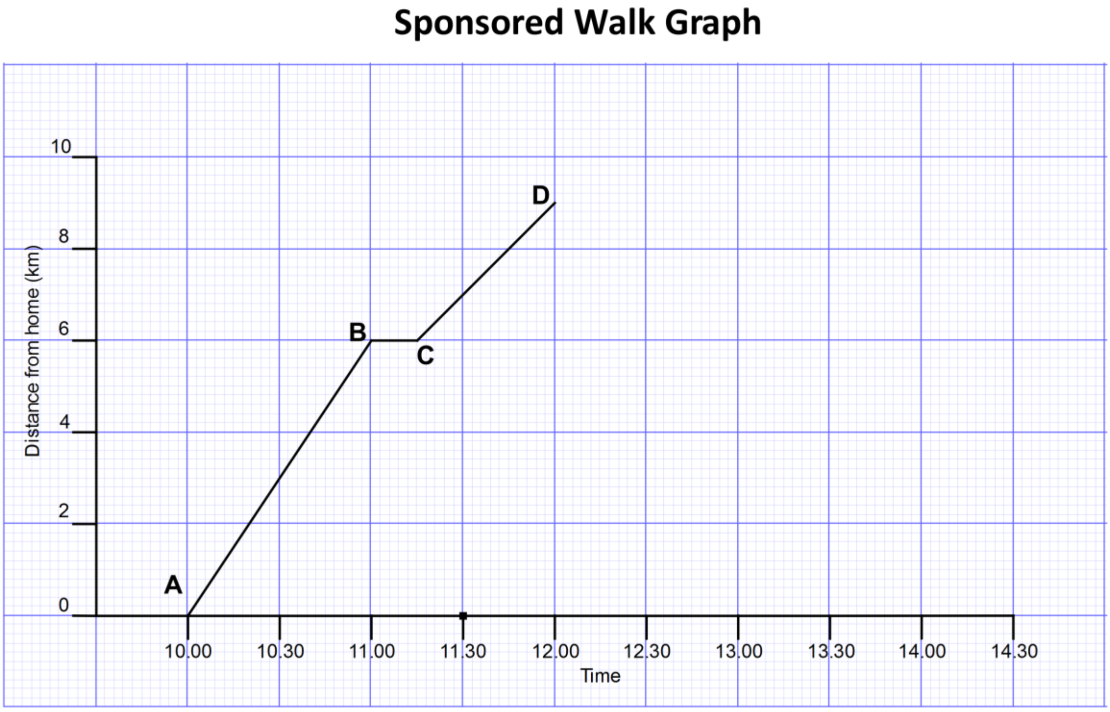


The section from **A** to **B** represents a rapid climb from ground level at Manchester to 5,000 metres in the space of five minutes, followed by a short five-minute spell at that intermediate height from **B** to **C**.

From **C** to **D** the plane gains height again in ten minutes, but at a slower rate than it did from **A** to **B**, as can be seen by the gentler slope. The next 30 minutes from **D** to **E** have the plane at its cruising height, until it begins its 20-minute descent into Paris at point **F**.

This was an example was of a distance-time graph with altitude on the *y*-axis.

Example (2): Sue took part in a sponsored walk and kept a record of her distance walked at various times. Part of her progress is shown on the displacement-time graph below.



i) Between which times did Sue take a rest break ?

ii) How far was Sue from home at point D ?

iii) Part of the walk was over hilly roads, where Sue had to slow down. Between which two points did this occur ? Explain your answer briefly.

i) Because the gradient of the graph represents speed, then Sue is not walking between the points where the graph is horizontal – i.e. between points **B** and **C** , or 11.00 to 11.15.

ii) Sue was 9 km from home at point **D**.

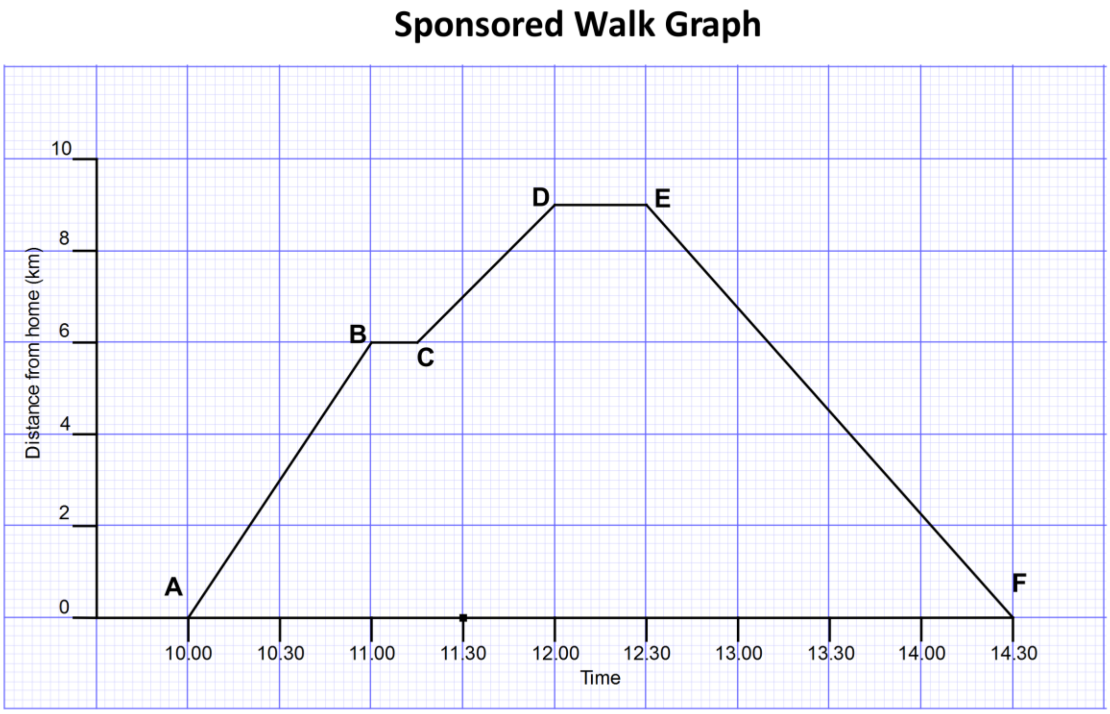
iii) The graph is less steep between points **C** and **D**, than it is between **A** and **B**. When the graph is less steep, we have a lower speed. Hence the hilly stretch of the walk is between **C** and **D**.

Example (2) continued :

iv) Sue took another break at 12 noon, lasting for half an hour. She then walked the return journey at a constant speed, arriving back at her starting point at 14.30. Complete the displacement/time graph.

v) What was the **total** distance walked by Sue ?

iv) We complete the graph by drawing a horizontal line from **D** to **E** to represent the rest break, followed by a line from **E** to the horizontal time axis where the time is 14.30.



v) Sue has walked 18 km in total – 9km on the outward and 9 km on the return walk.

In short, on a displacement-time graph:
an upward slope = moving in the forward direction
a downward slope = moving in the reverse direction
zero slope = stationary (no movement)
gradient = velocity

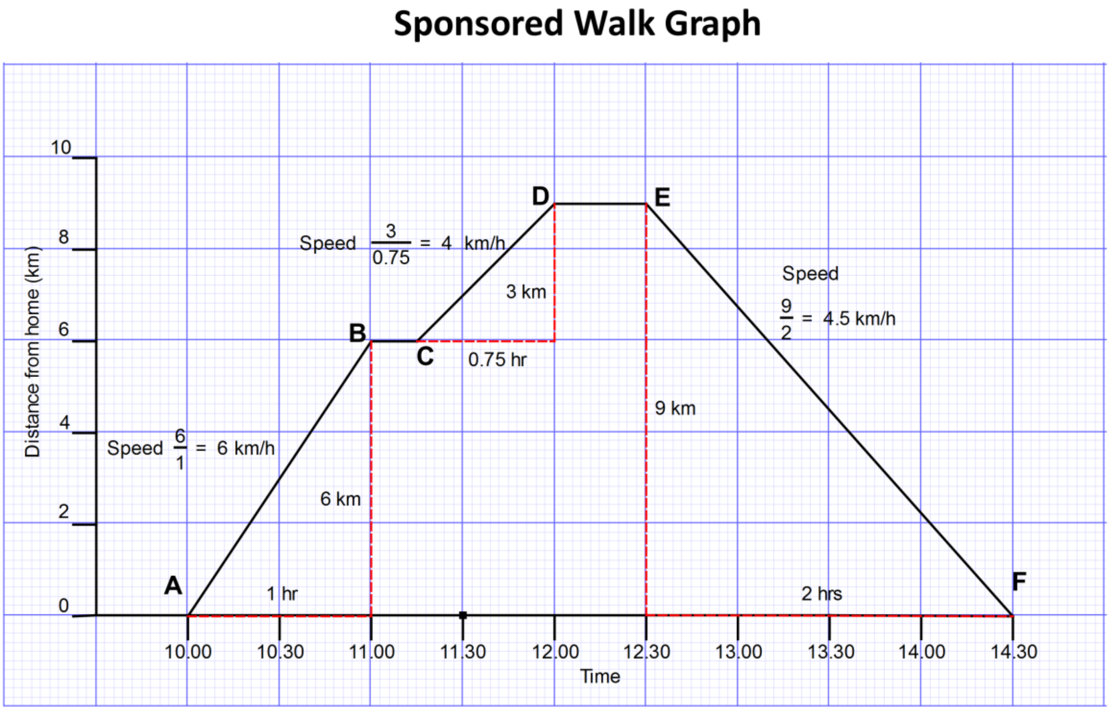
There is an important difference between **velocity** and **speed** (see later !), but it is not too vexing an issue at GCSE.

Finding speed from displacement-time graphs.

Example (3): Work out Sue's speed at each stage of her walk from Example (2), excluding the rest breaks. (Be careful with decimals and fractions of an hour !.)

Sue's speed over various sections of the walk can be worked out by finding gradients.
For example, she covers the 6 km between points **A** and **B** in the space of 1 hour, so her walking speed is simply $\frac{6}{1}$ km/h or 6 km/h.
The section from **C** to **D** takes Sue 45 minutes (0.75 hour) to walk, and is 3 km long, which makes her speed over that stretch equal to $\frac{3}{0.75}$ km/h or 4 km/h.

Finally, her speed over the section from **E** to **F** similarly works out at $\frac{9}{2}$ km/h or 4.5 km/h.



Difference between velocity and speed.

Although velocity and speed are often taken to be the same, there is a subtle difference between the two, and it is particularly important in physics.

Speed has **magnitude** but no **direction**, but velocity has both.

We had worked out Sue's speed between points **E** and **F** as being 4.5 km/h in the last example. At **E** she was 9 km from the start point, but at **F** she was already there. Her distance from home was decreasing between those two points, and so the gradient was actually $\frac{-9}{2}$ or –4.5, remembering that a downward-sloping line has a negative gradient.

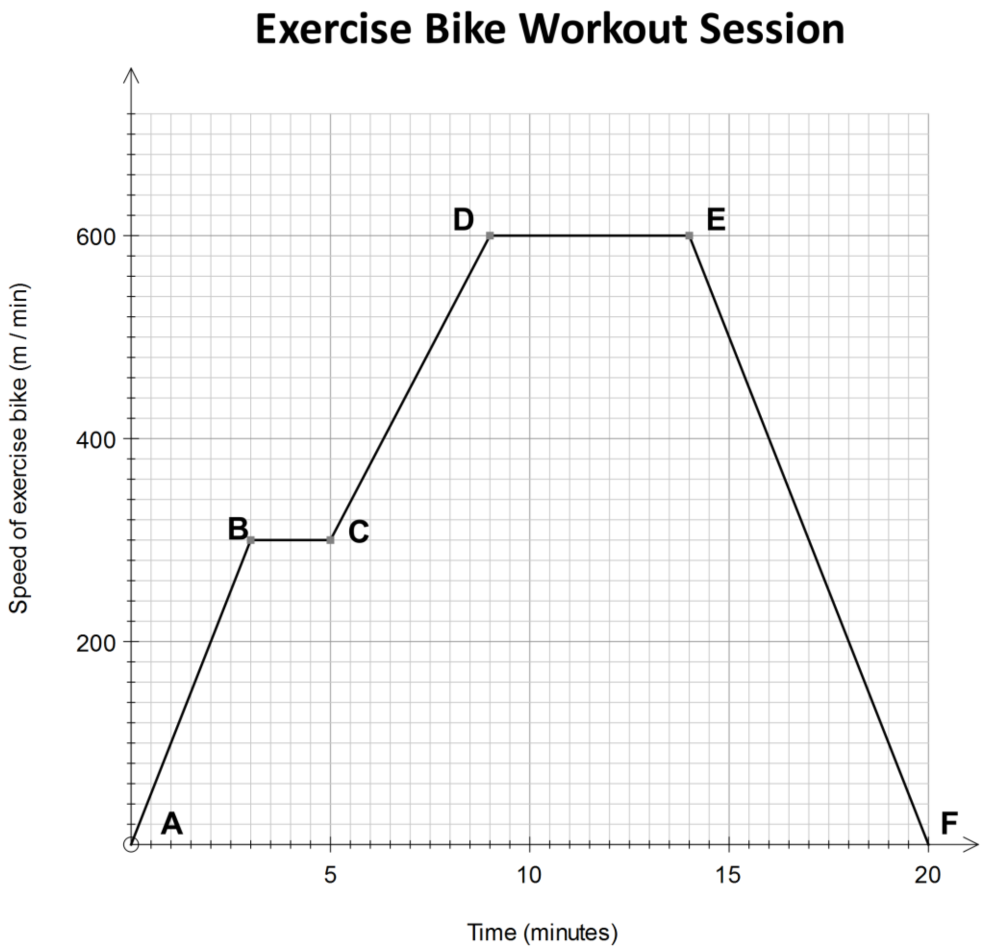
But because **speed** has no direction, we take it to be positive, i.e. 4.5 km/h in this case.

The **velocity**, on the other hand, must retain its sign. If we take velocities in the outward direction as positive, then those on the return journey will be negative.

Velocity-Time Graphs.

Recall the following characteristics of displacement-time graphs :
an upward slope = moving in the forward direction
a downward slope = moving in the reverse direction
zero slope = stationary (no movement)
gradient = velocity

Example (4a): Brad has spent a workout session on the exercise bike at his gym. His programme is shown on the speed / time graph below. (Take velocity and speed to be the same here.)

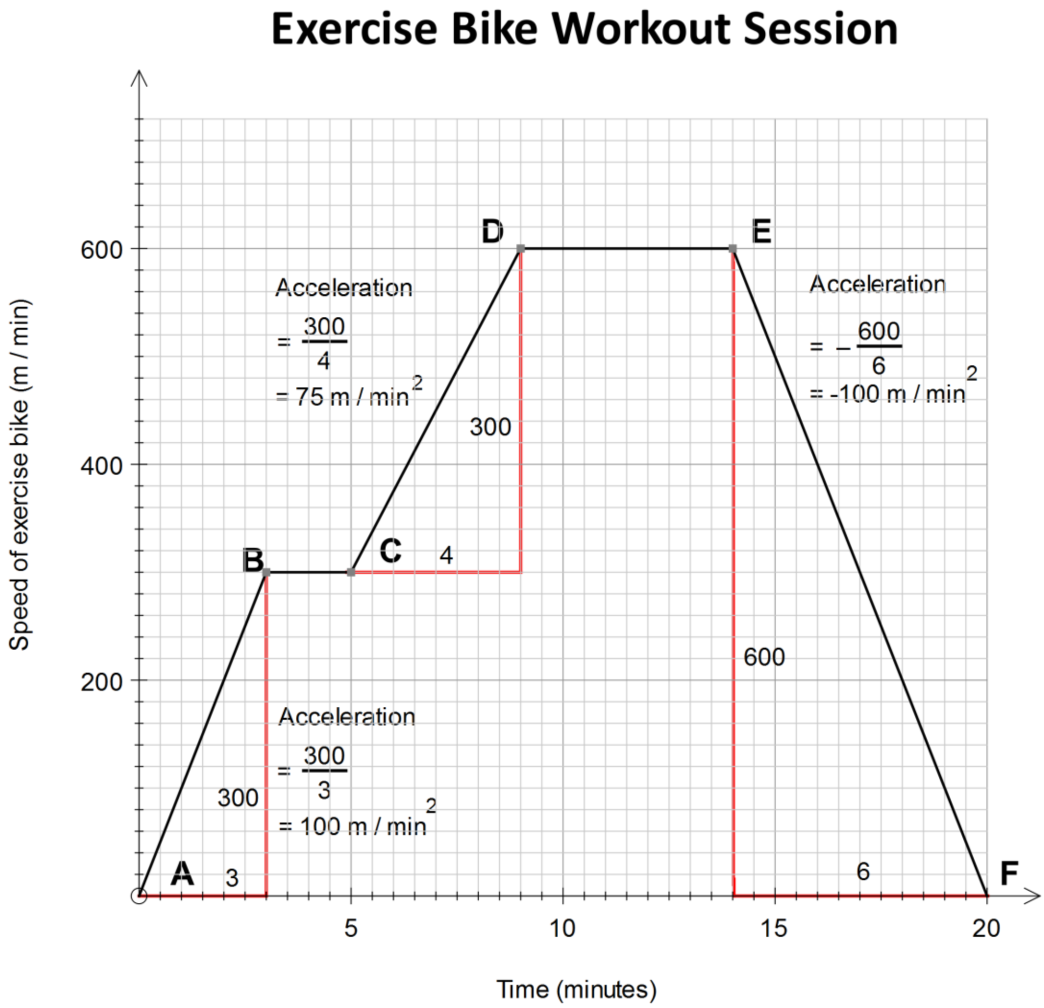


i) Describe Brad's programme by stages.
ii) Calculate the gradients between A and B, C and D and E and F. What do these gradients represent ?

The general shape of the graph is outwardly similar to that of the sponsored walk graph in the earlier example. The interpretation is however very different.

Brad's programme can be broken down into five sections.

He is stationary at **A** and accelerates uniformly to 300 m/min at **B** over the first three minutes.
For the next two minutes, between points **B** and **C**, his speed is a constant 300 m/min.
Over the next four minutes, between points **C** and **D**, he accelerates uniformly to 600 m/min.
His speed is constant at 600 m/min for the next 5 minutes, between points **D** and **E**.
Finally he decelerates uniformly from point **E** for the next 6 minutes until he is stationary at **F**.



ii) The gradient between **A** and **B** is $\frac{300}{3} = 100$, signifying an acceleration rate of 100 metres per minute per minute. (Remember, acceleration is the rate of change of speed). This can be written as 100 m /min^2 ("100 metres per minute squared").

The gradient between **C** and **D** is $\frac{300}{4} = 75$, or an acceleration rate of 75 m /min^2.

Finally the gradient between **E** and **F** is $-\frac{600}{6} = -100$.

This negative value for the acceleration means a slowdown, or *de*celeration.

Brad is therefore decelerating at a rate of 100 m /min^2.

It can be seen that, on a velocity-time graph:

an upward slope = speeding up, i.e. acceleration
a downward slope = slowing down, i.e. negative acceleration or deceleration
zero slope = constant velocity
gradient = acceleration if positive, deceleration if negative.

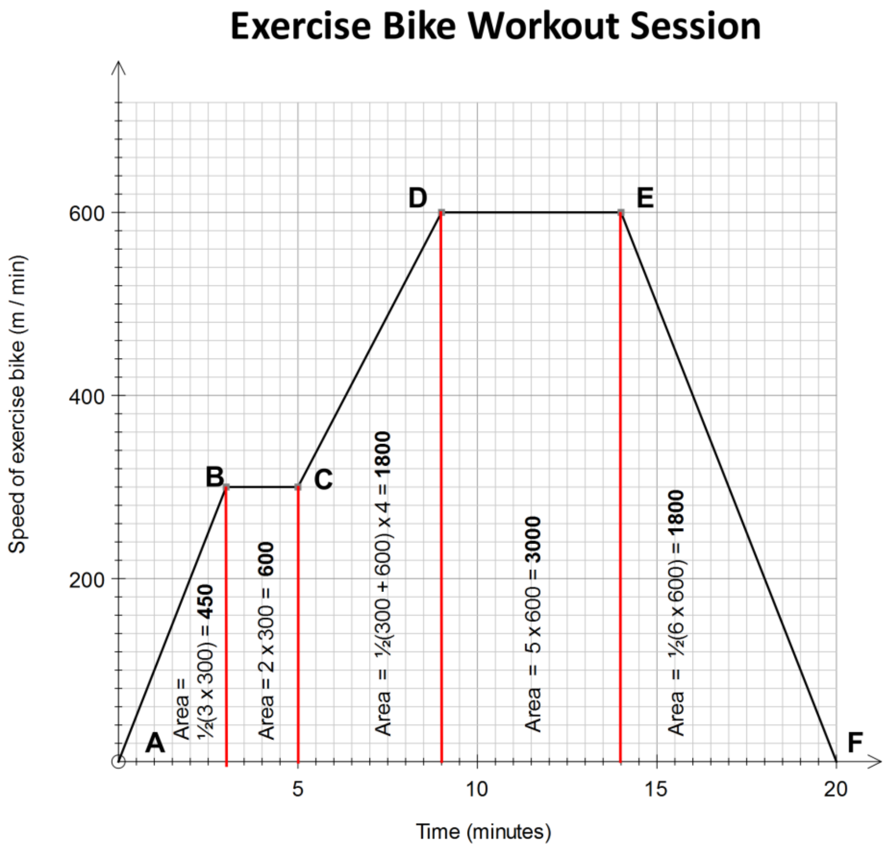
Finding the distance covered by using a velocity / time graph.

The velocity / time graphs in the last example can also be used to calculate the distance cycled by Brad in his exercise session.

To do this, we must **calculate the area between the graph and the horizontal axis.**

We can divide up the area into simpler shapes, and then add all the individual areas together.

Example (4b): Find the distance cycled by Brad on the exercise bike from Example 4(a). .



Distance cycled = (450 + 600 + 1800 + 3000 + 1800) metres = 7650 metres or 7.65 km.

The area under the graph has been divided up into two rectangles, two triangles and a trapezium. Visualise the graph rotated clockwise by 90° so that the red lines become bases of the given shapes .

We proceed to calculate the area of the individual sections:
Triangle from **A** to **B** : base 300, height 3 ; area = ½ (300 × 3) = 450
Rectangle from **B** to **C** : base 300, height 2 ; area = 600
Trapezium from **C** to **D** : height 4, parallel sides 300 and 600 ; area = ½(300 + 600) × 4 = 1800
Rectangle from **D** to **E** : base 600, height 5 ; area = 3000
Triangle from **E** to **F** : base 600, height 6 ; area = ½ (600 × 6) = 1800

The total area of the five sections is 7650, corresponding to the distance (in metres) cycled by Brad.

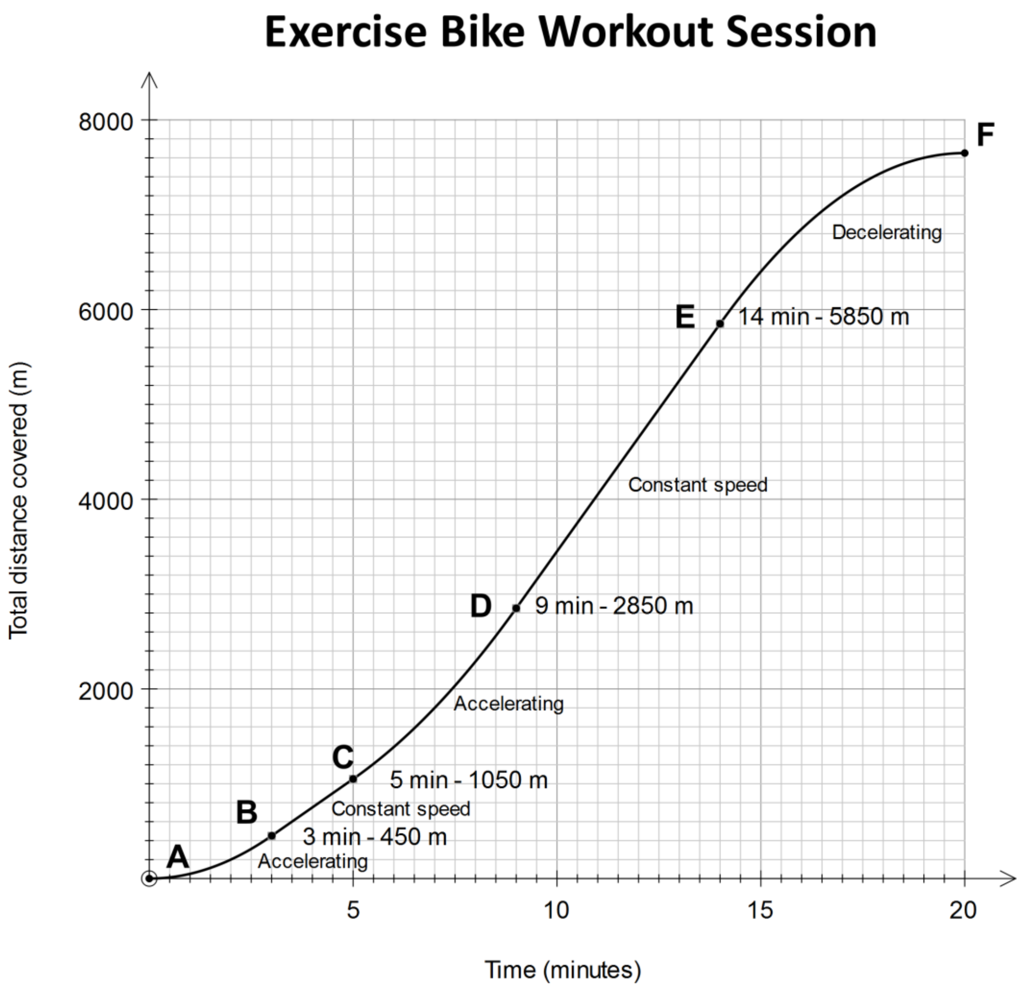
∴ Brad has cycled **7650 metres** or **7.65 km** in this exercise bike session.

N.B. The calculations are straightforward here because the units of distance and time are consistent.

Thus, if the speed axis were in metres per second, we would have to multiply all speeds by 60 to convert them into metres per minute. **Always use consistent units !**

The diagram below shows the velocity / time graph in Example (4) transformed into a displacement / time graph.

What do you notice about this graph, compared to the sponsored walk graph in Example (3) ?



This graph is entirely in the "forward" direction, so there is no "there and back" scenario.

The main difference, though, is in the presence of curved sections, and these occur whenever Brad is accelerating or decelerating, i.e. from **A** to **B**, from **C** to **D** and from **E** to **F**.

By contrast, the straight sections correspond to the sections where Brad's speed is constant, i.e. from **B** to **C** and from **D** to **E**.

Finally, on the curved sections where Brad is accelerating (**A** to **B**, **C** to **D**), there is an increasing gradient, but on the section between **E** and **F**, where Brad is decelerating, there is a decreasing gradient.

Therefore, on a **displacement-**time graph with curved sections:

an upward slope = moving in the forward direction
a downward slope = moving in the reverse direction
zero slope = stationary
gradient = velocity (if the section of the graph is curved, then use the gradient of the tangent)
increasing gradient = acceleration
decreasing gradient = deceleration

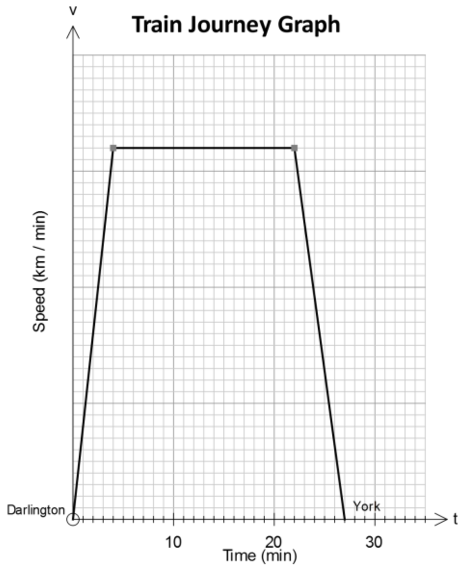
Example (5): George is travelling on an express train and has a chat with the guard about the train's performance. The guard says:

"This is a high-speed stretch of line from Darlington to York. It takes us 4 minutes to get to speed out of Darlington, and we hold this steady speed for 18 minutes. We then start to ease off and brake until we arrive at York 5 minutes later. "

George then proceeds to draw a velocity-time graph for the train's run from Darlington to York.

i) What major assumption has George made about the train's performance in that graph ?

ii) The distance from Darlington to York is 72 km. Use George's graph to find the steady speed of the train.



i) George's main assumption is that the train accelerates and decelerates at a constant rate, as those particular sections of the graph are straight line segments.

ii) We do not know the steady speed of the train (v), but we do know the timings of the three sections of the journey, as well as the total distance.

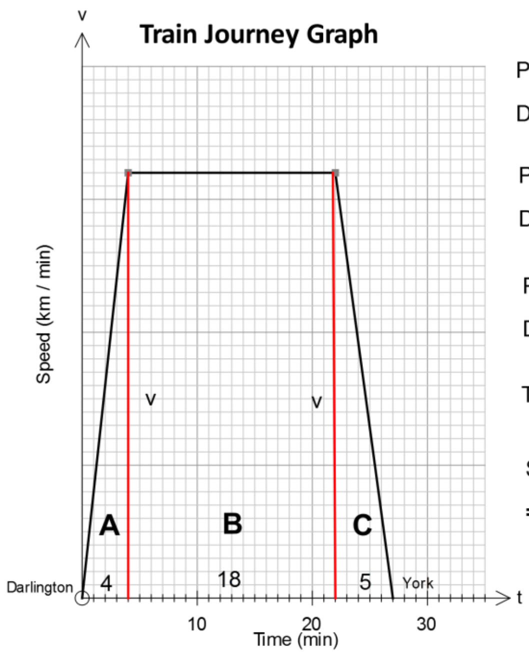
In part **A**, the train is accelerating from rest, so we find the area of the triangle, which is ½(base × height) = ½(4v) = 2v km.

Part **B** , shown as a rectangle, is the steady-speed section, so the distance covered is simply its area, i.e. 18v km.

In part **C**, the train is decelerating to a standstill, so again we have a triangle, of area ½(5v) = 2½v km.

The total area under the graph is 22½v units, but we are told that the total distance travelled is 72 km, so the steady speed v = $\frac{72}{22½} = 3.2$ km per minute,

or 3.2 × 60 = **192 km / h.**



Part A: acceleration

Distance : ½ (4v) = 2v km

Part B: steady speed

Distance : 18v km

Part C: deceleration

Distance : ½ (5v) = 2½v km

Total distance = 22½v km = 72 km

Steady speed, v = $\frac{72}{22½}$ = 3.2 km / min

= 192 km / h.

We could have also found v by using the trapezium area formula. The parallel sides are 18 and 27 units (time in minutes) and the area (distance in km) = 72 units^2.
The mean of the parallel sides is ½(18+27) = 22½ units, and dividing the area by the mean of the parallel sides gives the height of 3.2 units, corresponding to the steady speed v in km per minute.

Finding velocity from a non-linear displacement-time graph.

The last graph shown was an example of a displacement / time graph with curved sections. The question here is, how do we find the gradient of a curved graph ?

The next examples demonstrate the techniques, which form an introduction to **calculus,** a section of maths mostly beyond GCSE.

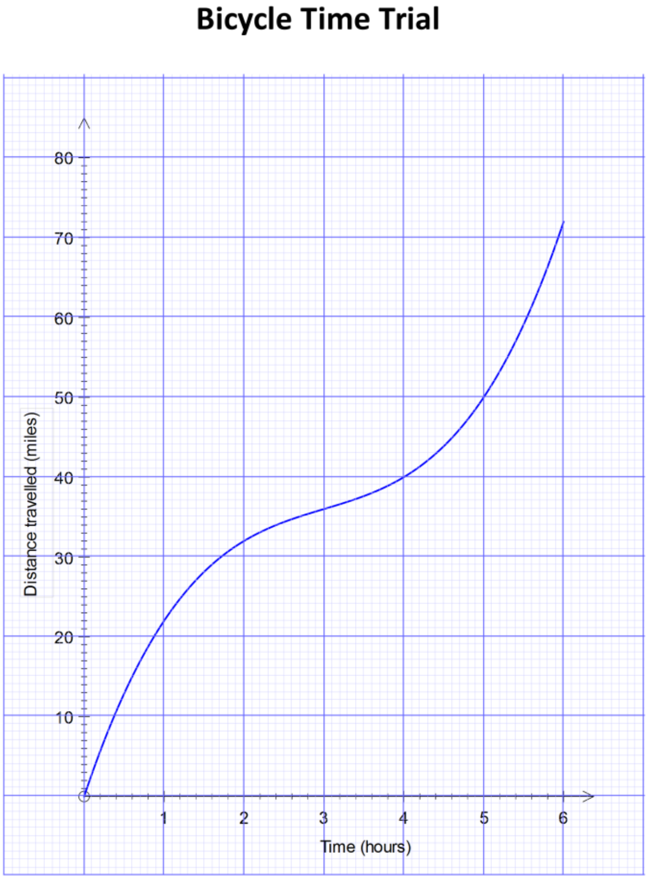
Example (6): After the comforts of working on the exercise bike on the gym, Brad sets off on a six-hour, non-stop cross-country cycle race. The course starts off level, but there is a very tough and hilly middle section until the road levels out again.

Brad uses the "target" distance/time graph on the right against which to measure his progress.

i) Explain how you can tell that the middle section is hilly.

ii) Calculate Brad's mean cycling speed during the fifth hour of the race according to the graph.

iii) Estimate Brad's cycling speed at the instant he has completed his fifth hour of the race.



i) It is much more difficult to cycle uphill than on the level, so Brad's speed will be much reduced. As this is a displacement / time graph, the gradient represents speed, and a reduced speed means a less steep gradient, as can be seen in the section between the second and fourth hours.

ii) The fifth hour of Brad's race takes place between the 4 and 5 hour marks.

By looking at the graph, Brad plans to have cycled 40 miles after 4 hours and 50 miles after 5 hours.

We can therefore draw a chord (i.e. a line) connecting the points (4, 40) and (5, 50) and work out its gradient to obtain Brad's mean speed between those two time intervals.

Recall the method of finding gradients :

$\frac{y_2 - y_1}{x_2 - x_1}$, or the change in the value of y divided by the change in the value of x.

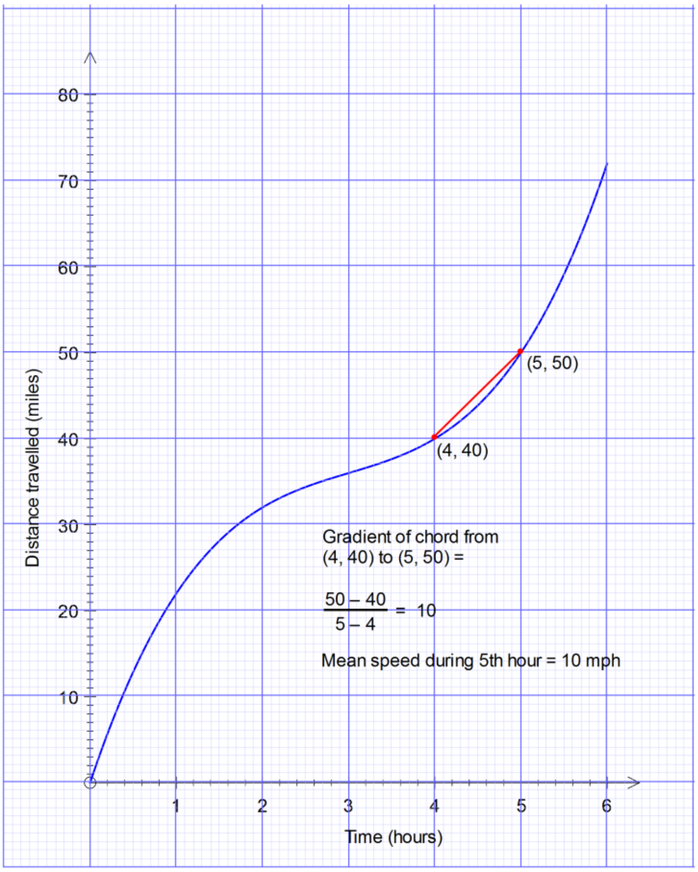
In this particular case,
$(x_1, y_1) = (4, 40)$ and $(x_2, y_1) = (5, 50)$

The gradient of this chord is

$$\frac{50-40}{5-4} = \frac{10}{1} = 10.$$

Since the time scale is in hours and the distance scale is in miles, Brad's mean cycling speed in that particular hour is **10 mph**.

Bicycle Time Trial



iii) In the last part, we found a mean speed using a chord joining two points on the distance/time graph, and then calculating the gradient of the chord.

To find an instantaneous speed, we can only estimate it by drawing a tangent to the curve as accurately as we can. A hint is to attempt to make the small angles on each side of the tangent as equal as possible.

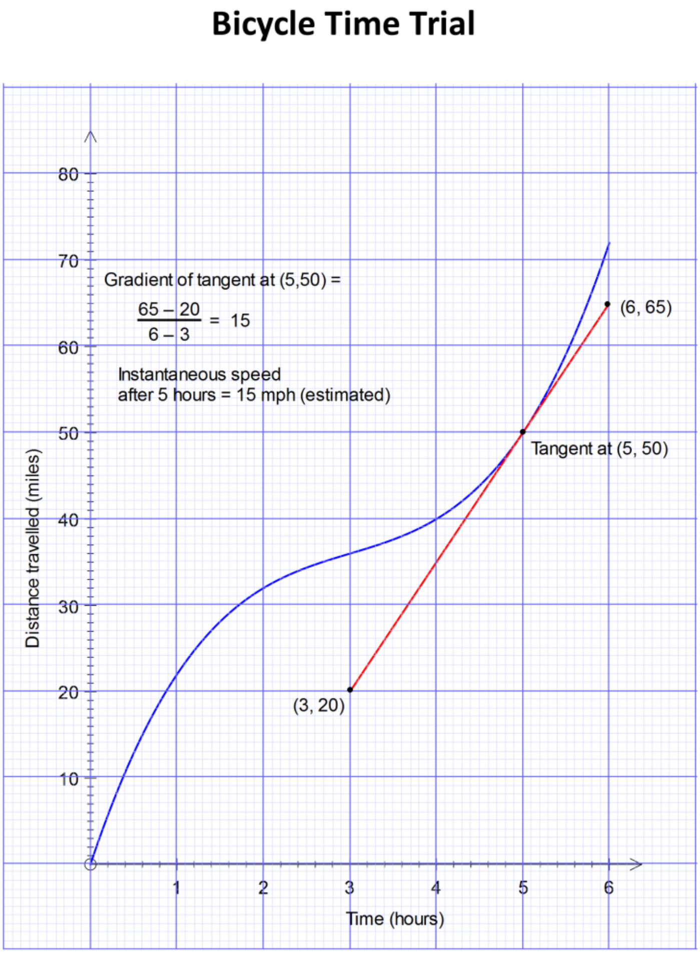
(Recall : a tangent is a line which touches a curve at one point)

Here, we have managed to draw a tangent to the curve at the point (5, 50), and this same tangent also passes through the points (3, 20) and (6, 65).

The gradient of the tangent as drawn is

$$\frac{65-20}{6-3} = \frac{45}{3} = 15.$$

Brad's instantaneous cycling speed at the 5-hour mark is therefore **15 mph** according to the estimated tangent drawn at the curve.



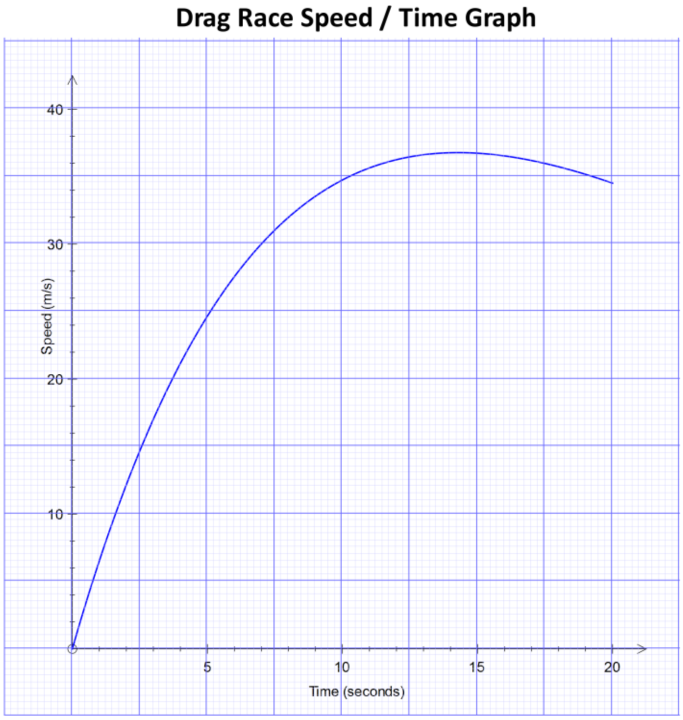
Finding the distance under a non-linear velocity / time graph.

In Example 4(a), we used the area under a linear velocity./ time graph to calculate the total distance, and we did that by dividing the area into triangles, rectangles and trapezia.

How do we extend this method to non-linear graphs ?

The graph on the right shows the progress of a car over a 20-second drag race, where the vertical axis is in metres per second.

There are no straight sections on this graph at all, so we can find the area under the curve only approximately.



Example (7a):

i) Use the speed / time graph on the previous page to find the speed of the car after 10 seconds, and after 20 seconds.

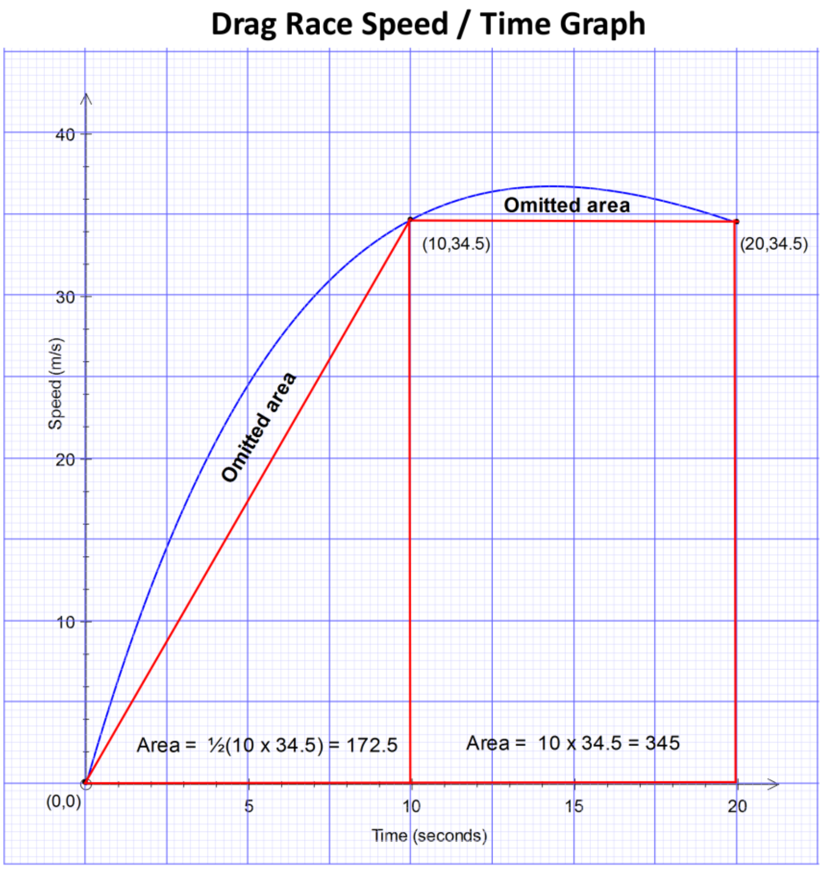
ii) Find the approximate area under the graph, and hence an estimate for the distance covered by the car in 20 seconds, giving the answer in metres.

i) The car's speed is 34.5 m/s after 10 seconds, and again 34.5 m/s after 20 seconds.

Additionally, the speed is zero at the start.

We have three points on the curve, and so we draw lines connecting them to each other, and also to the horizontal axis, as shown on the right.

What we have done is divide the area into two strips, one a triangle and the other a rectangle.



Estimated driven distance = 172.5 + 345 = 517.5 m = 518 m

As in Example 4(b) , we visualise the graph rotated clockwise by 90° so that the vertical red lines become bases of the given shapes .

We proceed to calculate the area of the strips from left to right :
a) a triangle for the "0 to 10 seconds" strip : base 34.5, height 10 ; area = ½ (34.5 × 10) = 172.5
b) a rectangle for the "10 to 20 seconds" strip: base 34.5, height 10 ; area = 345

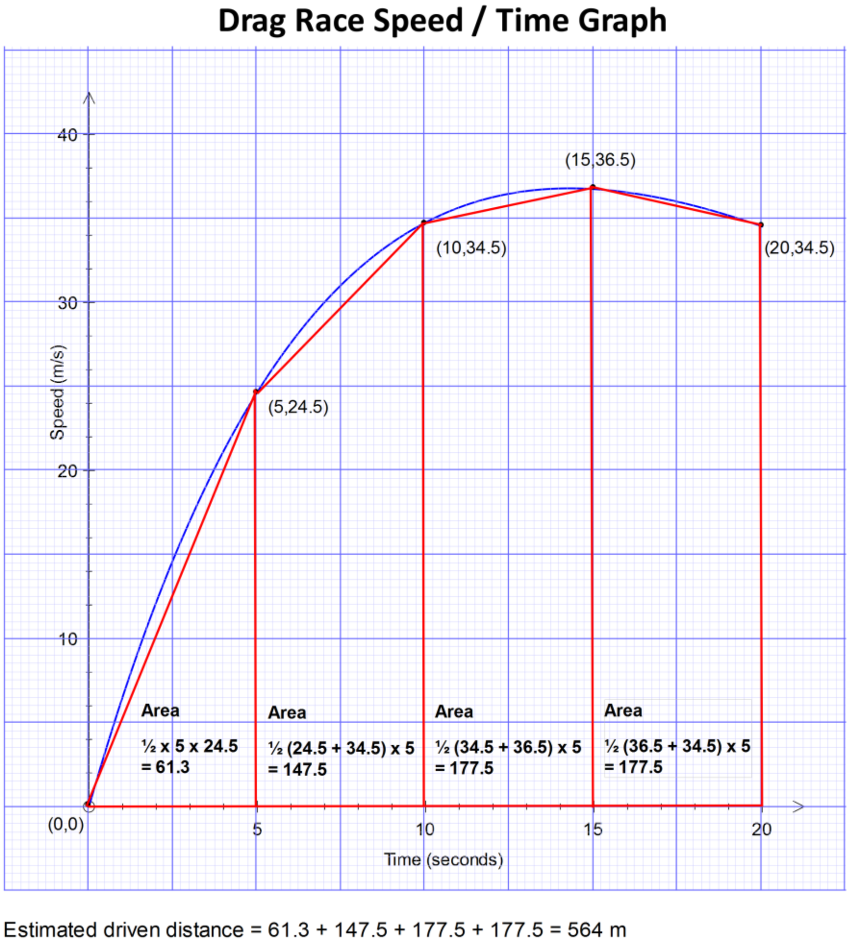
The total estimated area under the curve is 172.5 + 345, corresponding to a driven distance of **518 metres** to the nearest metre.

This estimate, using two strips, is not especially accurate, as there are two large areas under the curve omitted from the calculation. To reduce this error, and hence improve on the estimate, we must use more strips.

Example (7b):

i) Use the original speed / time graph on the right to find the speed of the car after 5, 10 15 and 20 seconds.

ii) Divide the graph up into four strips, and thus find the approximate area, and hence an improved estimate for the distance covered by the car in 20 seconds, giving the answer in metres.



i) The car's speed is :

0 m/s at 0 seconds,
24.5 m/s after 5 seconds,
34.5 m/s after 10 seconds,
36.5 m/s after 15 seconds
and 34.5 m/s after 20 seconds.

We now have five points on the curve and four strips, as on the right. The first strip is a triangle, and the other three are trapezia.

Again, we visualise the graph rotated clockwise by 90°.

Calculating the estimated area under the curve, we have

a) a triangle for the "0 to 5 seconds" strip : base 24.5, height 5 ; area = ½ (24.5 × 5) = 61.3

b) a trapezium for the "5 to 10 seconds" strip: height 5, parallel sides 24.5 and 34.5 ;
area = ½(24.5 + 34.5) × 5 = 147.5

c) a trapezium for the "10 to 15 seconds" strip: height 5, parallel sides 34.5 and 36.5 ;
area = ½(34.5 + 36.5) × 5 = 177.5

d) a trapezium for the "15 to 20 seconds" strip: height 5, parallel sides 36.5 and 34.5 ;
area = ½(36.5 + 34.5) × 5 = 177.5

The total area of the four strips is 61.3 + 147.5 + 177.5 + 177.5 = 564 to the nearest integer, corresponding to a driven distance of **564 metres** to the nearest metre.

We can also see that the omitted area, and hence the error in the estimate, is much lower. For more accurate estimates, all we need to do is keep increasing the number of strips.

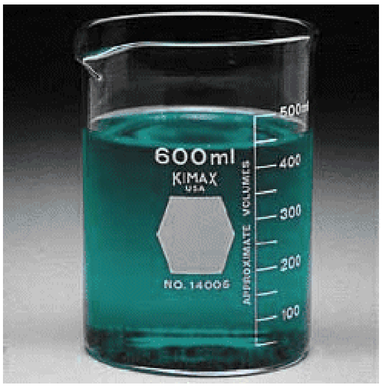
Other applications.

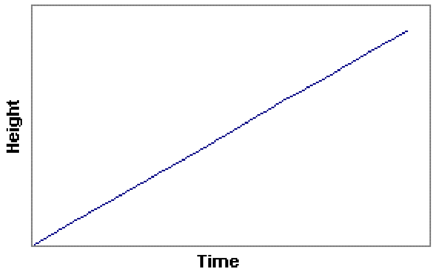
Another example of real-time graphs concerns rates of flow into vessels (such as flow of rain into a gauge). The example below illustrates the pattern:

Example (8): Water is poured at a constant rate into i) a beaker of constant width; ii) a conical flask; iii) a round-bottomed flask.

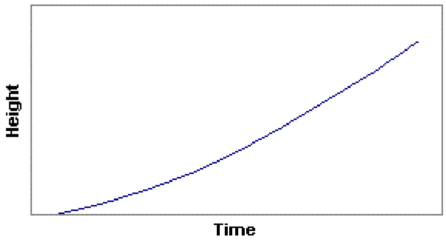
Sketch graphs showing the change in the height of the water in each container with increasing time.

The beaker has constant width, and so the increase in the height of the water follows a straight-line graph. There is no change in the surface area of the water / air boundary.





The conical flask narrows towards the neck, and therefore the water level increases at a faster rate as the surface area of the water / air boundary decreases. (We have not lncluded flow into the neck.)



With the round flask, the surface area of the water / air boundary starts off small, then grows to a maximum, and finally becomes small again. The flow rate therefore follows this trend in reverse; fast, then slow, then fast again. (Again, we have not included flow into the neck.)