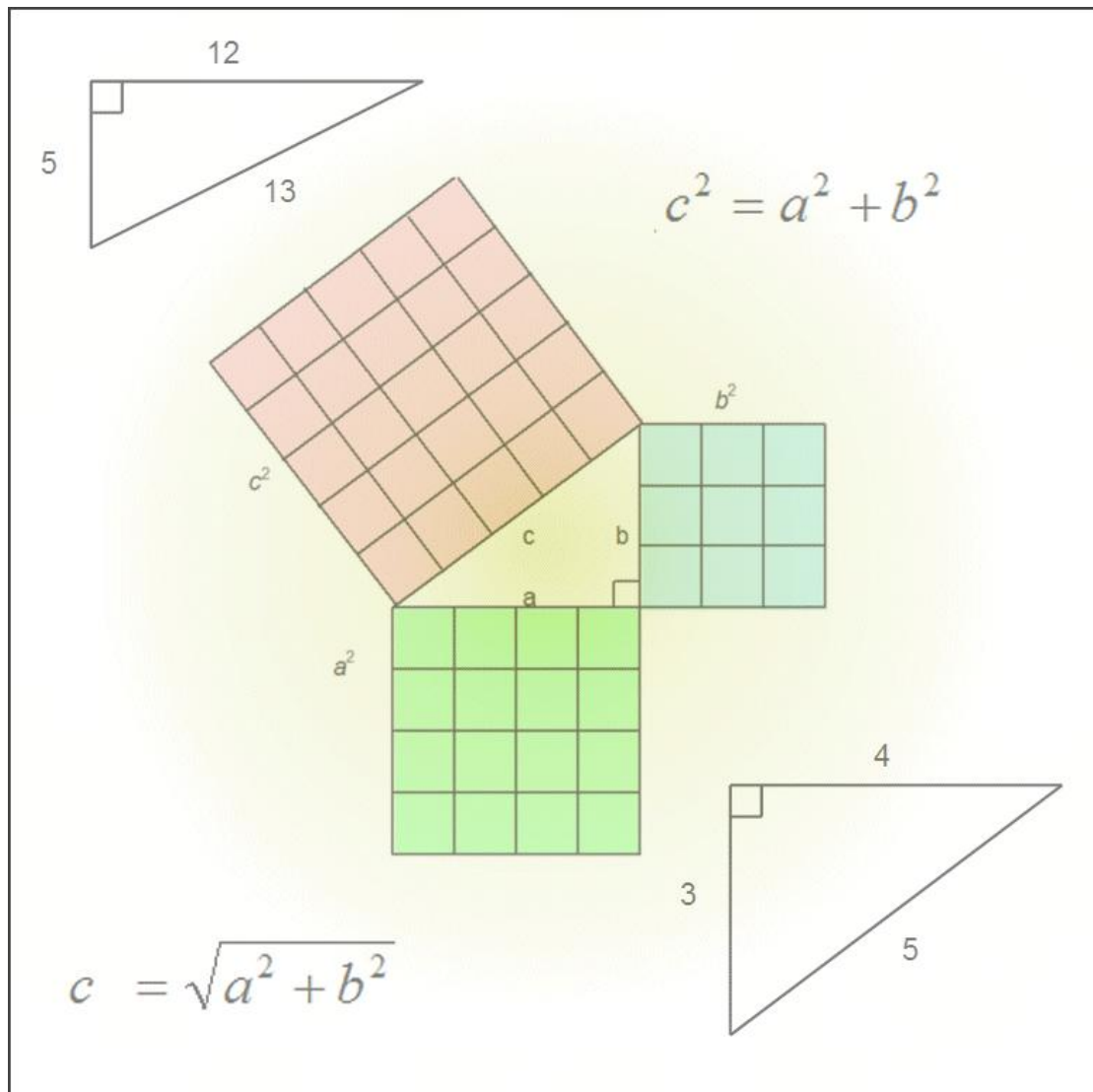


M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

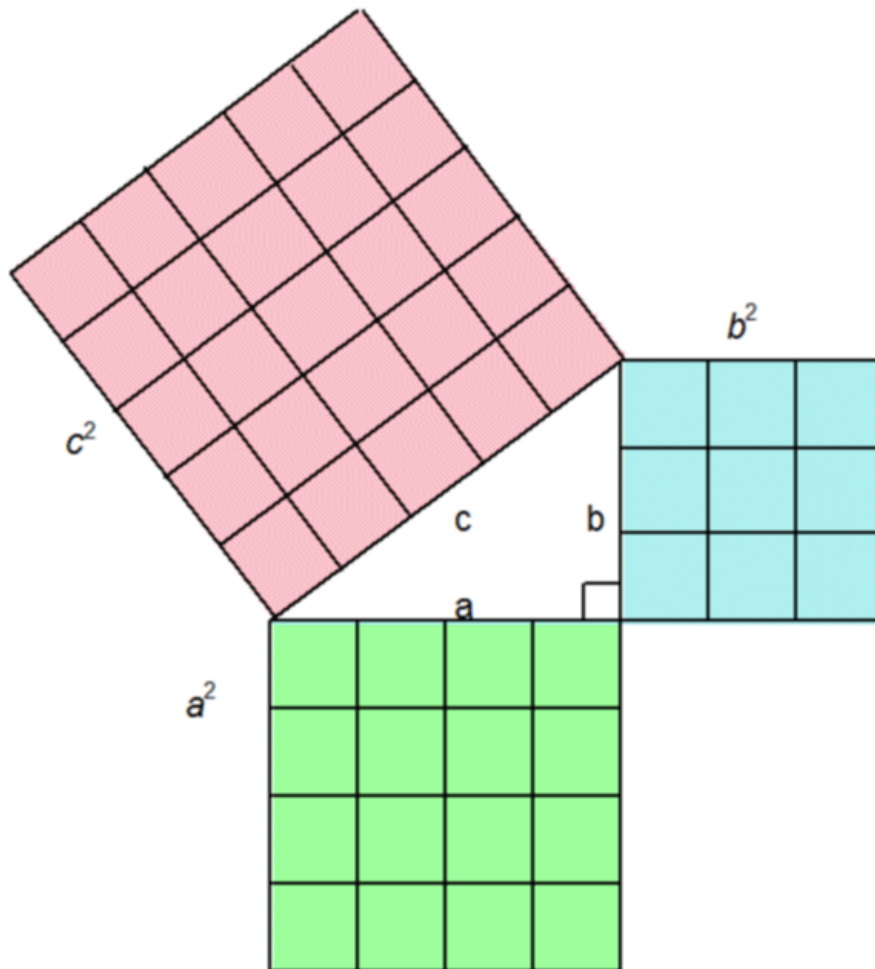
PYTHAGORAS' THEOREM



PYTHAGORAS' THEOREM.

Pythagoras' theorem states that for any right-angled triangle, **the square on the hypotenuse is equal to the sum of the squares of the other two sides.**

The hypotenuse is the side opposite the right angle, and is always the longest side.



In the diagram above, it thus follows that $c^2 = a^2 + b^2$,
or $c = \sqrt{a^2 + b^2}$, where c is the hypotenuse.

The above form is used when the hypotenuse is unknown, but it can be adapted to find an unknown side when the hypotenuse is known.

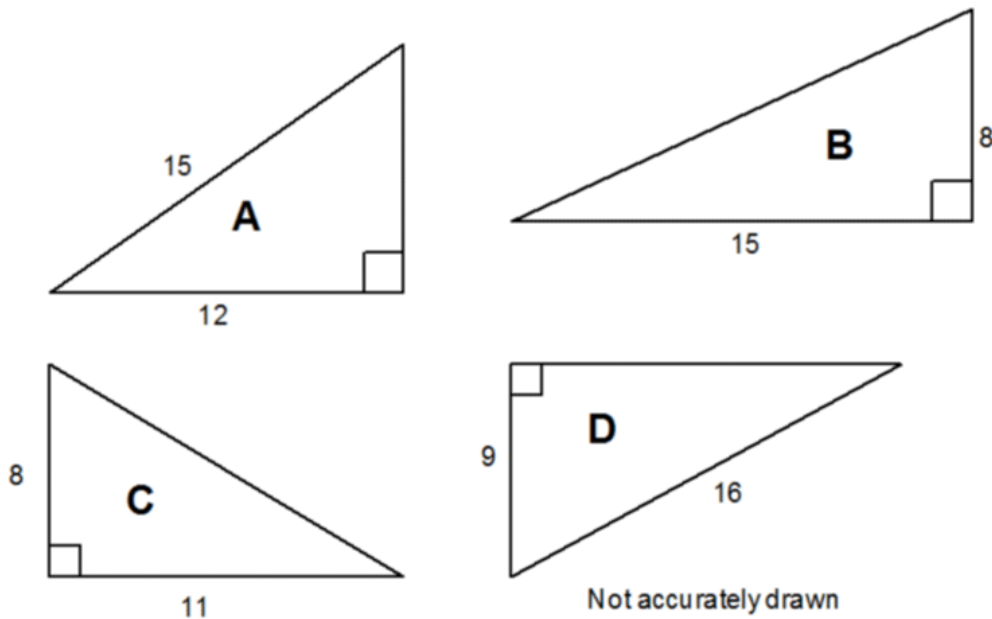
Then, $a^2 = c^2 - b^2$ or $a = \sqrt{c^2 - b^2}$ if a is unknown, and similarly

$b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ if b is unknown.

If the hypotenuse is unknown, we **add** the squares of the two other sides and take the square root.

If a shorter side is unknown, we **subtract** the smaller of the squares of the other two sides from the larger one and take the square root.

Example (1): Find the length of the missing side in each of the right-angled triangles below.



In triangle **A**, the hypotenuse is known, therefore the missing side has a length of

$$\sqrt{15^2 - 12^2} = \sqrt{225 - 144} = \sqrt{81} = 9 \text{ units}$$

In triangle **B**, the hypotenuse is unknown, therefore its length is

$$\sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17 \text{ units}$$

In triangle **C**, the hypotenuse is again unknown, therefore its length is

$$\sqrt{11^2 + 8^2} = \sqrt{121 + 64} = \sqrt{185} = 13.6 \text{ units, to 1 decimal place.}$$

In triangle **D**, the hypotenuse is known, therefore the missing side has a length of

$$\sqrt{16^2 - 9^2} = \sqrt{256 - 81} = \sqrt{175} = 13.2 \text{ units to 1 decimal place.}$$

Area of a right-angled triangle.

The formula for the area of a triangle is given by

$$\text{area} = \frac{1}{2}(\text{base} \times \text{height}).$$

This is particularly simple when the triangle is right-angled; all that is needed are the lengths of the sides containing the right angle.

Example (2): Find the area of triangles **A** and **B** from the previous example.

We are only given one side containing the right angle to begin with in the case of triangle **A**. After calculating the missing side as being 9 units long, we can work out the area as $\frac{1}{2}(9 \times 12)$ or 54 square units.

The two sides containing the right angle are already given in triangle **B**. The area is therefore $\frac{1}{2}(8 \times 15)$ or 60 square units.

Sometimes it might be necessary to apply Pythagoras’ theorem more than once.

Example (3): Find the length AD in the diagram shown on the right.

Here we have two right-angled triangles, ABC and ACD .

We therefore begin by finding side AC , which is the hypotenuse of triangle ABC .

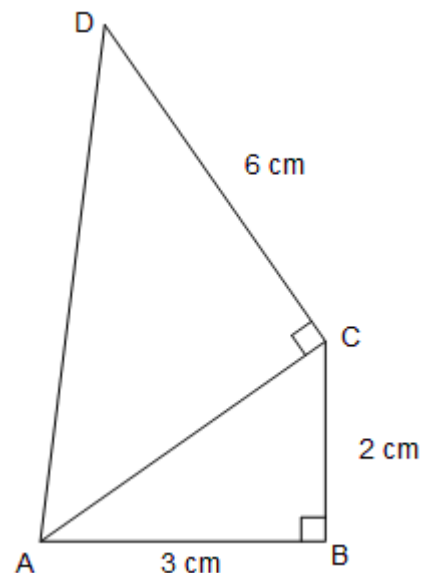
$$(AC)^2 = (AB)^2 + (BC)^2, \text{ i.e. } (AC)^2 = 2^2 + 3^2 = 13, \\ \text{so } AC = \sqrt{13}.$$

We leave the length of AC in square-root form rather than by using an approximate value, because we will only need to square it again in the next part of the question.

Since side AC is also one of the shorter sides of triangle ACD , and AD is its hypotenuse,

$$(AD)^2 = (AC)^2 + (CD)^2, \text{ or } (AD)^2 = 13 + 6^2 = 49, \\ \text{(remembering } (\sqrt{13})^2 \text{ is simply } 13).$$

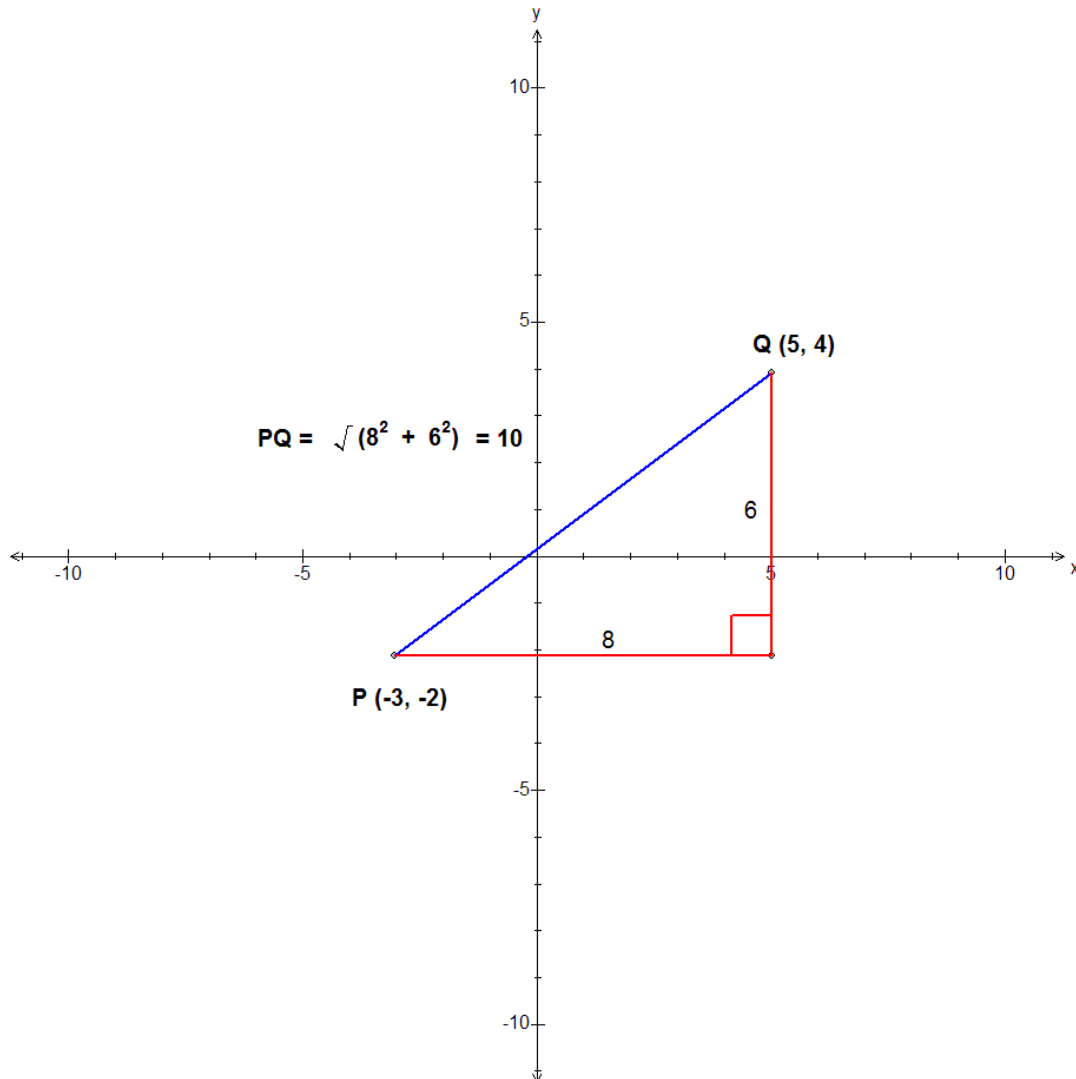
$$\text{Hence } AD = \sqrt{49} = 7 \text{ cm.}$$



Example (4): an important application to find the distance between two points.

Find the length of the line joining the points $(-3, -2)$ and $(5, 4)$.

The line joining the two points can be visualised as the hypotenuse of a right-angled triangle whose other two sides run parallel with the axes and whose right angle is at the point $(5, -2)$.



The lengths of the two sides are therefore:

8 units for the one parallel to the x -axis, obtained by subtracting the x -coordinate of the point $(-3, -2)$ from that of $(5, 4)$ - i.e. $(5 - (-3)) = 8$.

6 units for the one parallel to the y -axis, obtained by subtracting the y -coordinate of the point $(-3, -2)$ from that of $(5, 4)$ - i.e. $(4 - (-2)) = 6$.

The length of the hypotenuse, and the original line in question, is therefore $\sqrt{8^2 + 6^2} = \sqrt{100}$ units, or 10 units.

In general, the length of a line joining two points (x_1, y_1) and (x_2, y_2) on the plane is expressed as

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In other words, we square the difference between the x 's, then square the difference between the y 's, add the two squares and finally find the square root of the result.

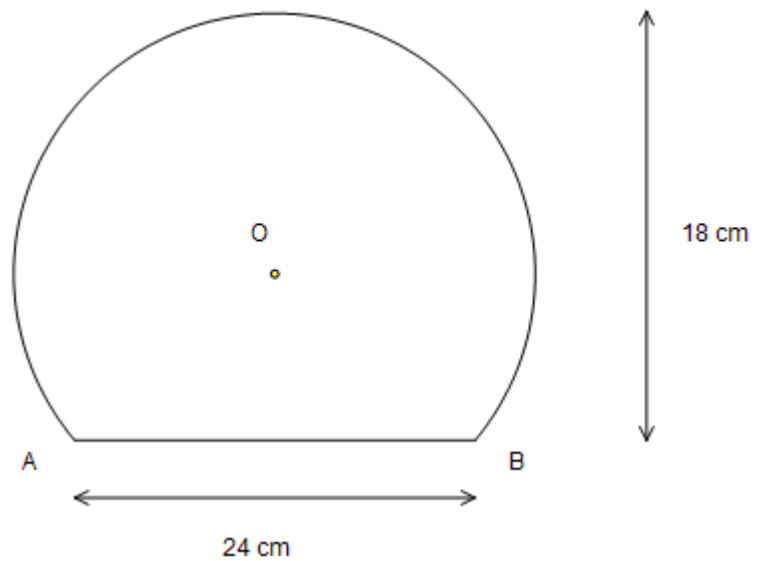
Example (5): This example combines properties of circles with an application of Pythagoras.

The diagram shows the major segment of a circle whose straight side AB is 24 cm long, and whose height is 18 cm. Find the diameter of the original circle.

This is no need to use trig ratios in this problem at all !

All we need to do is join points A and B to the centre of the circle, O .

The perpendicular from O then bisects the isosceles triangle OAB at point M , the midpoint of chord AB .



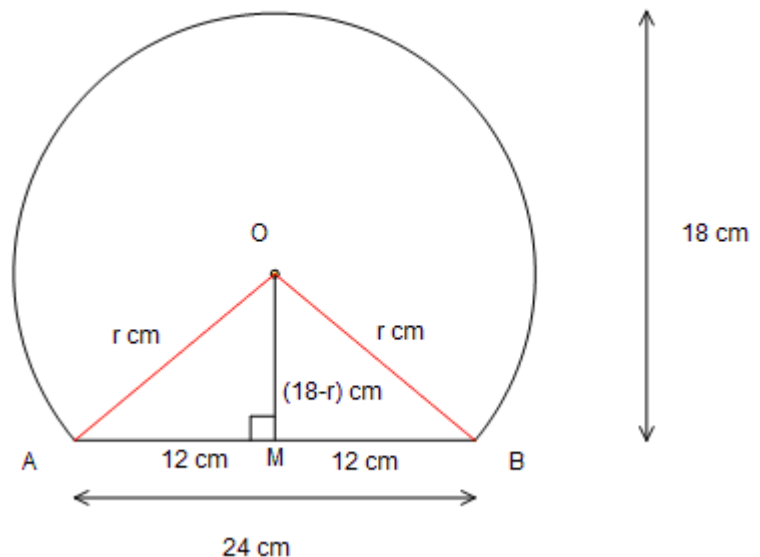
Hence $AM = MB = 12$ cm.

Let $AO = OB = r$ (the radius of the circle), and hence $OM = (18-r)$ cm.

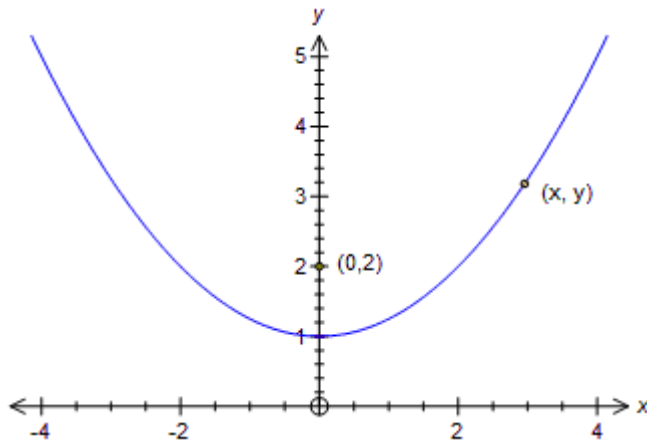
Since triangles AOM , BOM are right-angled, we can use Pythagoras to solve the equation

$$\begin{aligned} r^2 &= 12^2 + (18-r)^2 \\ \rightarrow r^2 &= 144 + 324 - 36r + r^2 \\ \rightarrow 468 - 36r &= 0 \\ \rightarrow r &= 13. \end{aligned}$$

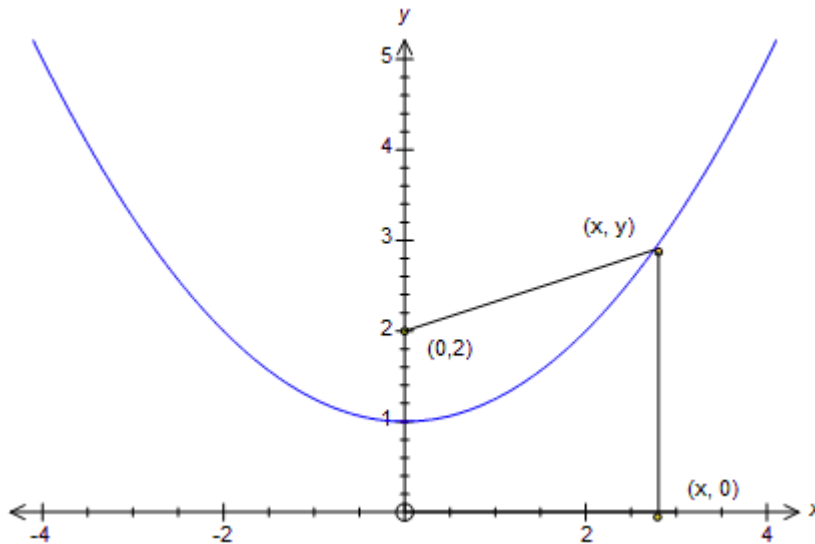
The radius of the circle is 13 cm, hence the diameter is 26 cm.



Example (6): The graph of a curve is shown below. Any point (x, y) on the graph is equidistant from both the x -axis and the point $(0, 2)$.



Show that the equation of the curve is $y = 1 + \frac{1}{4}x^2$.



The distance between (x, y) and the x -axis is $\sqrt{(x-x)^2 + (y-0)^2}$, or simply y .

The distance between (x, y) and the point $(0, 2)$ is $\sqrt{(x-0)^2 + (y-2)^2}$.

Squaring and expanding both expressions, we have :

$$y^2 = x^2 + (y-2)^2 \quad (\text{given distances are equal})$$

$$\rightarrow y^2 = x^2 + y^2 - 4y + 4$$

$$\rightarrow 4y + y^2 = x^2 + y^2 + 4 \quad (\text{bring } 4y \text{ over to LHS})$$

$$\rightarrow 4y = 4 + x^2 \quad (\text{cancel } y^2 \text{ from both sides})$$

$$\rightarrow y = 1 + \frac{1}{4}x^2$$