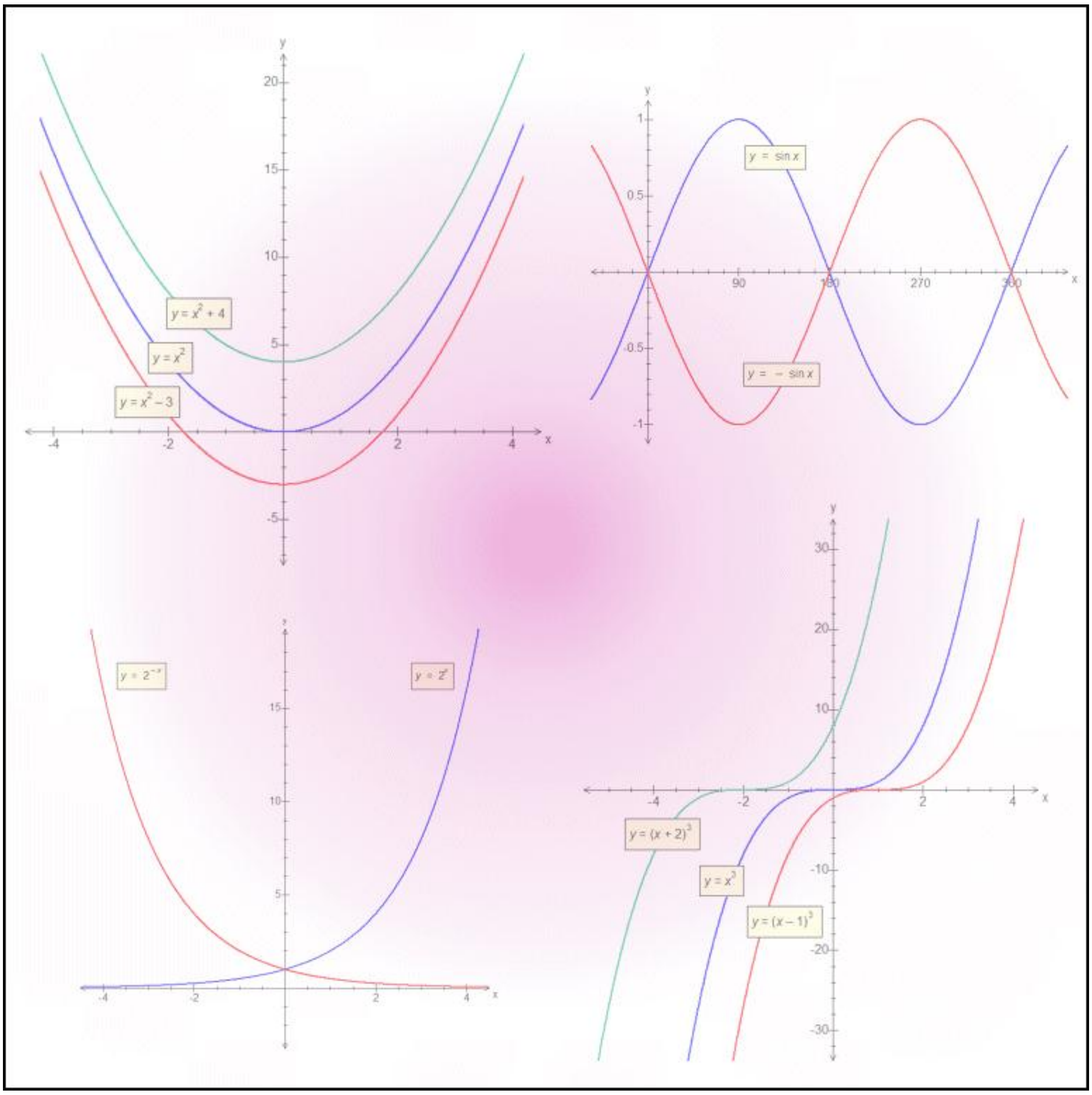
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

TRANSFORMATIONS OF GRAPHS



Version: 2.3 Date: 08-07-2016

TRANSFORMING GRAPHS.

It is possible to obtain a whole family of graphs from a single one by means of transformations.

There are four such transformations to remember: the *y*-translation, the *x*-translation, the *y*-stretch and the *x*-stretch. Of the last two, there are only two special cases to be concerned about at GCSE.

The *y*-translation.

This example will take the function $y = x^2$ and transform it into $y = x^2 + 4$ and $y = x^2 -3$.
The resulting set of graphs is shown below.

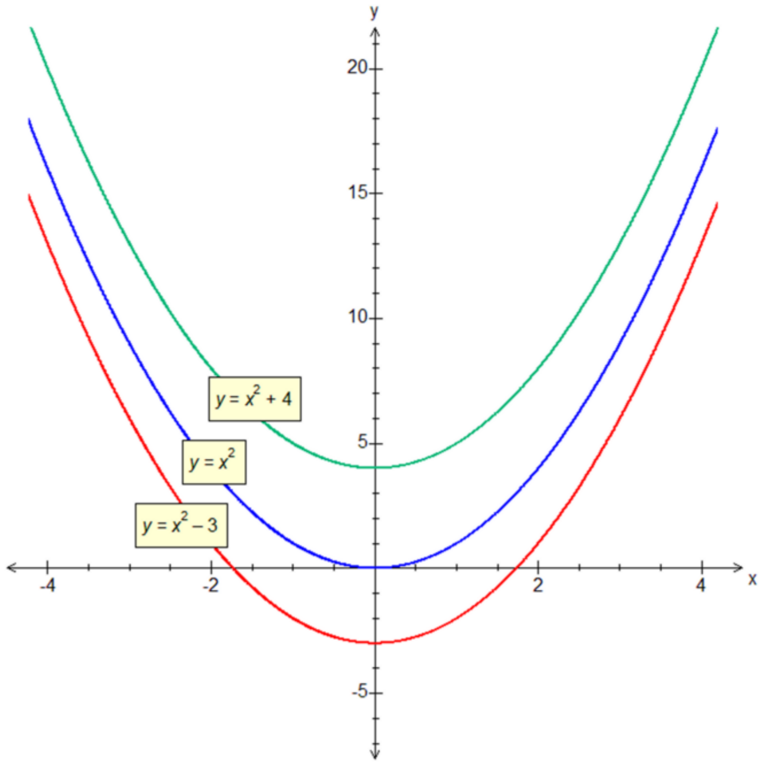
x	$y = x^2$	$y = x^2 + 4$	$y = x^2 - 3$
-3	9	13	6
-2	4	8	1
-1	1	5	-2
0	0	4	-3
1	1	5	-2
2	4	8	1
3	9	13	6

The graph of $y = x^2 + 4$ is obtained by moving that of $y = x^2$ upwards by 4 units.

In other words, we apply a translation of +4 units in the *y*-direction.

By contrast, we obtain the graph of $y = x^2 -3$ by moving that of $y = x^2$ downwards by 3 units.

This time, we apply a translation of -3 units in the *y*-direction.



For any function $y = f(x)$, the graph of the function $f(x) + k$ is the same as the graph of $f(x)$, but translated by k units in the *y*-direction. In column vector form, the transformation is $\begin{pmatrix} 0 \\ k \end{pmatrix}$.

In the examples above, the graph of $y = x^2$ is translated to that of $y = x^2 + 4$ by the vector $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, the and to that of $y = x^2 -3$ by the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

The *x*-translation.

This example will take the function $y=x^3$ and transform it into $y = (x + 2)^3$ and $y = (x - 1)^3$.

x	selected $y = x^3$	selected $y = (x + 2)^3$	selected $y = (x - 1)^3$
-5		-27	
-4		-8	
-3	-27	-1	
-2	-8	0	-27
-1	-1	1	-8
0	0	8	-1
1	1	27	0
2	8		1
3	27		8
4			27

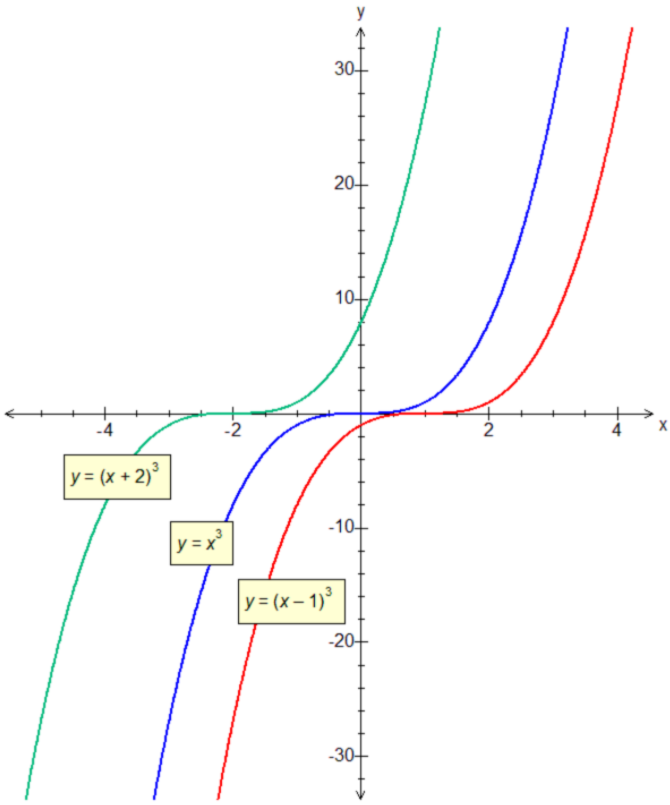
The graph of $y = (x + 2)^3$ is obtained by moving that of $y = x^3$ leftwards by 2 units.

In other words, we apply a translation of -2 units in the x-direction.

Similarly we obtain the graph of $y = (x - 1)^3$ by moving the graph of $y = x^3$ rightwards by 1 unit.

This time, we apply a translation of +1 unit in the x-direction.

This transformation may seem to work the 'wrong way' at first sight !



For any function $y = f(x)$, the graph of the function $f(x+k)$ is the same as the graph of $f(x)$, but translated by $-k$ units in the x-direction. In column vector form, the transformation is $\begin{pmatrix} -k \\ 0 \end{pmatrix}$.

In the examples above, the graph of $y=x^3$ is translated to that of $(x + 2)^3$ by the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, and to that of $(x - 1)^3$ by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The *y*-stretch.
This transformation is not studied at GCSE, save for one special case :

Reflection in the *x*-axis.

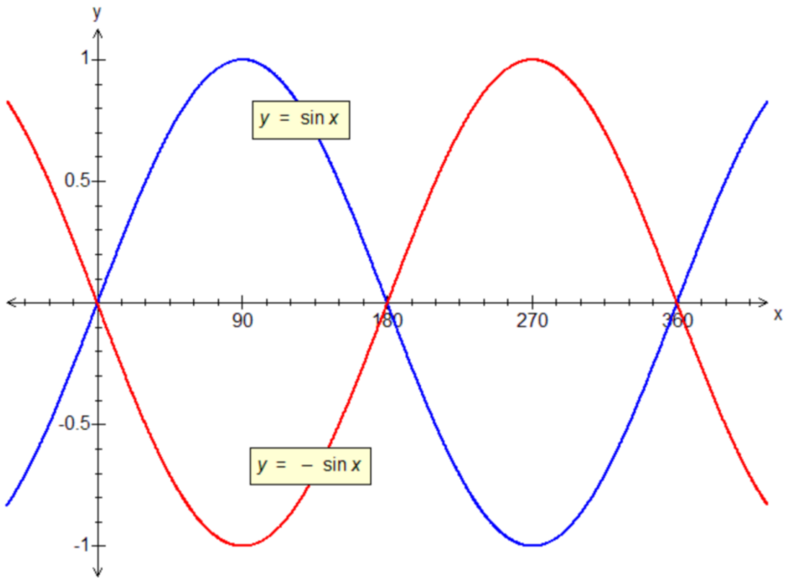
This example will take the function $y = \sin x°$ and transform it into $y = -\sin x°$.

$x°$	$y = \sin x°$	$y = -\sin x°$
0	0	0
30	0.5	-0.5
90	1	-1
150	0.5	-0.5
180	0	0
210	-0.5	0.5
270	-1	1
330	-0.5	0.5
360	0	0

If the graph of a function $f(x)$ is reflected in the *x*-axis, it is transformed to the graph of $-f(x)$.

The graphs shown here, where $f(x) = \sin x$, illustrate the situation .

(The *y*-values have had their sign reversed).



The x-stretch. (Phased out for 2017 exams)
Again, this transformation is not studied at GCSE, save for one special case :

Reflection in the y-axis.

This example will take the function $y = 2^x$ and transform it into $y = 2^{-x}$.

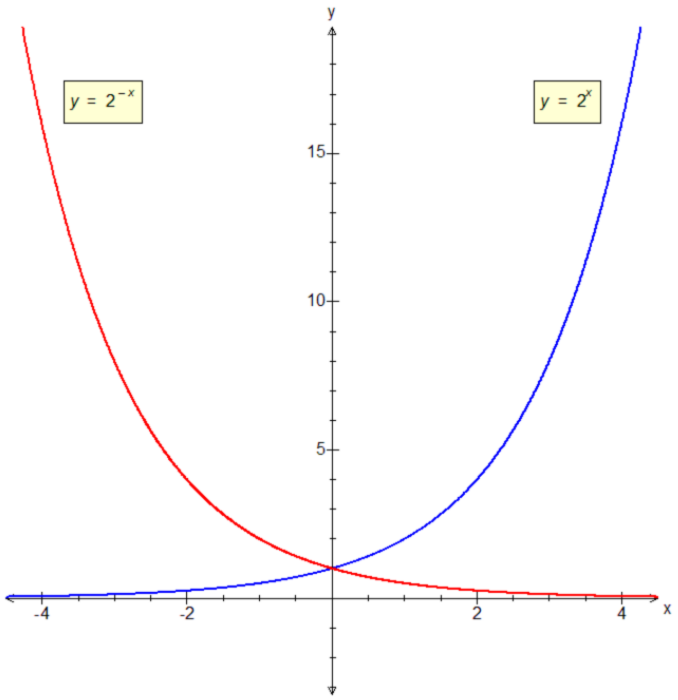
x	$y = 2^x$	$y = 2^{-x}$
-4	$\frac{1}{16}$	16
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
4	16	$\frac{1}{16}$

If the graph of a function $f(x)$ is reflected in the y-axis, it is transformed to the graph of $f(-x)$.

The graphs shown here, where $f(x) = 2^x$, illustrate the situation .

The transformed graph is that of $y = 2^{-x}$.

(The x-values have had their sign reversed).

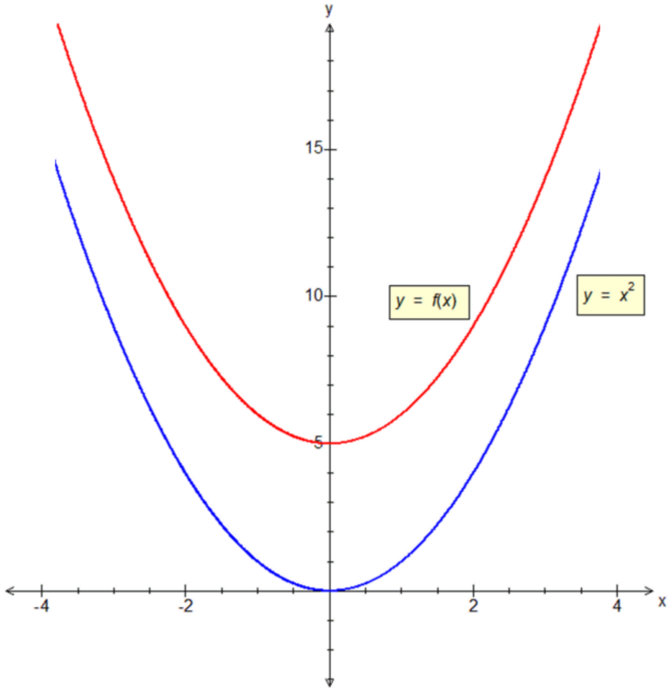


Example (1): Describe the transformation mapping the graph of $y = x^2$ to $y = f(x)$.

Hence find the equation of $f(x)$.

We see that the graph of $y = f(x)$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$, or a y-shift of +5 units.

Therefore $f(x) = x^2 + 5$.

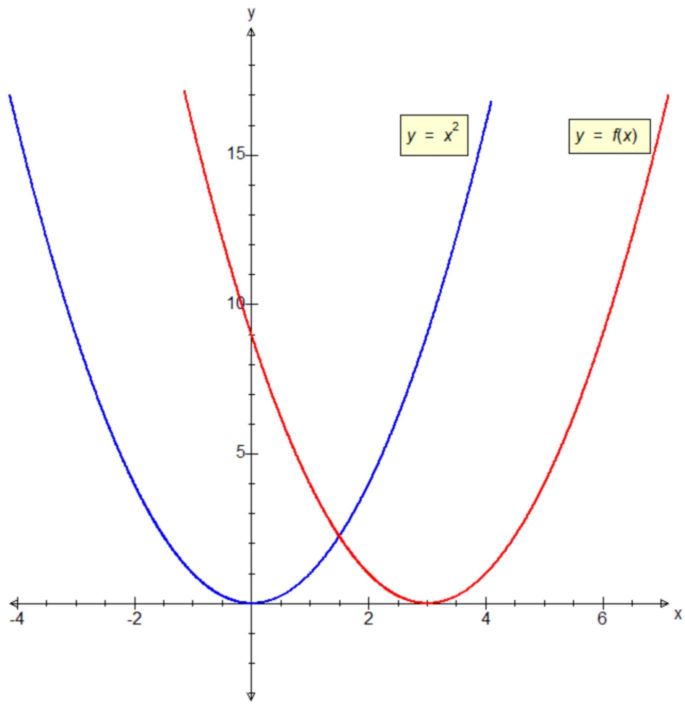


Example (2): Describe the transformation mapping the graph of $y = x^2$ to $y = f(x)$.

Hence find the equation of $f(x)$.

We see that the graph of $y = f(x)$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or an x-shift of +3 units.

Therefore $f(x) = (x - 3)^2$.



Examples (3): Describe the transformations required to map:

i) $f(x) = x^2$ to $g(x) = (x+4)^2$

ii) $f(x) = \sqrt{x}$ to $g(x) = -\sqrt{x}$

iii) $f(x) = \sin x$ to $g(x) = \sin(-x)$

i) $f(x) = x^2$ is transformed to $g(x) = (x + 4)^2$ by a translation using vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

ii) $f(x) = \sqrt{x}$ is transformed to $g(x) = -\sqrt{x}$ by reflection in the x-axis.

iii) $f(x) = \sin x$ is transformed to $g(x) = \sin(-x)$ by reflection in the y-axis.

Examples (4): The transformations in each case map $f(x)$ to $g(x)$. Find $g(x)$.

i) $f(x) = \sqrt{x}$ by a translation with vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

ii) $f(x) = x^3$ by a translation with vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

i) Translating $f(x) = \sqrt{x}$ with the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ gives $g(x) = \sqrt{x} + 2$.

ii) By translating $f(x) = x^3$ with the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, we obtain $g(x) = (x - 3)^3$.

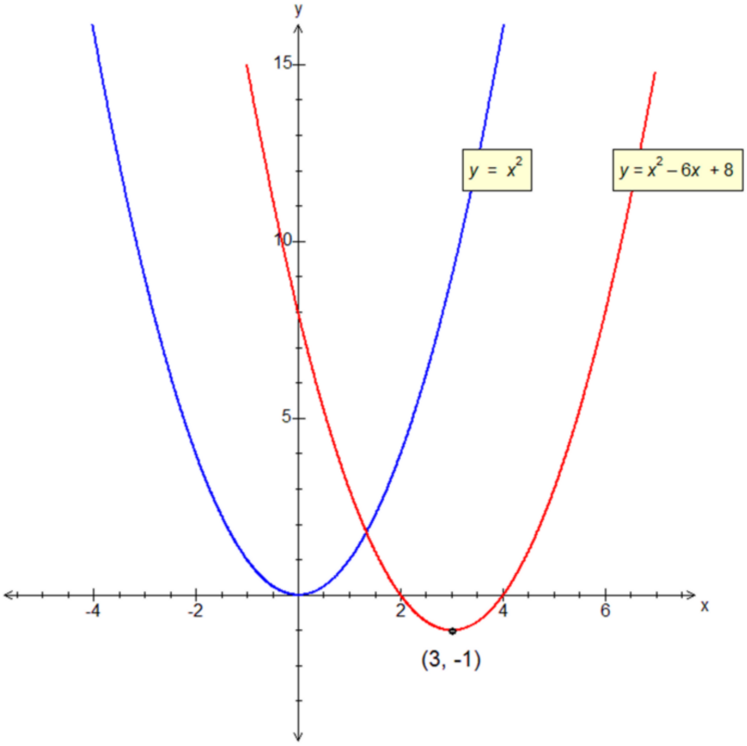
Composite graph transformations (Grade 9 only).

It is possible to combine transformations of graphs, as the next examples show.

Example (5): How can we transform the graph of $y = x^2$ to the graph of $y = x^2 - 6x + 8$?

When we look at the two graphs, we can see that they have the same shape, and that the second is a translation of the first.

By inspecting the minimum points of the two graphs, we can see that $y = x^2 - 6x + 8$ is a translation of $y = x^2$ using the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.



In other words, we have translated the graph of $y = x^2$ by 3 units right and 1 unit down.

How could we obtain the vector without plotting the graph ?

The solution lies in completing the square !

$x^2 - 6x + 8 = (x - 3)^2 - 9 + 8$

$= (x - 3)^2 - 1.$

(We are not asked to solve the equation.)

This completed-square expression therefore gives us information about the required transformations.

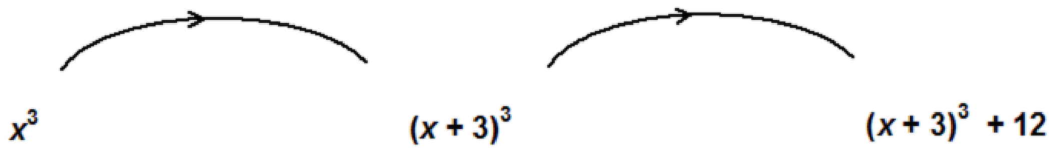
Starting with $y = x^2$, we begin with an x-translation using the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. This produces an intermediate function of $y = (x - 3)^2$.

Next, we translate in the y-direction via the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ to obtain the final function of $y = (x - 3)^2 - 1$.

The two translations can be combined in a single vector, namely $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Example (6): The graph of $y = x^3$ can be transformed to $y = (x + 3)^3 + 12$ in two separate transformations. Describe them and sketch the graphs.

For compound transformations of this kind, it is often helpful to use function diagrams to break down the process into steps.

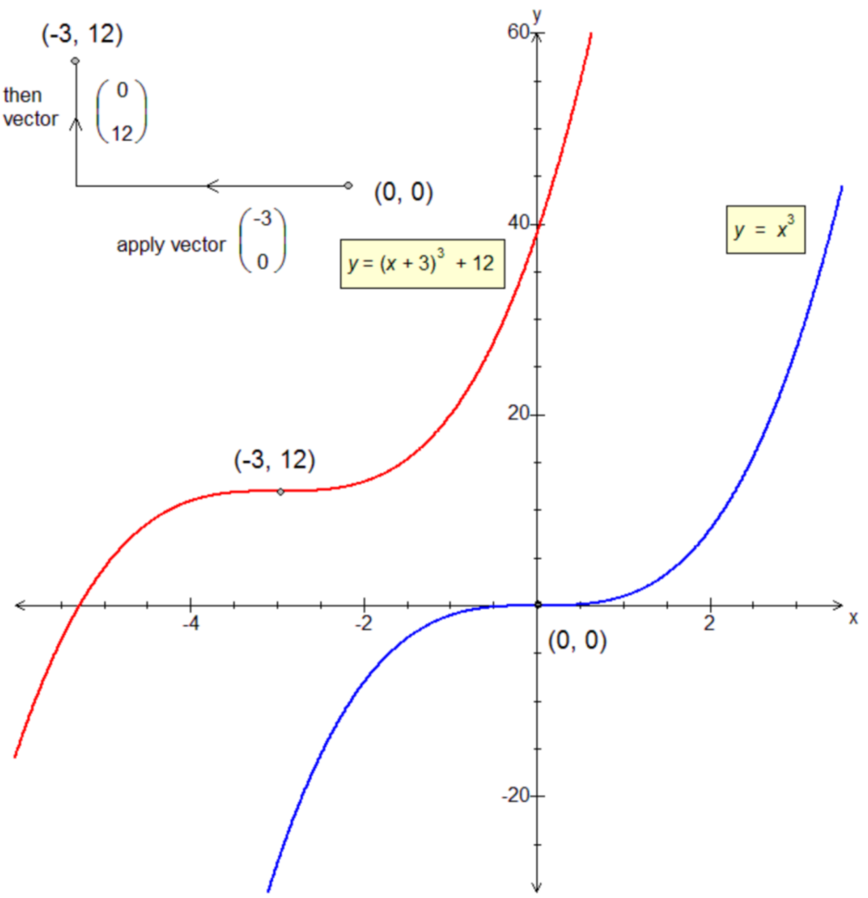


The first step is to get from x^3 to $(x + 3)^3$, and this means a translation using the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

The second is to get from $(x + 3)^3$ to $(x + 3)^3 + 12$. This requires a translation using the vector $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$.

The two translations can be combined in one vector, namely $\begin{pmatrix} -3 \\ 12 \end{pmatrix}$.

The graphs are shown below with the point (0,0) on the original for reference. (The intermediate function of $(x + 3)^3$ is not shown.)



These graphs are drawn accurately here, but a rougher sketch would be acceptable in an examination provided the general shape is recognisable and the vector(s) noted.