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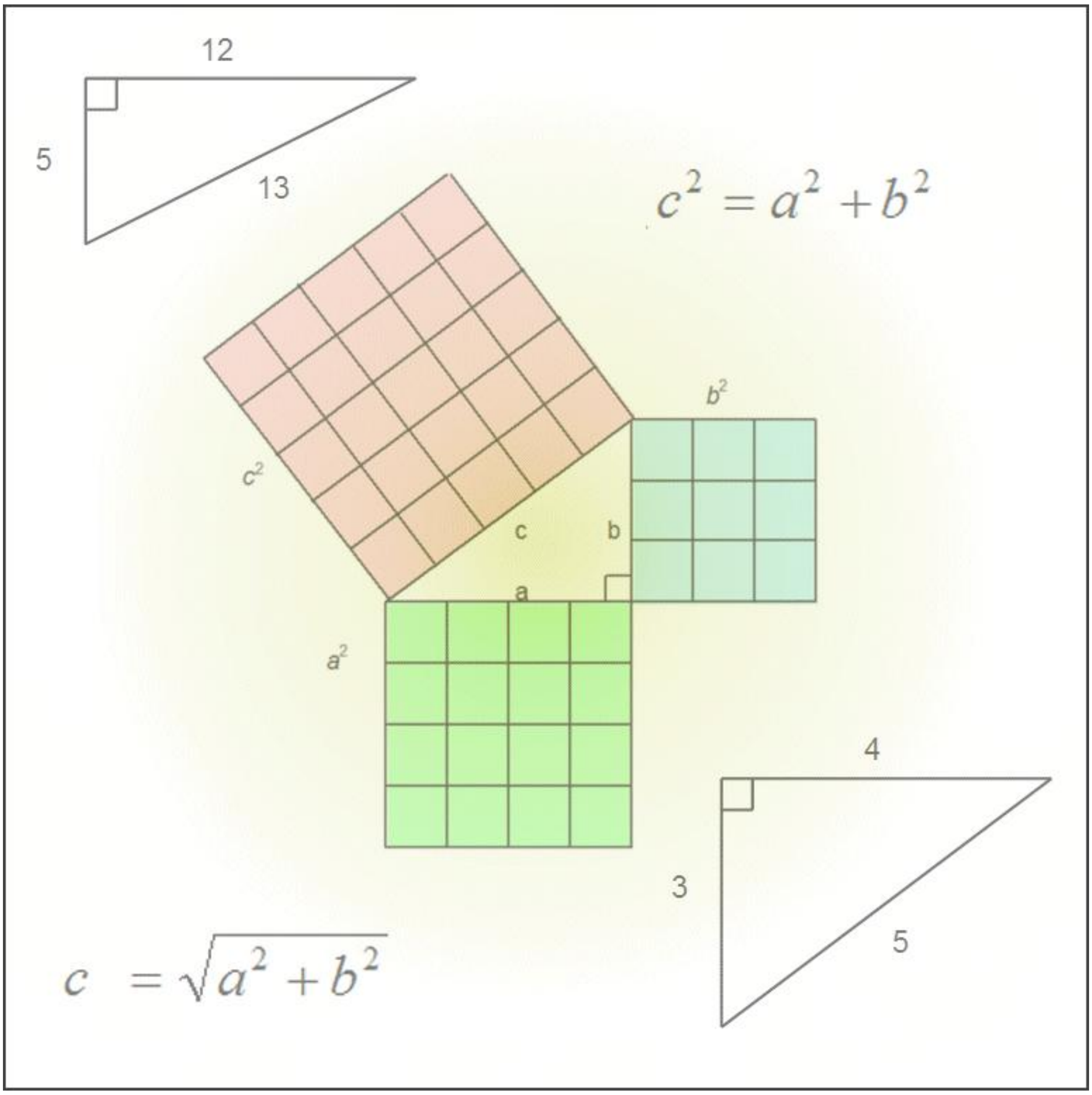
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

PYTHAGORAS' THEOREM



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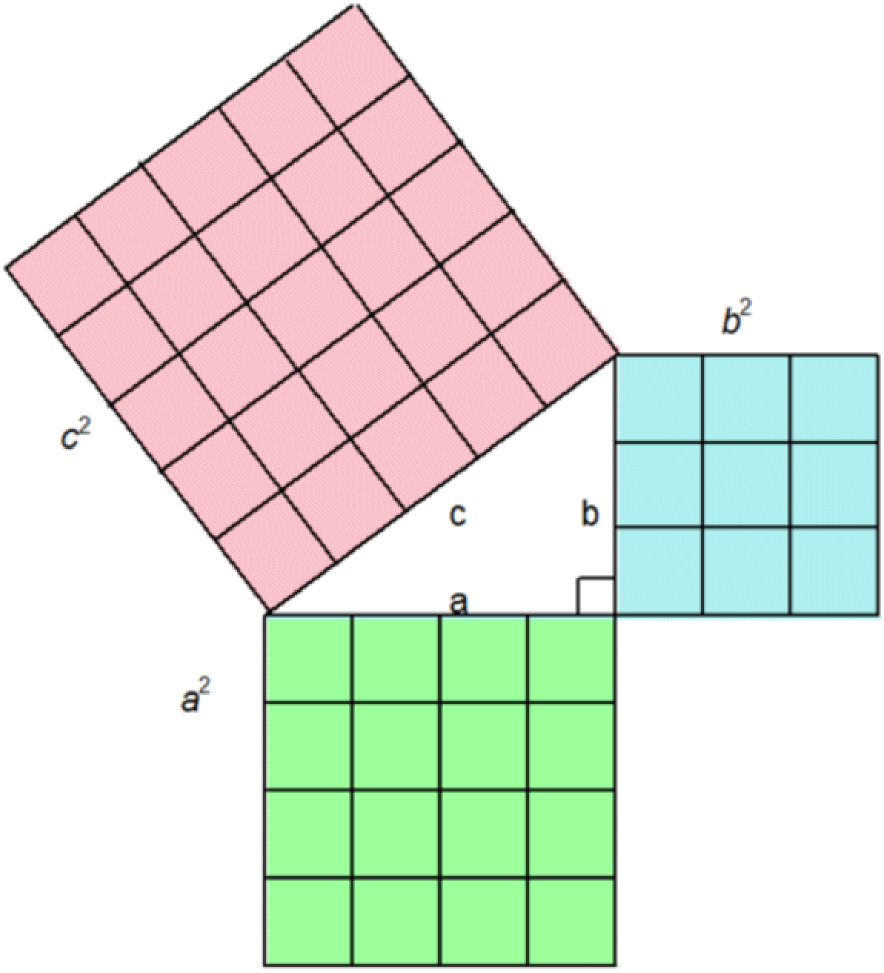
PYTHAGORAS' THEOREM.

Pythagoras' theorem states that for any right-angled triangle, **the square on the hypotenuse is equal to the sum of the squares of the other two sides.**

The hypotenuse is the side opposite the right angle, and is always the longest side.



In the diagram above, it thus follows that $c^2 = a^2 + b^2$,

or $c = \sqrt{a^2 + b^2}$, where *c* is the hypotenuse.

The above form is used when the hypotenuse is unknown, but it can be adapted to find an unknown side when the hypotenuse is known.

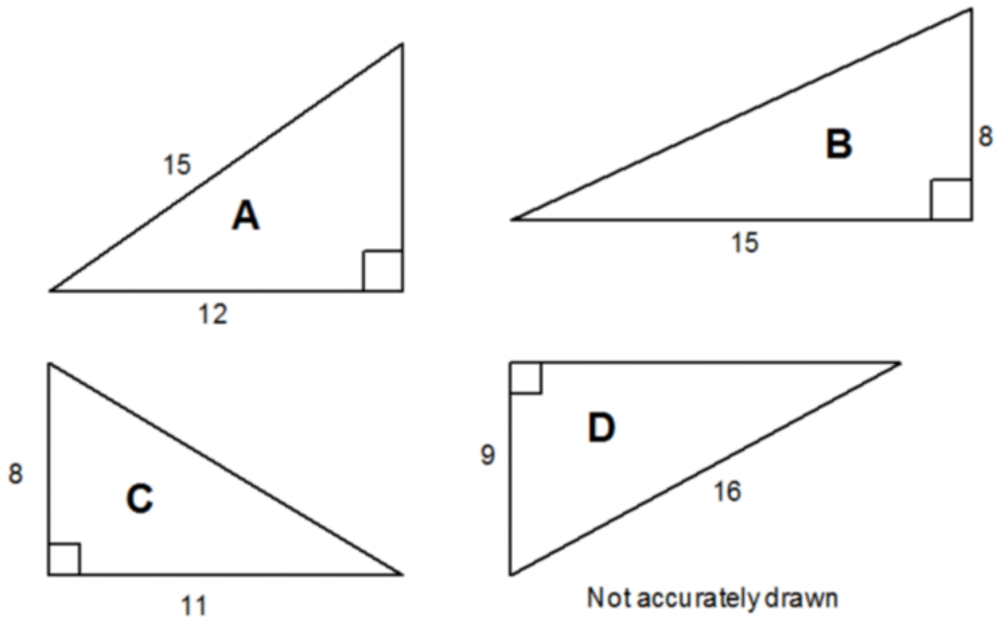
Then, $a^2 = c^2 - b^2$ or $a = \sqrt{c^2 - b^2}$ if *a* is unknown, and similarly

$b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ if *b* is unknown.

If the hypotenuse is unknown, we **add** the squares of the two other sides and take the square root.

If a shorter side is unknown, we **subtract** the smaller of the squares of the other two sides from the larger one and take the square root.

Example (1): Find the length of the missing side in each of the right-angled triangles below.



In triangle **A**, the hypotenuse is known, therefore the missing side has a length of

$\sqrt{15^2 - 12^2} = \sqrt{225 - 144} = \sqrt{81} = 9$ units

In triangle **B**, the hypotenuse is unknown, therefore its length is

$\sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$ units

In triangle **C**, the hypotenuse is again unknown, therefore its length is

$\sqrt{11^2 + 8^2} = \sqrt{121 + 64} = \sqrt{185} = 13.6$ units, to 1 decimal place.

In triangle **D**, the hypotenuse is known, therefore the missing side has a length of

$\sqrt{16^2 - 9^2} = \sqrt{256 - 81} = \sqrt{175} = 13.2$ units to 1 decimal place.

Area of a right-angled triangle.

The formula for the area of a triangle is given by

area = ½(base × height).

This is particularly simple when the triangle is right-angled; all that is needed are the lengths of the sides containing the right angle.

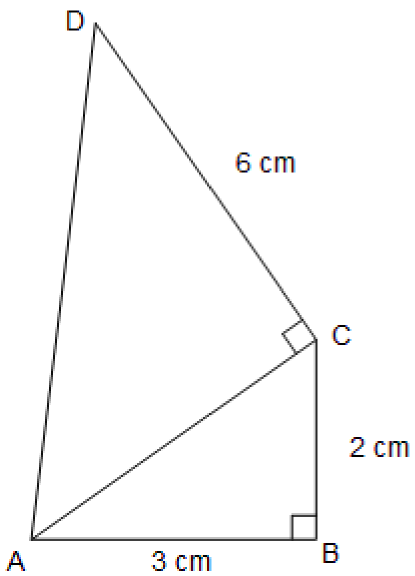
Example (2): Find the area of triangles **A** and **B** from the previous example.

We are only given one side containing the right angle to begin with.in the case of triangle **A**. After calculating the missing side as being 9 units long, we can work out the area as ½ (9 × 12) or 54 square units.

The two sides containing the right angle are already given in triangle **B**. The area is therefore ½ (8 × 15) or 60 square units.

Sometimes it might be necessary to apply Pythagoras' theorem more than once.

Example (3): Find the length *AD* in the diagram shown on the right.



Here we have two right-angled triangles, *ABC* and *ACD.*

We therefore begin by finding side *AC*, which is the hypotenuse of triangle *ABC.*

$(AC)^2 = (AB)^2 + (BC)^2$, i.e. $(AC)^2 = 2^2 + 3^2 = 13$,
so $AC = \sqrt{13}$.

We leave the length of *AC* in square-root form rather than by using an approximate value, because we will only need to square it again in the next part of the question.

Since side *AC* is also one of the shorter sides of triangle *ACD*, and *AD* is its hypotenuse,

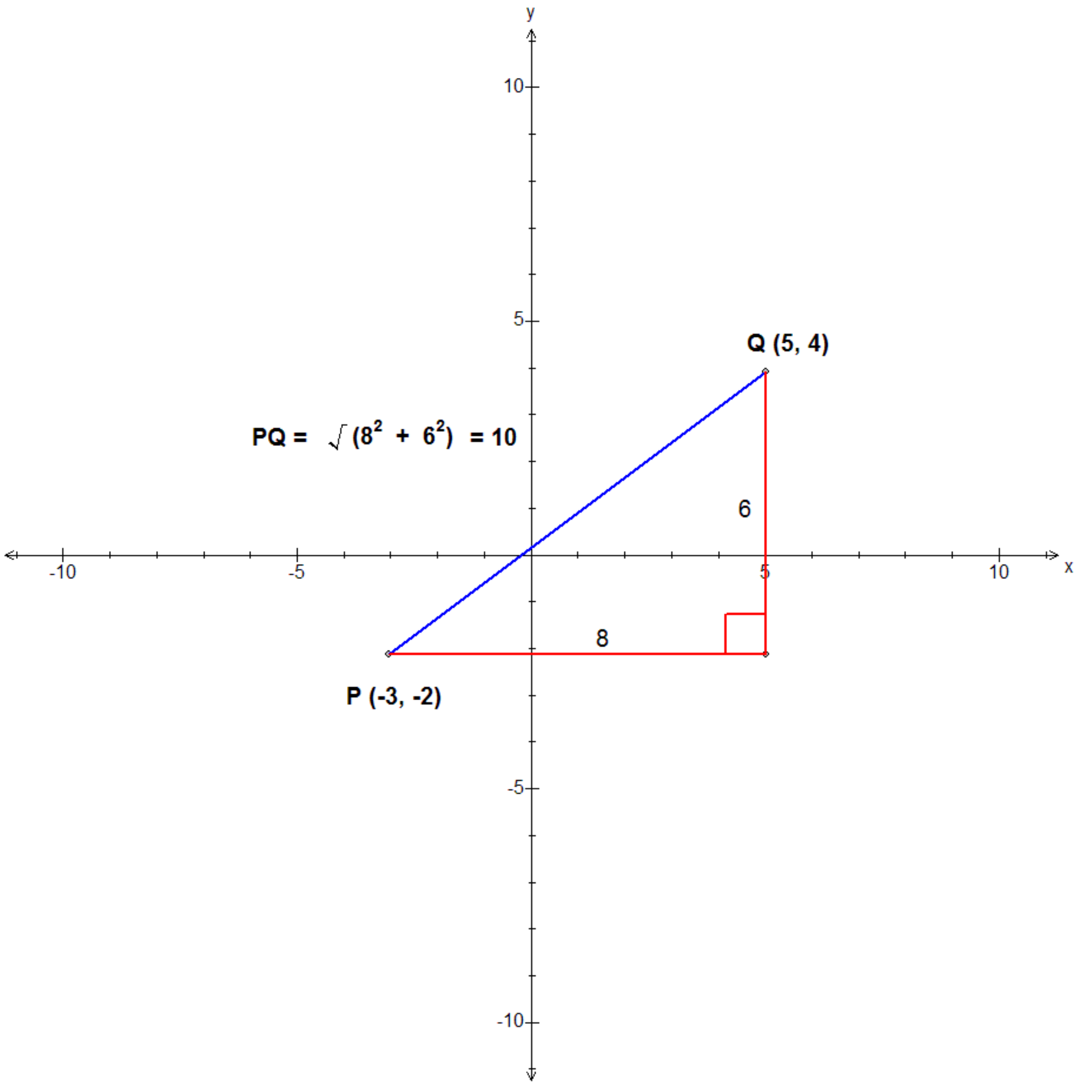
$(AD)^2 = (AC)^2 + (CD)^2$,or $(AD)^2 = 13 + 6^2 = 49$,
(remembering $(\sqrt{13})^2$ is simply 13).

Hence $AD = \sqrt{49} = 7$ cm.

Example (4): an important application to find the distance between two points.

Find the length of the line joining the points (-3, -2) and (5, 4).

The line joining the two points can be visualised as the hypotenuse of a right-angled triangle whose other two sides run parallel with the axes and whose right angle is at the point (5, -2).



The lengths of the two sides are therefore:
8 units for the one parallel to the *x*-axis, obtained by subtracting the *x*-coordinate of the point (-3 ,-2) from that of (5, 4) - i.e. (5 - (-3)) = 8.
6 units for the one parallel to the *y*-axis, obtained by subtracting the *y*-coordinate of the point (-3 ,-2) from that of (5,4) - i.e. (4 - (-2)) = 6.

The length of the hypotenuse, and the original line in question, is therefore $\sqrt{8^2+6^2} = \sqrt{100}$ units, or 10 units.

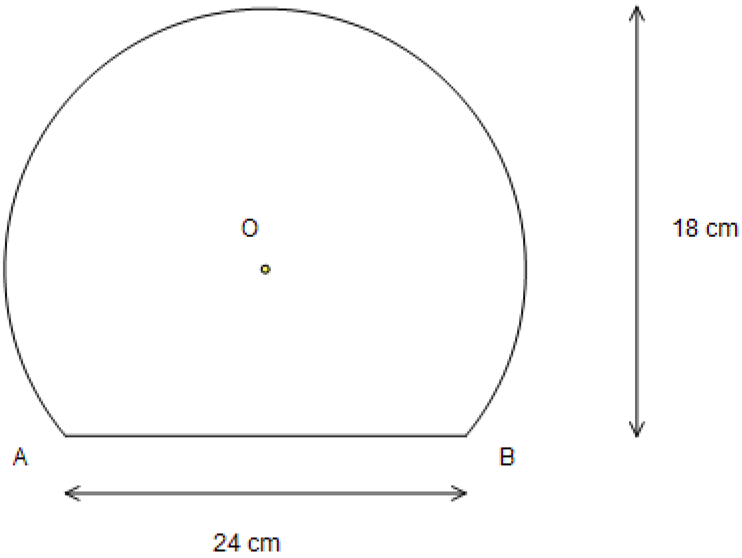
In general, the length of a line joining two points (x_1, y_1) and (x_2, y_2) on the plane is expressed as

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In other words, we square the difference between the *x*'s, then square the difference between the *y*'s, add the two squares and finally find the square root of the result.

Example (5): This example combines properties of circles with an application of Pythagoras.

The diagram shows the major segment of a circle whose straight side *AB* is 24 cm long, and whose height is 18 cm. Find the diameter of the original circle.

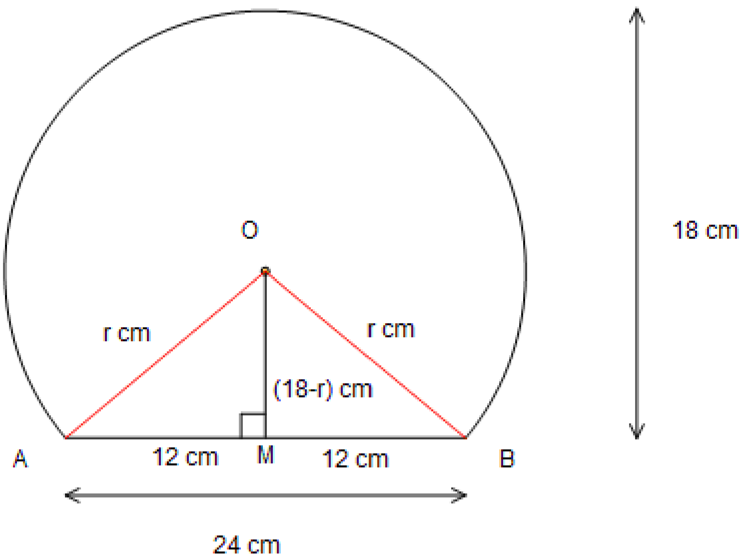


This is no need to use trig ratios in this problem at all !

All we need to do is join points *A* and *B* to the centre of the circle, *O*.

The perpendicular from *O* then bisects the isosceles triangle *OAB* at point *M*, the midpoint of chord *AB*.

Hence *AM* = *MB* = 12 cm.



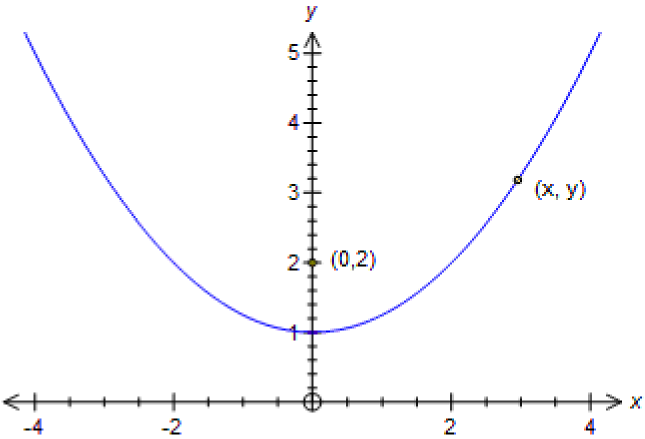
Let *AO* = *OB* = *r* (the radius of the circle), and hence *OM* = (18-*r*) cm.

Since triangles *AOM*, *BOM* are right-angled, we can use Pythagoras to solve the equation

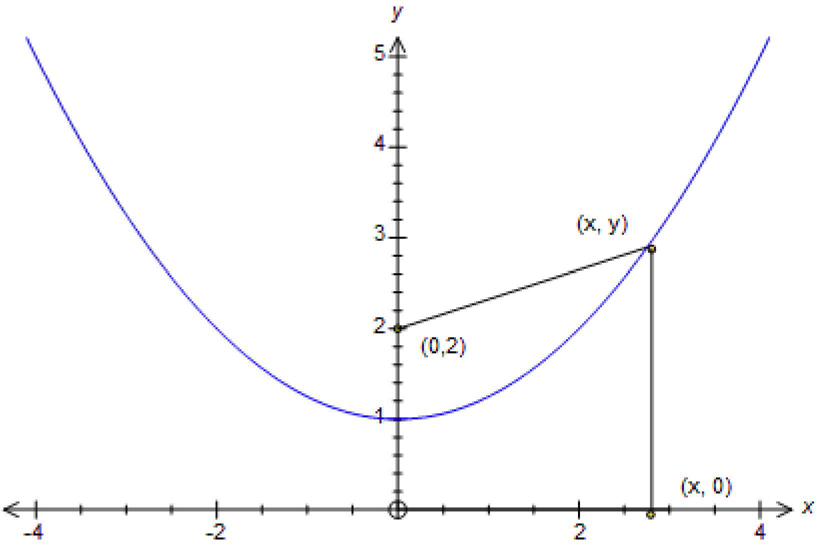
$r^2 = 12^2 + (18-r)^2$
$\rightarrow\ r^2 = 144 + 324 - 36r + r^2$
$\rightarrow\ 468 - 36r = 0$
$\rightarrow\ r = 13.$

The radius of the circle is 13 cm, hence the diameter is 26 cm.

Example (6): The graph of a curve is shown below. Any point (x, y) on the graph is equidistant from both the x-axis and the point (0, 2).



Show that the equation of the curve is $y = 1 + \frac{1}{4}x^2$.



The distance between (x, y) and the x-axis is $\sqrt{(x-x)^2 + (y-0)^2}$, or simply y.

The distance between (x, y) and the point (0, 2) is $\sqrt{(x-0)^2 + (y-2)^2}$.

Squaring and expanding both expressions, we have :

$y^2 = x^2 + (y\text{-}2)^2$ (given distances are equal)

$\rightarrow\ y^2 = x^2 + y^2 - 4y + 4$

$\rightarrow\ 4y + y^2 = x^2 + y^2 + 4$ (bring $4y$ over to LHS)

$\rightarrow\ 4y = 4 + x^2$ (cancel y^2 from both sides)

$$\rightarrow\ y = 1 + \frac{1}{4}x^2$$