

M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SOLVING RIGHT-ANGLED AND ISOSCELES TRIANGLES

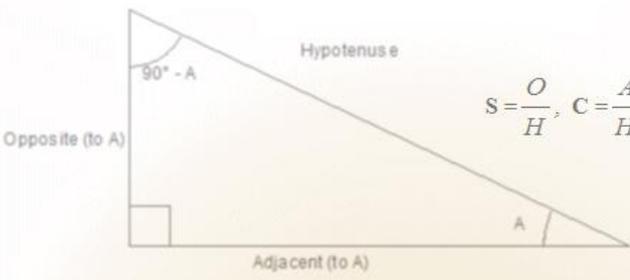
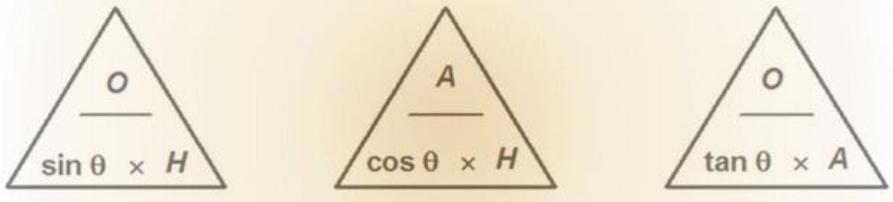


Diagram of a right-angled triangle with a right angle at the bottom-left corner. The top angle is labeled $90^\circ - A$. The bottom-right angle is labeled A . The hypotenuse is labeled "Hypotenuse". The side opposite to angle A is labeled "Opposite (to A)". The side adjacent to angle A is labeled "Adjacent (to A)".

$S = \frac{O}{H}$, $C = \frac{A}{H}$, $T = \frac{O}{A}$ or 'SOHCAHTOA'.



Three small triangles illustrating the trigonometric ratios:

- Triangle 1: $\frac{O}{\sin \theta \times H}$
- Triangle 2: $\frac{A}{\cos \theta \times H}$
- Triangle 3: $\frac{O}{\tan \theta \times A}$

O = opposite (to θ); A = adjacent (to θ); H = hypotenuse.

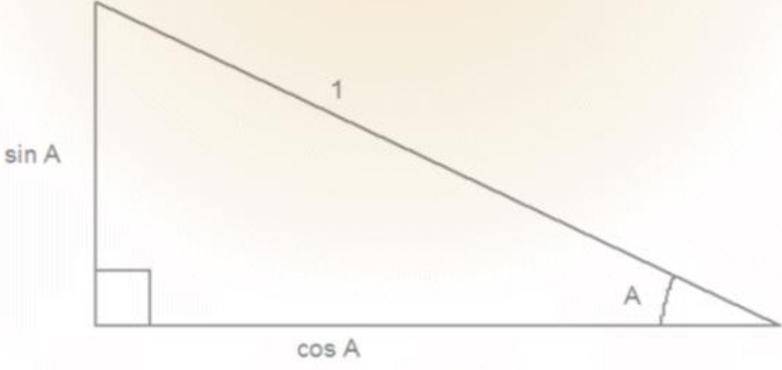


Diagram of a right-angled triangle with a right angle at the bottom-left corner. The hypotenuse is labeled "1". The side opposite to angle A is labeled "sin A". The side adjacent to angle A is labeled "cos A".

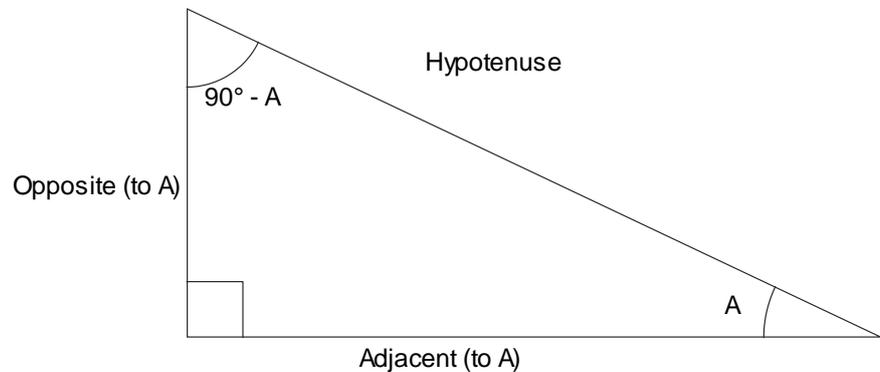
THE RIGHT-ANGLED TRIANGLE AND TRIGONOMETRIC RATIOS.

The examples in Pythagoras' Theorem showed how to find one side given the two others.

Here, we will show how to solve any right-angled triangle given sufficient information.
 The general right-angled triangle looks like this:

The hypotenuse is the longest side of the triangle, and is the side opposite the right angle.

Using angle A as the reference, the other two sides of the triangle are the 'opposite' which is opposite A, and the 'adjacent' which runs between A and the right angle.



The three sides are all related by the following ratios; the **sine**, **cosine** and **tangent** of A. They are abbreviated to **sin**, **cos** and **tan**.

The sine of angle A (sin A) is the ratio: opposite ÷ hypotenuse

The cosine of angle A (cos A) is the ratio: adjacent ÷ hypotenuse

The tangent of angle A (tan A) is the ratio: opposite ÷ adjacent

In brief, $S = \frac{O}{H}$, $C = \frac{A}{H}$ and $T = \frac{O}{A}$ or 'SOHCAHTOA'.

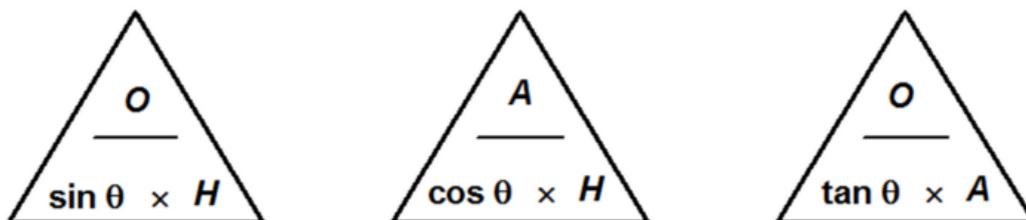
The formulae above can also be rearranged to find missing sides by changing the subject.

$$\text{Hypotenuse} = \frac{\text{opposite}}{\sin A} \text{ or } \frac{\text{adjacent}}{\cos A}$$

$$\text{Opposite} = \text{hypotenuse} \times \sin A \text{ or } \text{adjacent} \times \tan A$$

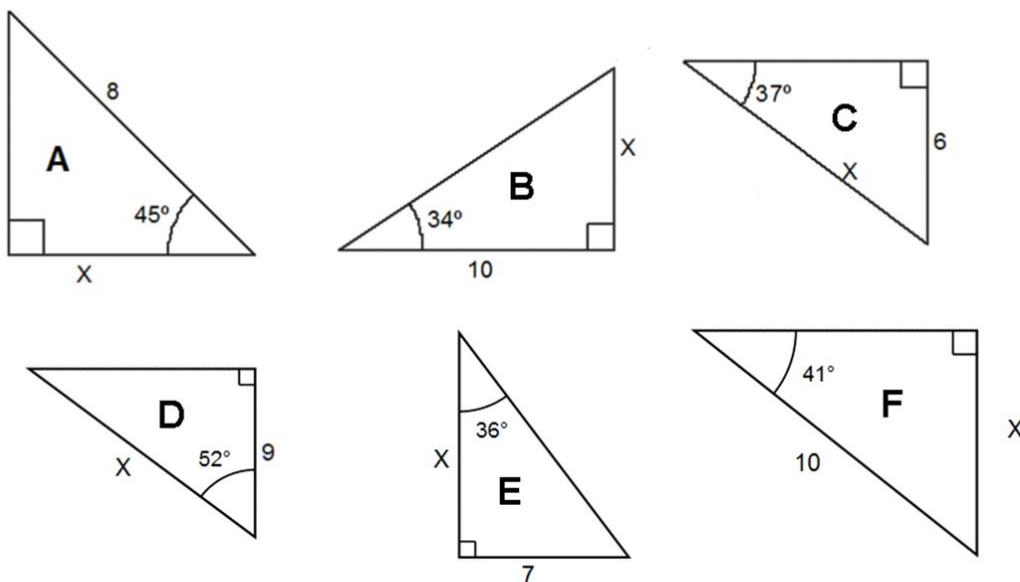
$$\text{Adjacent} = \text{hypotenuse} \times \cos A \text{ or } \frac{\text{opposite}}{\tan A}$$

Or we can use formula triangles as below: (the symbol θ = Greek 'theta' denotes the angle)



Here O = opposite (to θ) ; A = adjacent (to θ); H = hypotenuse.

Example (1a): Find the sides labelled X in the triangles below. All lengths are in centimetres. Give results to 2 decimal places.



Triangles not accurately drawn !

In triangle **A**, we are given the angle of 45° and the hypotenuse of 8 cm. Side X is not opposite the 45° angle, so it is the adjacent.

The length of X (the adjacent) is therefore (hypotenuse $\times \cos 45^\circ$), or $8 \cos 45^\circ$, or 5.66 cm.

We have the angle and the adjacent in triangle **B**, but we need to find the opposite. Use the formula
 opposite = (adjacent $\times \tan 34^\circ$), giving the length of X as $10 \tan 34^\circ$ or 6.75 cm.

In triangle **C**, we are given the angle (37°) and the opposite (6 cm), but we are required to work out the hypotenuse X .

Hence the hypotenuse = $\frac{6}{\sin 37^\circ}$, so $X = 9.97$ cm.

In triangle **D**, we are given the angle of 52° and the adjacent (9 cm),

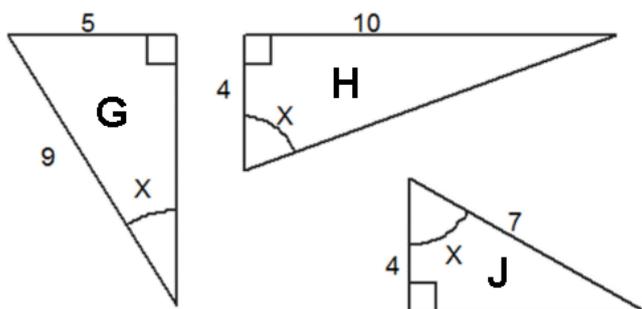
so $X = \text{hypotenuse} = \frac{9}{\cos 52^\circ} = 14.62$ cm.

Triangle **E** has the opposite (7 cm) and the angle (36°) given,

so $X = \text{adjacent} = \frac{7}{\tan 36^\circ} = 9.63$ cm.

Finally, triangle **F** has the angle (41°) and the hypotenuse (10 cm) given,
 so $X = \text{opposite} = 10 \sin 41^\circ = 6.56$ cm.

Example (1b): Find the angles labelled X in the triangles below. All lengths are in centimetres. Give results to 1 decimal place.



Triangles not accurately drawn !

This time we are looking for angles, not sides, so we make use of **inverse trig functions**.

These functions are used to find an unknown angle given one of its ratios. They are $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. For example, $\sin^{-1}(0.5) = 30^\circ$ and $\tan^{-1}(1) = 45^\circ$.

The input to $\sin x$, $\cos x$ and $\tan x$ is in each case an angle, and the output a number. With the inverse functions, the input is a number and the output is an angle.

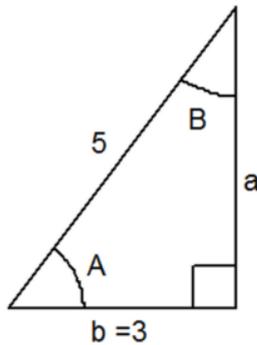
Triangle **G** has the opposite (5 cm) and hypotenuse (9cm) known, therefore $\sin X = \frac{5}{9}$,
in other words $X = \sin^{-1}\left(\frac{5}{9}\right) = 33.7^\circ$.

The opposite (10cm) and adjacent (4cm) are known in triangle **H**, so $\tan X = \frac{10}{4}$,
or $X = \tan^{-1}\left(\frac{10}{4}\right) = 68.2^\circ$.

The adjacent and hypotenuse are known in triangle **J**, so $\cos X = \frac{4}{7}$,
i.e. $X = \cos^{-1}\left(\frac{4}{7}\right) = 55.2^\circ$.

Generally, it is possible to use the above formulae in conjunction with Pythagoras' theorem to solve a right-angled triangle in its entirety.

In the triangle shown left (sides in cm), we can work out side a , the angles A and B , and the area using both Pythagoras and the trig ratios. (Note also the convention: side a is opposite angle A , and side b opposite angle B)



Side a is $\sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$ cm long.

Now that we have both the base and the height, the area can be easily worked out as $\frac{1}{2} (3 \times 4)$ or 6 cm^2 .

Since side b is opposite angle B , we have $\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$

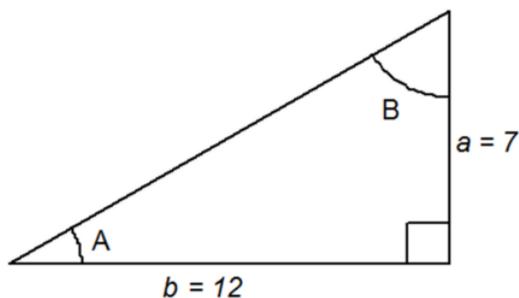
and therefore angle $B = 36.9^\circ$.

Angle A can be worked out by realising that side b is adjacent to it, and using

$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$, giving $B = 53.1^\circ$.

Alternatively, we could have realised that angle $B = 90^\circ - A$ (angles of a triangle add up to 180°)

Example (2): Find the unknown sides (in cm) and angles in the triangle below, together with its area.



The area can be worked out at once as we have the base and height given – it is $\frac{1}{2} (7 \times 12)$ or 42 cm^2 . Side a is opposite angle A , and side b is adjacent to angle A ,

$$\therefore \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{12}$$

giving $A = 30.3^\circ$, and thus $B = 59.7^\circ$.

The hypotenuse, c , can be worked out either by

i) using the value of A obtained earlier and the formula : $\text{hypotenuse} = \frac{\text{opposite}}{\sin A}$

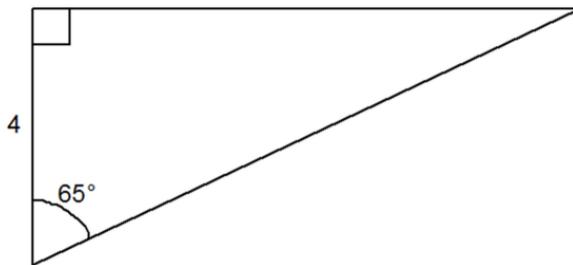
giving $c = \frac{7}{\sin 30.3^\circ}$ or 13.9 cm.

ii) using Pythagoras:

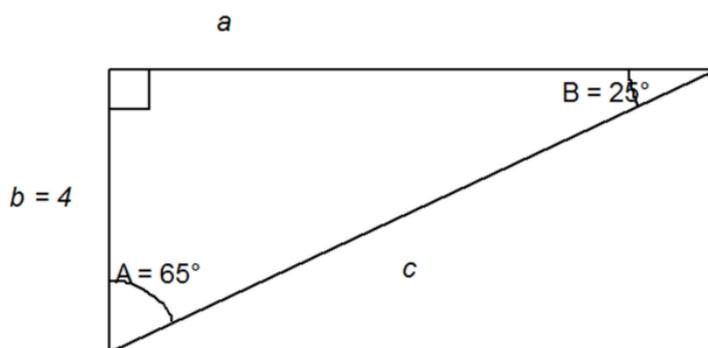
$$c = \sqrt{a^2 + b^2} = \sqrt{49 + 144} = \sqrt{193} = 13.9 \text{ cm}$$

(Here, Pythagoras gives a more accurate result, as the value of 30.3° has been rounded.)

Example (3): Find the unknown sides and angles in the triangle below along with its area, giving the results to two decimal places. (Lengths in cm.)



This time, we are given just one angle and one side, and neither the sides nor the angles have been lettered. The first step is to label the side and angles using side a opposite angle A and so forth. Angle B can be worked out at once by subtracting 65° from 90° .



We have the length of the adjacent to A , namely 4, but not that of the opposite, a .

We use the formula $\tan A = \frac{\text{opposite}}{\text{adjacent}}$ and rewrite it as

$$\text{opposite} = \tan A \times \text{adjacent}.$$

This gives $a = \tan 65^\circ \times 4$ cm, or 8.58 cm. (Use more decimal places, i.e. 8.578 or full calculator accuracy when working out intermediate results).

We now have the required information to work out the area - it is half the product of the opposite and the adjacent, or $\frac{1}{2} (4 \times 8.578)$, or 17.16 cm^2 .

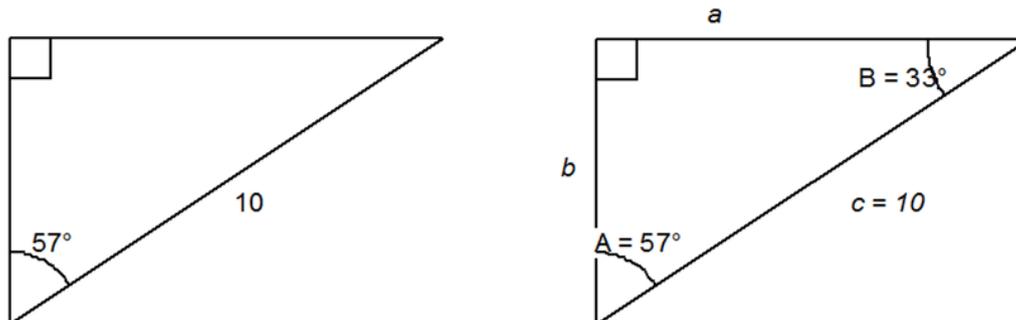
(We could have equally used angle B and treated b as the opposite, but this time the tangent formula would be rewritten as

$$\text{adjacent} = \frac{\text{opposite}}{\tan B}, \text{ giving } a = \frac{4}{\tan 25^\circ}, \text{ or } 8.58 \text{ cm as before.})$$

From this, we can calculate the length of the hypotenuse:

$$c = \sqrt{a^2 + b^2} = \sqrt{8.578^2 + 16} = \sqrt{89.58} = 9.46 \text{ cm}.$$

Example (4): Find the unknown sides and angles in the triangle below, along with its area, giving the results to two decimal places. (Lengths in cm).



Label sides and angles, remembering that $57^\circ + 33^\circ = 90^\circ$

This time, we know the length of the hypotenuse c , so we use the formula $\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$ and rewrite it as

$$\text{opposite} = \sin A \times \text{hypotenuse}.$$

This gives $a = \sin 57^\circ \times 10$ cm, or 8.39 cm.

There are three ways to work out side b :

i) We can treat it as the opposite side to angle B and apply the same formula as before, i.e. gives $b = \sin 33^\circ \times 10$ cm, or 5.45 cm.

ii) We can treat it as the adjacent side to angle A and use $\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$ and rewrite it as

$$\text{adjacent} = \cos A \times \text{hypotenuse}, \text{ giving } b = \cos 57^\circ \times 10 \text{ cm or } 5.45 \text{ cm}.$$

iii) Use Pythagoras after having worked out the length of side a .

$$b = \sqrt{c^2 - a^2} = \sqrt{100 - 8.387^2} = \sqrt{29.66} = 5.45 \text{ cm}$$

The area is half the product of a and b , or $\frac{1}{2} (10 \sin 57^\circ \times 10 \cos 57^\circ)$ or $(50 \times 0.8387 \times 0.5446) = 22.84 \text{ cm}^2$.

Three important properties can be gleaned from the last example:

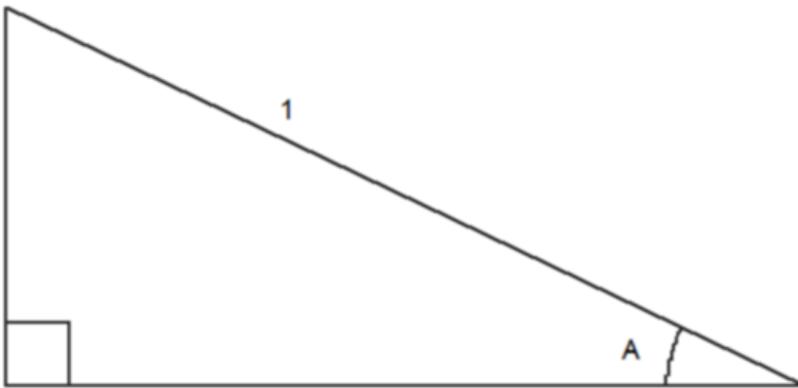
Given the hypotenuse c and an angle A , another formula for the area is $\frac{1}{2}(c^2 \sin A \cos A)$.

The sine of an acute angle is equal to the cosine of its difference from 90° , i.e. $\sin A = \cos (90^\circ - A)$.

Also, the tangent of an acute angle is equal to the reciprocal of the tangent of its difference from 90° (excluding multiples of right angles), i.e. $\tan A \times \tan (90^\circ - A) = 1$.

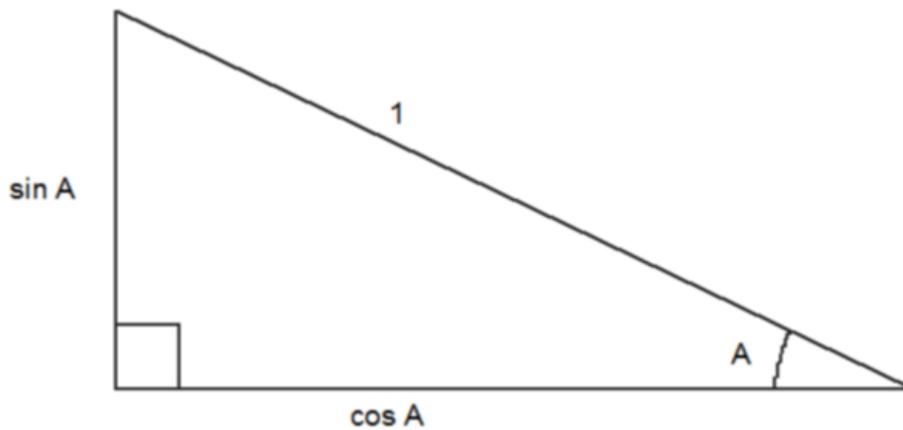
Example (5): An important trig identity.

Find a formula relating $\sin A$ and $\cos A$ in the triangle below. Use Pythagoras' Theorem here.



The opposite side has a length of $(\sin A) \times \text{hypotenuse}$, but since the hypotenuse is 1 unit long, its length is simply $(\sin A)$ units.

Similarly, the adjacent has a length of $(\cos A)$ units.



Pythagoras' theorem states that the square of the hypotenuse is equal to the sum of the squares on the other two sides, therefore

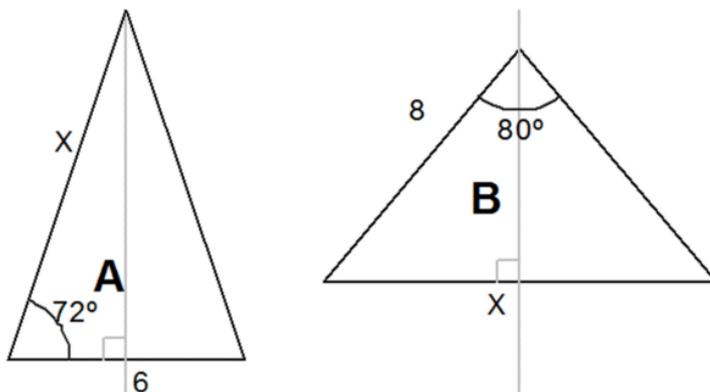
$$\cos^2 A + \sin^2 A = 1.$$

(This identity holds for all angles A .)

Solving isosceles triangles.

It is possible to modify the methods used for right-angled triangles to find the area, unknown sides and angles in an isosceles triangle, because it has a line of symmetry bisecting it into two identical right-angled triangles.

Examples (6). Find the sides labelled **X** in the triangles below, and additionally calculate their areas. (All lengths are in centimetres).



Triangle **A** is bisected to give two right-angled triangles. We need to find the hypotenuse of one of those halves, being given the adjacent (here 3 - half the base).

$$\text{hypotenuse} = \frac{\text{adjacent}}{\cos 72^\circ}, \text{ or } \frac{3}{\cos 72^\circ} \text{ which gives the length of } \mathbf{X} \text{ as } 9.71 \text{ cm.}$$

(Do not forget to divide the base of the triangle by 2 !)

We are required to find the base of triangle **B**, namely **X**. Here we are given the hypotenuse and the angle, but we need to find the opposite. The line of symmetry bisects the angle of 80°, so we use 40° in the calculation.

opposite = hypotenuse \times sin 40°, therefore the length of **X** is double that (by symmetry), namely 16 sin 40° or 10.28 cm.

(Don't forget to double the length of the opposite !)

To obtain the areas of the triangles **A** and **B**, we will need to find their perpendicular heights.

The height of **A** can be worked out as opposite = adjacent \times tan 72°, or 3 tan 72° or 9.23 cm, and thus the area of the whole triangle is 3 \times 9.23 or 27.70 cm². The ½ has been taken out because we are measuring the area of **two** combined right-angled triangles.

The height of **B** can be worked out as adjacent = hypotenuse \times cos 40°, or 8 cos 40° or 6.13 cm. This gives the area of the whole triangle as ½ \times 10.28 \times 6.13 or 31.51 cm².

Another way of working out the area is to use the general formula:

Area = ½ab sin C where *a* and *b* are two sides and C the included angle. This formula will be used under 'Solving General Triangles', but we can use it here.

Using the base and the one other side as sides a and b , and the angle of 72° between them, we have

$$\text{Area} = \frac{1}{2} \left(6 \times \frac{3}{\cos 72^\circ} \times \sin 72^\circ \right) = \frac{9 \sin 72^\circ}{\cos 72^\circ} \text{ or } 27.70 \text{ cm}^2.$$

The height of **B** can be worked out as adjacent = hypotenuse \times $\cos 40^\circ$, or $8 \cos 40^\circ$ or 6.13 cm. This gives the area of the whole triangle as $\frac{1}{2} \times 10.28 \times 6.13$ or 31.51 cm^2 .

Again, the formula gives a quicker result, if we use the two sides of 8 units and the angle of 80° between them.

$$\text{Area} = \frac{1}{2} (8^2 \times \sin 80^\circ) = 32 \sin 80^\circ = 31.51 \text{ cm}^2.$$

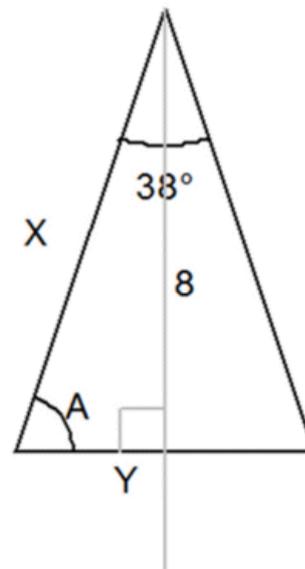
Sometimes an isosceles triangle might have an angle and its perpendicular height known, but none of its sides.

Example (7). Find angle **A**, side **X** and side **Y** in the isosceles triangle on the left, as well as its area..

To find angle **A**, subtract half of 38° from 90° (by symmetry). **A** is therefore 71° .

To work out the sides, we can either use angle **A** or half of 38° (19°) for reference. Here angle **A** will be used.

The height of the triangle is therefore the opposite, but we need to find the adjacent and the hypotenuse, not forgetting to double the adjacent to give side **Y** by symmetry.



$$\text{hypotenuse} = \frac{\text{opposite}}{\sin 71^\circ}, \text{ or } \frac{8}{\sin 71^\circ} \text{ which gives the}$$

length of **X** as 8.46 cm.

$$\text{adjacent} = \frac{\text{opposite}}{\tan 71^\circ}, \text{ or } \frac{8}{\tan 71^\circ}, \text{ giving the length of } \mathbf{Y} \text{ as } \frac{16}{\tan 71^\circ} \text{ units, or } 5.51 \text{ cm}$$

Now that **Y** has been worked out, the area is $\frac{1}{2} (5.51 \times 8)$ or 22.04 cm^2 .

Example (8): A ship leaves port on a bearing of 160° , and there is also a stationary tug anchored 5 km south of the port. Calculate the shortest distance between the moving ship and the tug, and also the ship's distance from port when this situation occurs.

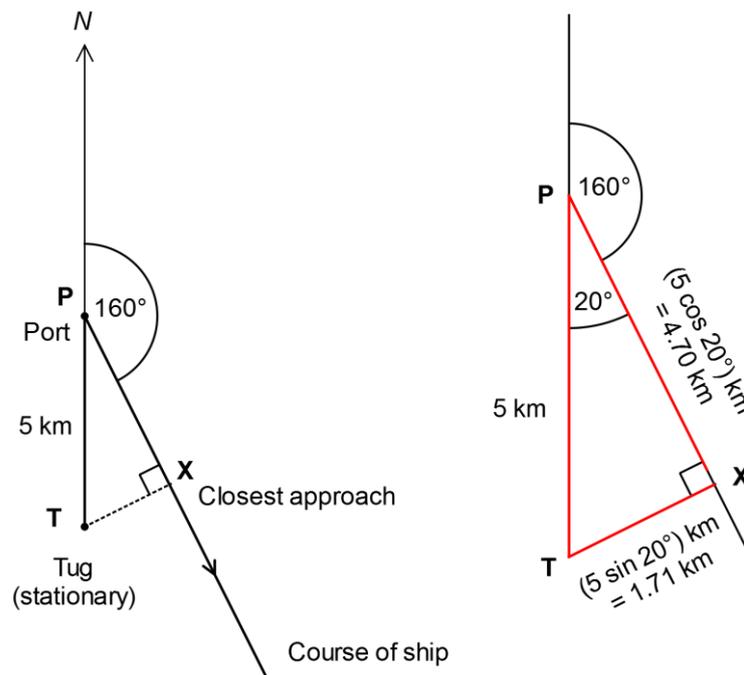
Let the port be at **P** and the tug at point **T**, 5 km due south of the port. The ship's course is in the direction of the arrow at a bearing of 160° , and the point **X** is where the ship and the tug are at their closest to each other.

The shortest path from a point to a line is also perpendicular to the line, so the angle between **PX** and **XT** is a right angle. Additionally, the angle **TPX** = 20° (180° in a straight line).

PT is the hypotenuse of the triangle **PXT**, **TX** is opposite the 20° angle and **PX** is adjacent to it.

Therefore the shortest distance between the ship and the tug, **TX** = $(5 \sin 20^\circ)$ km = **1.71 km**.

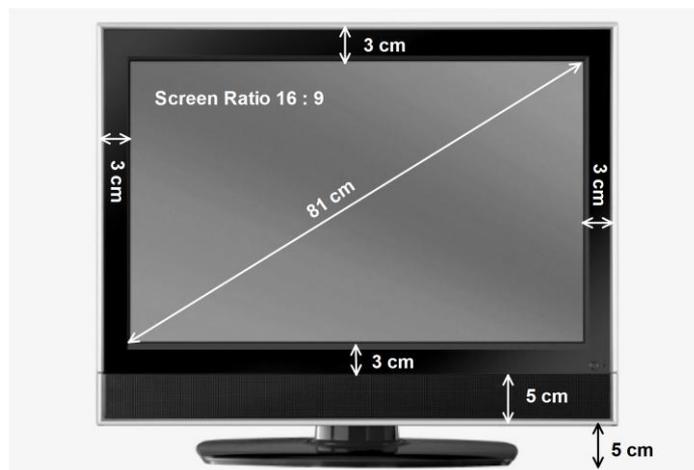
When the ship and the tug are at their closest, the distance between the port and the ship is $PX = (5 \cos 20^\circ)$ km = **4.70 km**.



Example (9): Jim is looking at a television set but is not sure whether it would fit inside his cabinet. The interior dimensions of the cabinet are 80 cm by 60 cm.

The diagonal of the visible screen of the set is 81 cm, and the side lengths are in the ratio 16 : 9. There is a bezel (border) of 3 cm around all the edges of the screen and a 5 cm loudspeaker area below the lower edge bezel. There is also a stand which adds a further 5 cm to the overall height.

Will the TV set fit inside Jim's cabinet? Justify your answer, showing all calculations.



We do not know the length or the width of the visible screen, but we are given their ratio, 9 : 16.

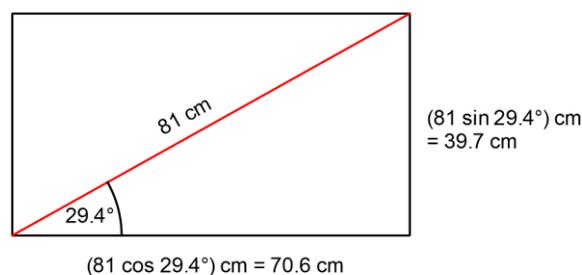
The tangent of the angle between the horizontal and the diagonal is therefore $\frac{9}{16}$,

i.e. the angle is $\tan^{-1}\left(\frac{9}{16}\right) = 29.4^\circ$.

The screen diagonal of 81 cm is therefore the hypotenuse of the right-angled triangle (right), the length of the screen is the adjacent side, and the height of the screen is the opposite side.

Length of screen = $(81 \cos 29.4^\circ)$ cm = **70.6 cm.**

Height of screen = $(81 \sin 29.4^\circ)$ cm = **39.7 cm.**



Screen ratio 9 : 16

Angle between diagonal and horizontal = $\tan^{-1}\left(\frac{9}{16}\right) = 29.4^\circ$

We need to add the two bezel widths of 3 cm each to obtain the total length of the set, namely **76.6 cm.**

Similarly we need to add the two bezel heights of 3 cm each, the loudspeaker area of 5 cm, and the stand of 5 cm – a total of 16 cm – to the height of the screen. The total height of the set is **55.7 cm.**

The dimensions of the TV set as a whole are 76.6 cm by 55.7 cm, so the set will fit inside Jim's TV cabinet with a few centimetres to spare either way. .