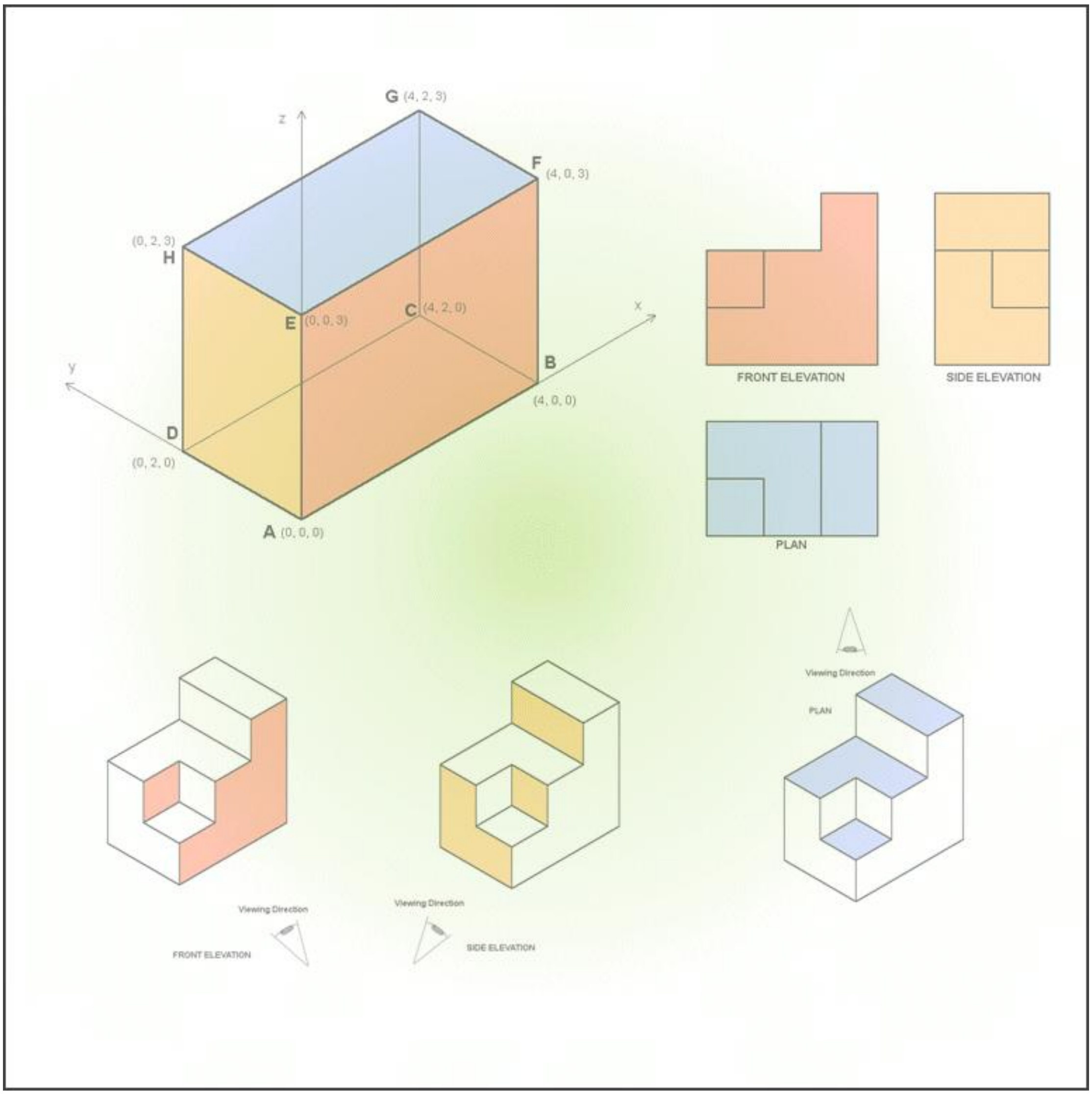
# M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

# THREE-DIMENSIONAL GEOMETRY

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# THREE-DIMENSIONAL GEOMETRY

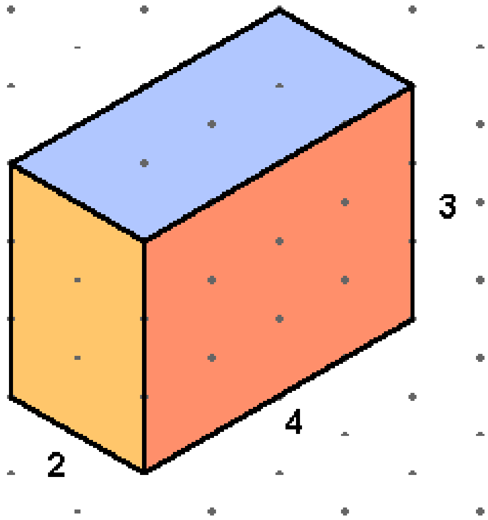
There are various ways of representing a three-dimensional figure in two dimensions.

One method is to display it in **isometric projection**, where lengths are true to scale and the object is viewed from equal angles to all the three axes. The dots on an isometric grid coincide with the corners of the tessellation of equilateral triangles.

**Example (1):** Draw a 4 × 3 × 2 cuboid using isometric projection. (The blank grid would normally be supplied with the exam question.)

The diagram on the right shows one possible alignment.

Note how *lengths* are preserved, but *angles* are not; right angles in nature do not appear as right angles in the projection. .



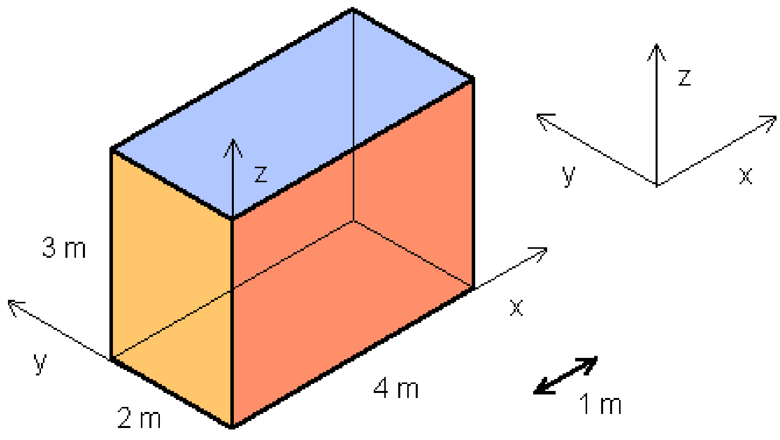

**The three-dimensional coordinate system.**

The coordinate system in two-dimensional space makes use of *two* axes at right angles to one another. Extending to three dimensions, we make use of a third axis, the *z*-axis, perpendicular to both the *x*- and *y*-axes.

In the example on the right, the lowermost corner of the cuboid is taken to be the origin of the coordinate system.

If we were to view the cuboid by looking directly at the blue face, we would find the *x*- and *y*-axes in their expected position , whilst the *z*-axis would appear to point out vertically from the origin.

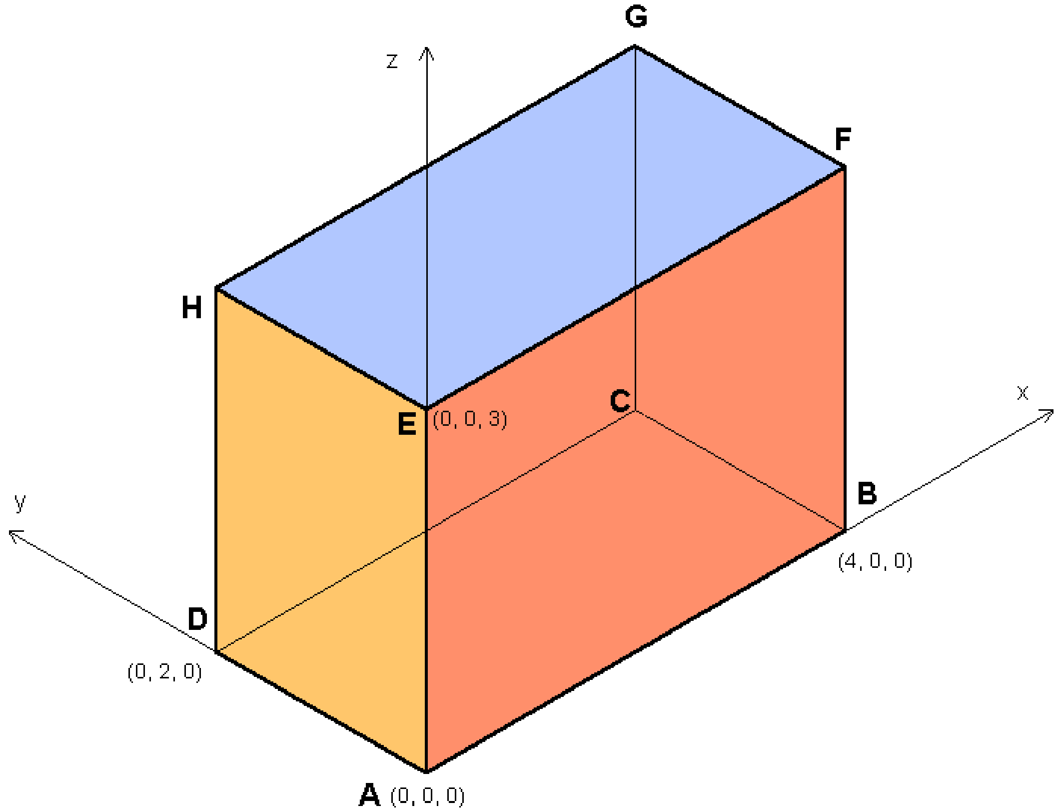
The ***positive*** *z*-axis points ***outwards*** from the paper towards the eye. Conversely, the negative *z*-axis points 'into the depth' of the paper.




Points in three-dimensional space have ($x$, $y$, $z$) coordinates, where the origin is taken as (0, 0, 0).

The next examples demonstrate the 3-D coordinate system.

**Example (2):** Find the coordinates of the vertices *C*, *F*, *G* and *H* of the cuboid shown below**.**




We begin by finding the coordinates of *C*. All the faces of a cuboid are rectangles, so *DC* is parallel to *AB*, and *BC* to *AD*.

Since *BC* and *AD* are parallel, and *D* is a translation of *A* by 2 units in the *y*-direction, *C* is a translation of *B* (4, 0, 0) by the same distance.

Hence the coordinates of *C* are (4, 2, 0) – the *y*-coordinate of *B* is increased by 2.

We could have applied a similar argument to sides *DC* and *AB***:**

*DC* and *AB* are parallel, and *B* is a translation of *A* by 4 units in the *x*-direction, *C* is a translation of *D* (0, 2, 0) by the same distance.

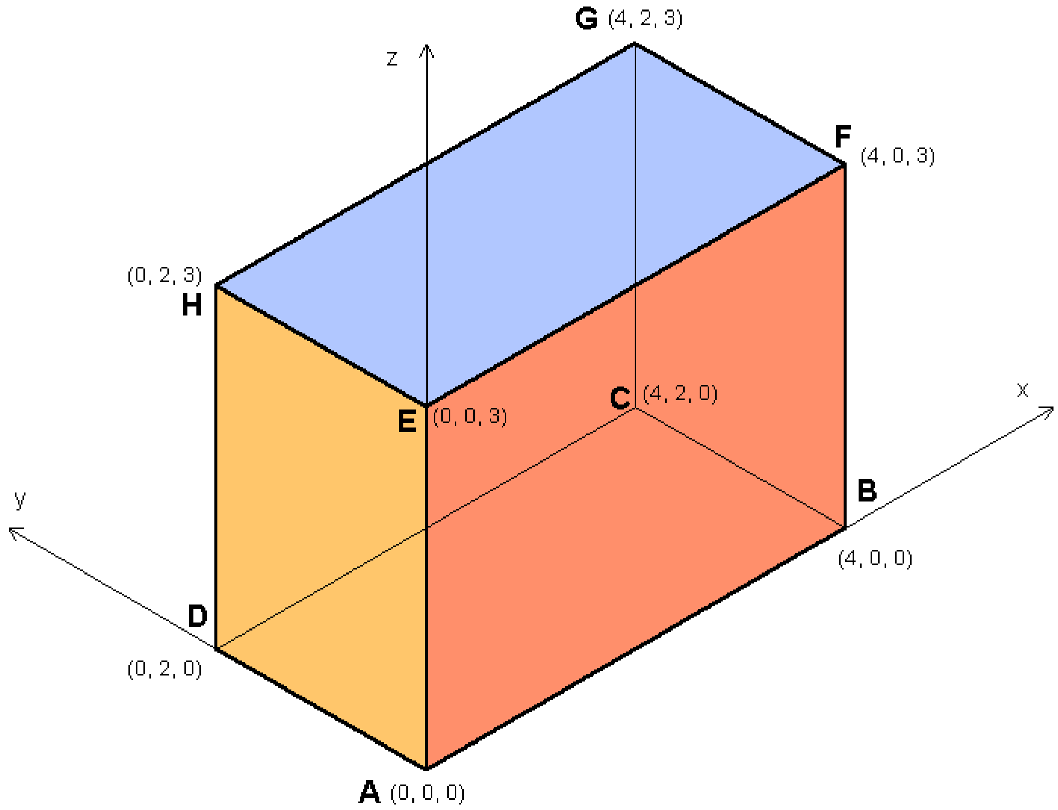
Hence the coordinates of *C* are (4, 2, 0) – the *x*-coordinate of *D* is increased by 4.

The next step is to find the coordinates of points *F*, G and *H*. The points *A* and *E* provide the necessary reference here - point *E* (0, 0, 3) being a translation of *A* (0, 0, 0) by 3 units in the *z*-direction.

Since *BF* is parallel to *AE*, it follows that *F* is also a translation of *B* by 3 units in the *z*-direction.
The coordinates of *F* are (4, 0, 3) – the *z*-coordinate of *B* is increased by 3.

We can apply the same reasoning to find the coordinates of *G* from *C*, and *H* from *D***.** For the sake of completeness, *G* has coordinates (4, 2, 3) and *H* has coordinates (0, 2, 3).

See the completed diagram on the next page.




Notice the pattern in the coordinates of each face of the cuboid:

All the points on the front face *ABFE* (in red) have their *y*-coordinate fixed at zero.
(On the hidden rear face *DCGH*, all points have *y*-coordinates of 2).

All the points on the side face *AEHD* (in gold) have their *x*-coordinate fixed at zero.
(On the hidden opposite side face *BFGC*, all points have *x*-coordinates of 4)

All the points on the top face *EFGH* (in blue) have their *z*-coordinate fixed at 3.
(On the hidden bottom face *ABCD*, all points have *z*-coordinates of 0).

**Orthographic Projection.**

The isometric view of the cuboid in the last example displayed three faces in one diagram, from one particular angle.

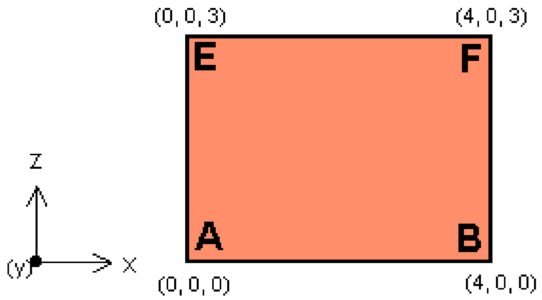
The idea of the orthographic projection method is to view the cuboid from three different angles – giving varying views, i.e. the front elevation, the side elevation and the plan. .

**Example (3):** Draw the front elevation, side elevation and plan of the cuboid in Example (2).

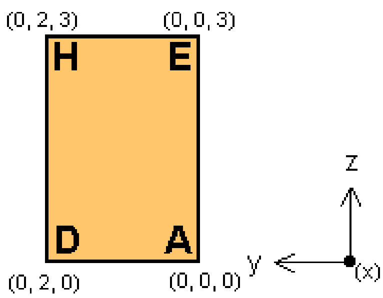
In the **front elevation**, the eye is looking directly at the front of the cuboid (red), with the *y*-axis foreshortened to a point.

In the **side elevation**, the eye is looking at the visible side of the cuboid (gold), with the *x*-axis now 'invisible'.

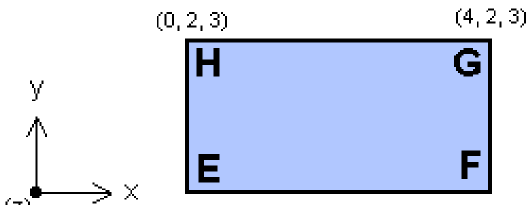
Finally, in the **plan,** the eye is looking at the cuboid from above (blue). Here it is the turn of the *z*-axis to become foreshortened out of view.




FRONT ELEVATION




SIDE ELEVATION




PLAN

The coordinates of each point have been given here to match with the isometric projection, but again this detail is not needed in exams.

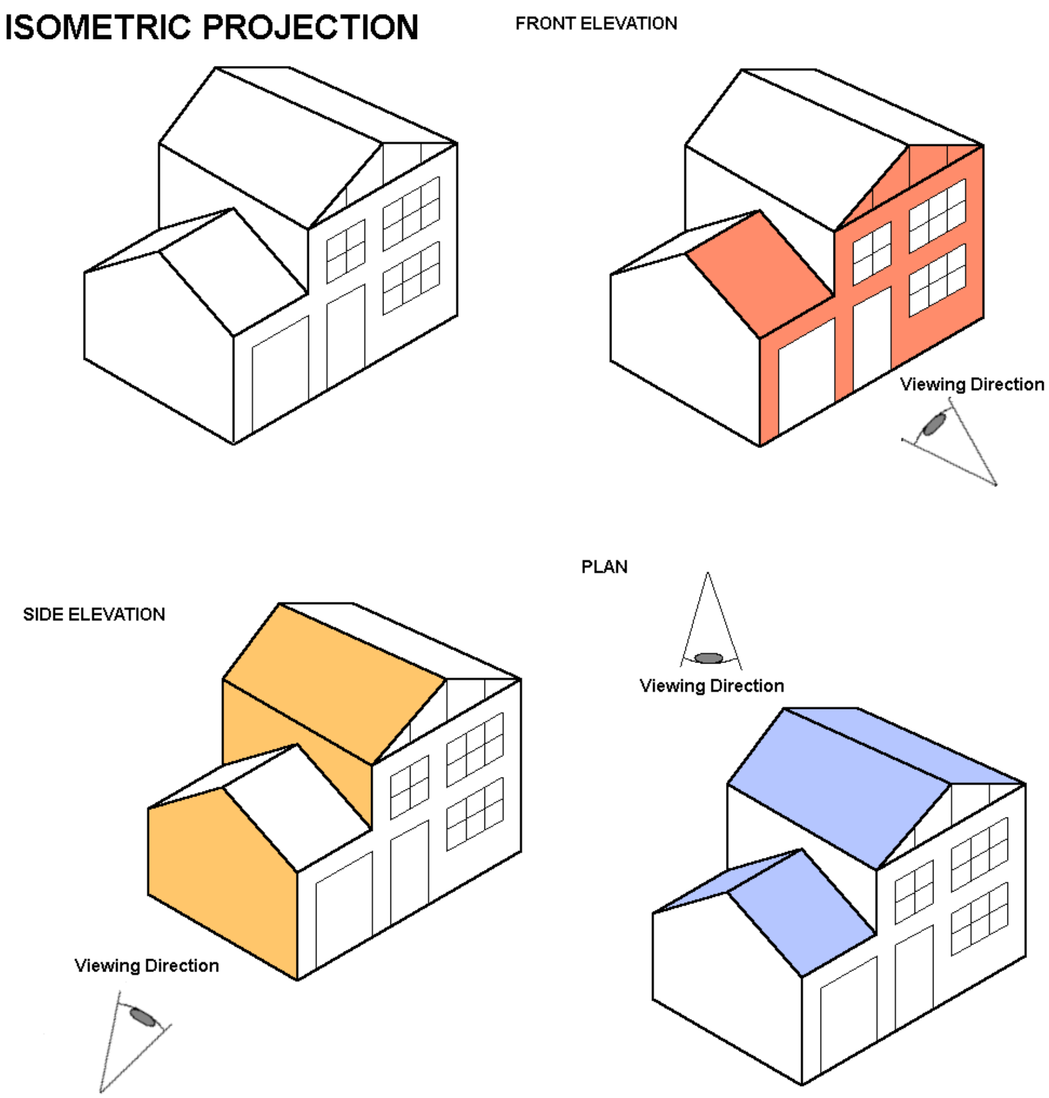
Notice how point *E* appears in all the projections, but point *C* in none.

This example was rather boring, to say the least, but the next one is a bit more interesting.

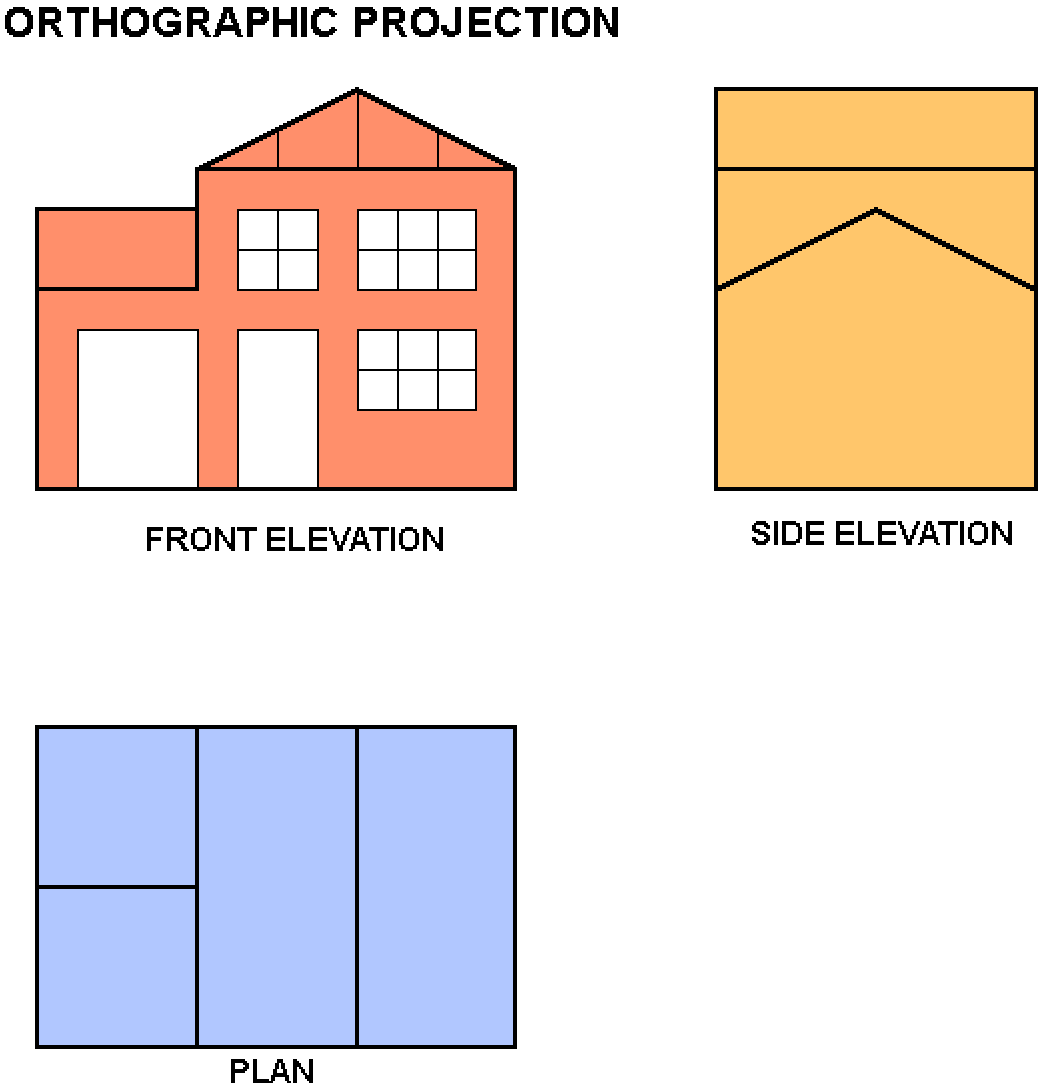
**Example (4):** Study the isometric projections of the house, and use the diagrams to construct orthographic projections of the house.

Although the figure of the house is more complex than the earlier example, the same ideas apply when it comes to drawing the orthographic projections.

Again, the eye is looking 'face-on' for the front elevation (in red), 'side-on' for the side elevation (in gold), and from 'up above' for the plan (in blue).



The orthographic projections of the house are shown below.




Notice how some features, i.e. the roofs, are visible on more than one projection.

For instance, the front garage roof is visible on both the plan and the front elevation, and the left side house roof on both the plan and the side elevation.

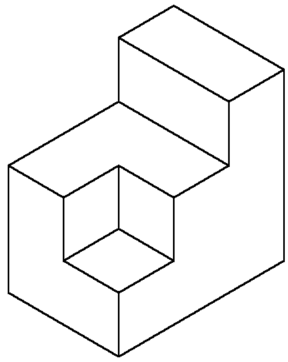
Notice also how there is no visible sense of 'depth' through perspective.

Thus, on the plan, there is no way of seeing that the main roof is on a higher level than the garage roof, for example.

Examination questions on projections will usually be restricted to simpler arrangements of cubes, as in the next example.

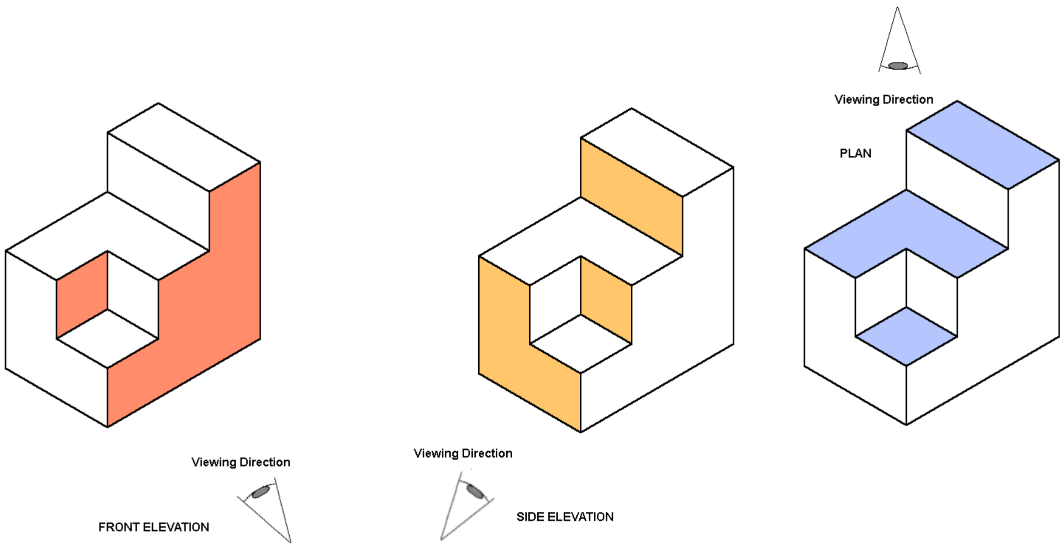
**Example (5):** The figure on the right is composed entirely of unit cubes, and shown in isometric projection.

Draw the front elevation, side elevation and plan of the figure.
(An exam question would generally be accompanied by a square grid.)



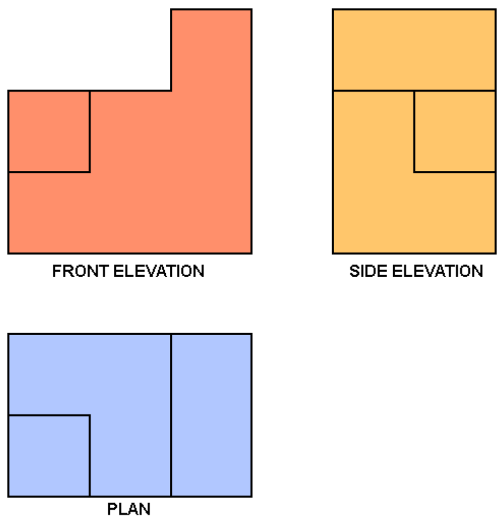
The diagrams below show the parts of the figure in view for each particular viewing angle.

The eye is looking 'face-on' for the front elevation (in red), 'side-on' for the side elevation (in gold), and from 'up above' for the plan (in blue).




The orthographic projections of the figure are shown on the right.

Notice again how there is no visible sense of 'depth' through perspective.

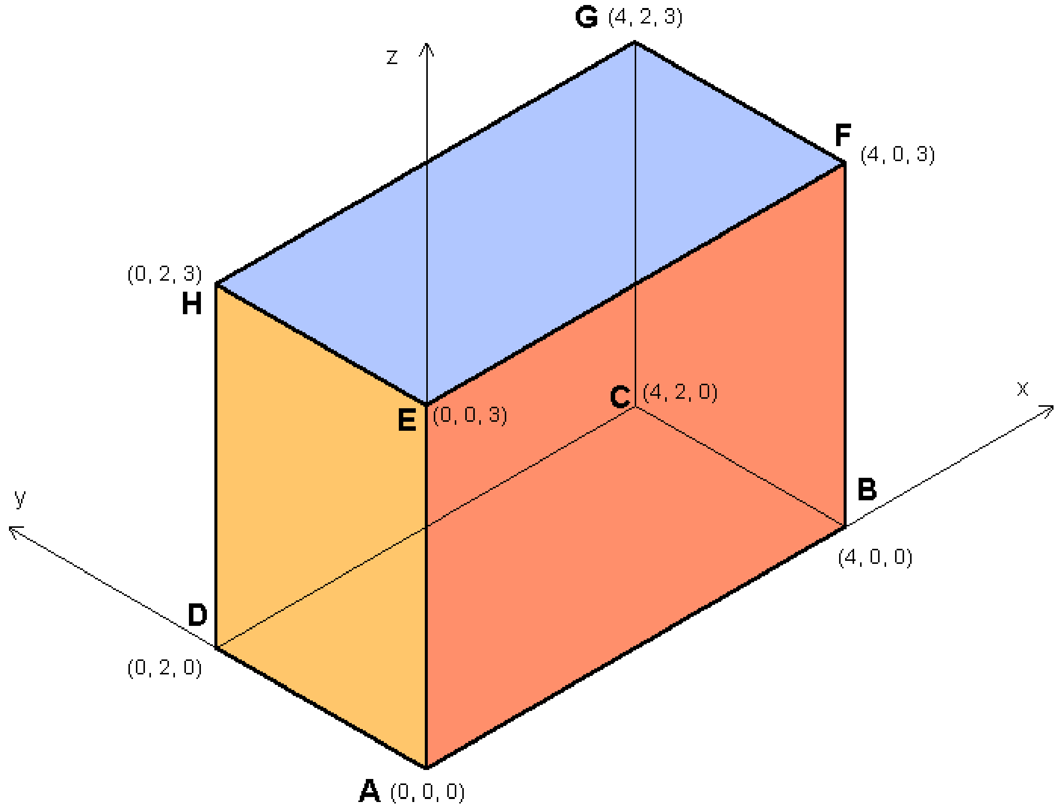


Several methods in two-dimensional geometry can be extended to three dimensions.

**Midpoint of a line in three dimensions.**

If a point $P$ has coordinates $(x_1, y_1, z_1)$ and a point $Q$ has coordinates $(x_2, y_2, z_2)$ , then the midpoint of the line $PQ$ has the coordinates

$$(x_m, y_m, z_m) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$




**Example (6):** Find the coordinates of the midpoints of i) side $GH$ ; ii) side $FG$ ; iii) diagonal $DF$ of the cuboid shown above.

i) Since $G$ has coordinates of (4, 2, 3) and $H$ has coordinates of (0, 2, 3), the midpoint's coordinates are $\left(\frac{4+0}{2}, \frac{2+2}{2}, \frac{3+3}{2}\right)$, or **(2, 2, 3)**.

ii) Similarly, the midpoint of the line joining $F$ (4, 0, 3) and $G$ (4, 2, 3) is $\left(\frac{4+4}{2}, \frac{0+2}{2}, \frac{3+3}{2}\right)$, or **(4, 1, 3)**.

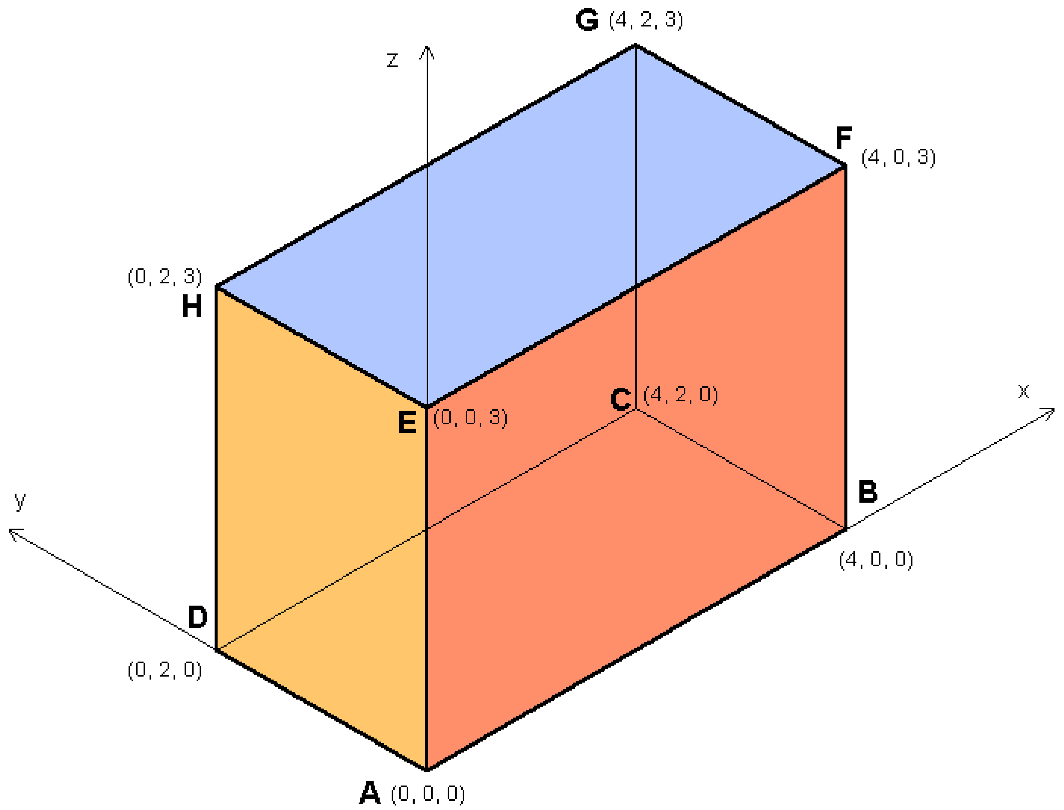
iii) Again, the midpoint of the line joining $D$ (0, 2, 0) and $F$ (4, 0, 3) is $\left(\frac{0+4}{2}, \frac{2+0}{2}, \frac{0+3}{2}\right)$, or **(2, 1, 1½)**.

**The distance between two points in three dimensions.**

This is another straightforward modification, still using Pythagoras' theorem.

The length of a line joining two points ($x_1$, $y_1$, $z_1$) and ($x_2$, $y_2$, $z_2$) in three-dimensional space is expressed as

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$




**Example (7):** Find the distance between i) the points *A* and *G*; ii) the point *G* and the point *P* (6, 5, 9).

i) Taking *A* (0, 0, 0) and *G* (4, 2, 3), the length of the line joining the two points is

$$\sqrt{(4-0)^2 + (2-0)^2 + (3-0)^2}$$

or $\sqrt{16+4+9} = \sqrt{29}$ units.

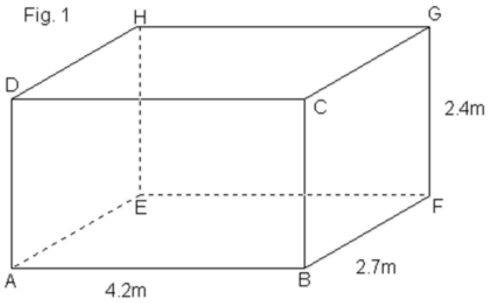
ii) Similarly, the distance between *G* (4, 2, 3) and *P* (6, 5, 9) is $\sqrt{(6-4)^2 + (5-2)^2 + (9-3)^2}$

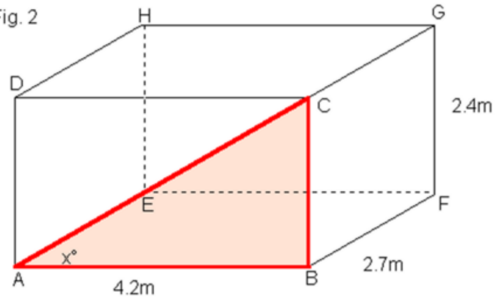
or $\sqrt{4+9+36} = \sqrt{49} = 7$ units.

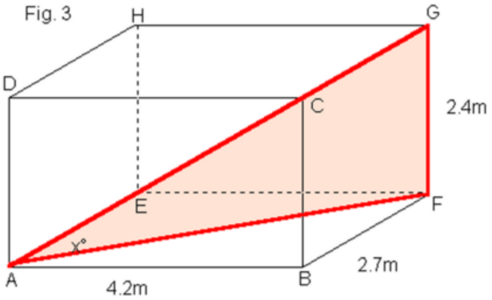
Other three-dimensional problems may require more use of trigonometry for their solution.

The best way to tackle such problems is to 'pick out' appropriate right-angled triangles from the 2-D diagrams. Remember that projective distortion can make right angles look acute or obtuse !

**Example (8).**




A cuboidal room is 4.2 metres long, 2.7 metres wide and 2.4 metres high. (See Fig. 1)

Find:
i) the distance *AC*, and the angle between *AC* and the floor of the room, *ABFE*.
ii) the diagonal distance, *AG*, between the opposite corners *A* and *G*.
iii) the angle *GAF* between *AG* and the floor *ABFE*.

i) As Fig. 2 shows, the distance *AC* can be calculated as being the hypotenuse of the right-angled triangle whose other two sides are *AB* and *BC*. The triangle is easy to find here as it is in the plane of the front face.

The distance *AC* is therefore $\sqrt{4.2^2+2.4^2} = \sqrt{17.64+5.76} = \sqrt{23.4} = 4.837$ m, or 4.84m to 3 s.f.

The angle, *CAB* or *x*°, can be found by the formula $\tan x° = \frac{2.4}{4.2}$ giving *x* = 29.7 °.

ii) To find the diagonal distance between the two corners *A* and *G* as per Fig.3, we need to pick out the right-angled triangle *AFG*, where the right angle appears obtuse due to perspective.

We apply Pythagoras twice here. Firstly, the distance *AF* (the floor diagonal) can be worked out as the hypotenuse of the right-angled triangle whose other two sides are *AB* and *BF*.

The distance *AF* is therefore $\sqrt{4.2^2+2.7^2} = \sqrt{17.64+7.29} = \sqrt{24.93}$ m.
(Leave this result as a surd, since it will only need to be squared again in the next stage.)

Applying Pythagoras again, we use the squares of the distances *AF* and *FG***.**

$AG = \sqrt{24.93+2.4^2} = \sqrt{24.93+5.76} = \sqrt{30.69}$ m, or 5.54m to 3 s.f.

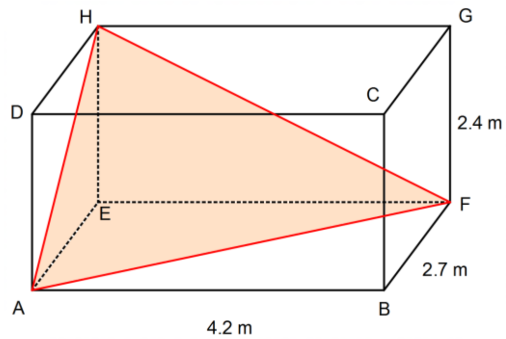
iii) The angle *GAF* or *x*°, can be found by the formula $\tan x° = \frac{2.4}{\sqrt{24.93}}$ giving *x* = 25.7°.

**Example (8a):** Calculate the angles of the triangle *AFH* in the cuboid from Example (8).

This calls for the use of the sine and cosine rules in three dimensions.

Firstly, we find the side lengths by Pythagoras:

$AF = \sqrt{4.2^2 + 2.7^2} = \sqrt{17.64 + 7.29} = \sqrt{24.93}$ m.
$AH = \sqrt{2.4^2 + 2.7^2} = \sqrt{5.76 + 7.29} = \sqrt{13.05}$ m.
$FH = \sqrt{2.4^2 + 4.2^2} = \sqrt{5.76 + 17.64} = \sqrt{23.40}$ m.

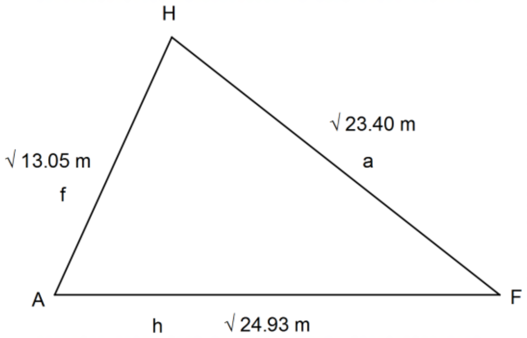



Using the convention of lower-case letters for sides and upper-case letters for angles, we label the triangle *AFH* as per the diagram below (leaving side lengths in exact form), and then substitute into the cosine formula.

We will find the angle opposite the longest side first (to take care of a possible obtuse angle),

$\cos H = \frac{a^2 + f^2 - h^2}{2af}$ , or $\frac{23.40 + 13.05 - 24.93}{2\sqrt{23.40}\sqrt{13.05}}$, or 0.3296, thus $H$ = 70.8° to 1 d.p.

(Keep the more accurate value of 70.75° for future working)




Next, we can find angle *A* using the sine rule:

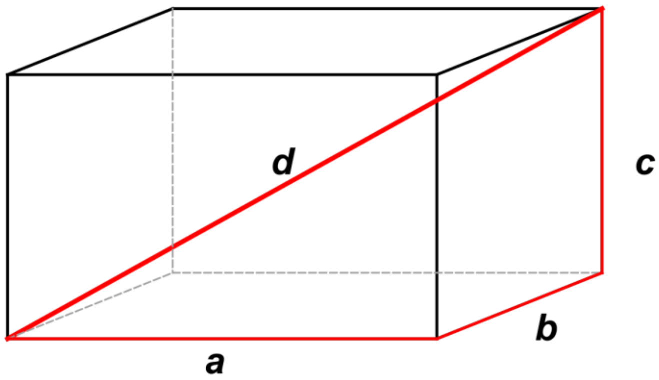
$\sin A = \frac{a \sin H}{h} = \; = \frac{\sqrt{23.40}\sin 70.75°}{\sqrt{24.93}} = 0.9147$ , and thus $A$ = 66.2° to 1 d.p. Only the acute angle is valid here, as it is smaller than angle *H*. (We will keep the more accurate value of 66.16°)

To find *F*, we subtract the sum of *A* and *H* from 180°, so *F* = 180 – (70.75+66.16)° = 43.1° to 1 d.p.

The angles of triangle *AFH* are **70.8°, 66.2° and 43.1°** to 1 d.p.

(Rounding issues have given an angle sum of 180.1°).

In Example (8), we calculated the distance between the diagonally opposite corners **A** and **G** of the cuboid by applying Pythagoras' theorem twice.




Let *a*, *b* and *c* be the linear dimensions of the cuboid, and *d* the diagonal between the opposite corners.

The length of the diagonal *d* can be calculated in one step by adapting Pythagoras' theorem as

$d^2 = a^2 + b^2 + c^2$ , or taking square roots, $d = \sqrt{a^2 + b^2 + c^2}$ .

Hence the length of the diagonal $AG = \sqrt{4.2^2 + 2.7^2 + 2.4^2} = \sqrt{30.69} = 5.54$ m to 3 s.f.

**Example (9):** Find the longest diagonals of the following cuboids:

i) Total surface area 216 cm$^2$, volume 140 cm$^3$, height 2 cm.
ii) Total surface area 100 cm$^2$, volume 40 cm$^3$, height 2 cm.

Recall the formulae:

Volume of cuboid = $abc$ where $a$, $b$ and $c$ are the length, depth and height.
Surface area of cuboid = $2(ab + ac + bc)$; again $a$, $b$ and $c$ are the length, depth and height.

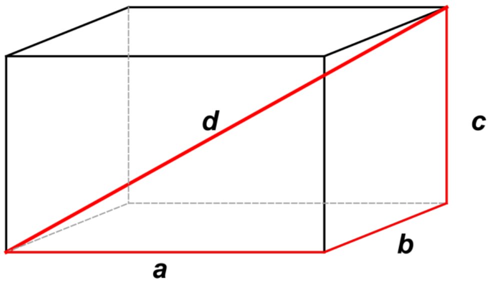
i) The height of the cuboid, $c$, = 2 cm, so $abc = 2ab = 140 \rightarrow ab = 70$.

Also, the surface area formula $2(ab + ac + bc)$ can be rewritten as $2(ab + (a + b)\,c)$.

Substituting $c = 2$ gives us $2(ab + 2(a + b)) = 216 \rightarrow ab + 2(a + b) = 108$.
As $ab = 70$, $2(a + b) = 38 \rightarrow a + b = 19$.

The length $a$ and depth $b$ must therefore have a product of 70 and a sum of 19, bringing to mind factorisation of quadratic expressions.

Hence $a$ and $b$ are 14 cm and 5 cm. (order does not matter).




The dimensions of the cuboid are 14 cm × 5 cm × 2 cm, and the longest diagonal is

$$d = \sqrt{a^2 + b^2 + c^2} = \sqrt{196 + 25 + 4} = \sqrt{225} \text{ cm} = 15 \text{ cm.}$$

ii) The height of the cuboid, $c$, = 2 cm, so $abc = 2ab = 40 \rightarrow ab = 20$.
Substitute $c = 2$ into the surface area formula: $2(ab + 2(a + b)) = 100 \rightarrow ab + 2(a + b) = 50$.
As $ab = 20$, $2(a + b) = 30 \rightarrow a + b = 15$.

The length $a$ and depth $b$ must therefore have a product of 20 and a sum of 15, but unfortunately there is no integer pair satisfying the conditions, recalling the case of a quadratic expression that cannot be factorised.

There are two possible methods of finding the long diagonal:

**1) By linear / quadratic simultaneous equations**

We could form a pair of simultaneous equations by substitution :

$ab = 20$ ; $a + b = 15$

Substituting $b = \frac{20}{a}$ into the second equation we have $a + \frac{20}{a} = 15 \rightarrow a^2 + 20 = 15a$
$\rightarrow a^2 - 15a + 20 = 0$. Using the general quadratic formula, we have the solutions of

$a = \frac{15 \pm \sqrt{225 - 80}}{2} \ \therefore \ a = \frac{15 \pm \sqrt{145}}{2}$ . Thus $a$ = 13.52 and $b$ = 1.48 or $a$ = 1.48 and $b$ = 13.52.

As $c$ = 2 cm, the longest diagonal of the cuboid is $d = \sqrt{a^2 + b^2 + c^2} = \sqrt{13.52^2 + 1.48^2 + 2^2}$ , or 13.7 cm to 1 decimal place.

**2) By using quadratic identity $(a + b)^2 = a^2 + 2ab + b^2$**

The question only asks for the length of the longest diagonal, without needing to find the actual height $a$ or depth $b$ at all, but only the sum of their squares. Instead we use the quadratic identity

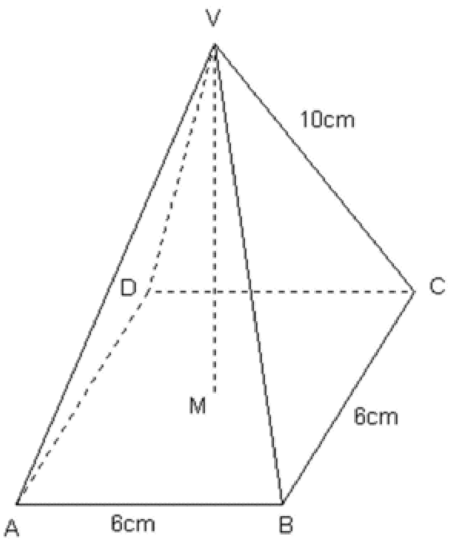
$(a + b)^2 = a^2 + 2ab + b^2$ and rearrange it into the form $a^2 + b^2 = (a + b)^2 - 2ab$.

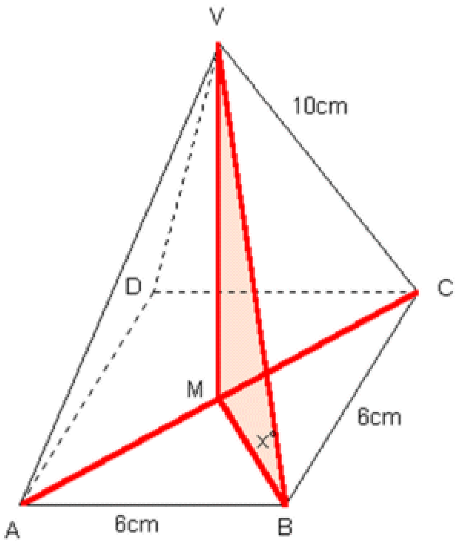
Substituting $a + b = 15$ and $ab = 20$, we have $a^2 + b^2 = 15^2 - (2 \times 20) = 185$.

As $c$ = 2 cm, the longest diagonal of the cuboid is $d = \sqrt{a^2 + b^2 + c^2} = \sqrt{185 + 4} = \sqrt{189}$ cm, or 13.7 cm to 1 decimal place.

Method (2) has the advantage of giving the answer in an exact form more easily.

**Example (10).**




The diagrams above are of a pyramid with a square base of side 6cm, and a slant height, *VB*, of 10cm. The vertex of the pyramid, *V*, is directly above the centre of the base, *M*.

Find : i) the length of the diagonal of the base, *AC*
ii) the height of the pyramid *VM*
iii) the angle between *VB* and the base of the pyramid.

i) The distance *AC* can be calculated as being the hypotenuse of the right-angled triangle whose other two sides are *AB* and *BC*, or alternatively, the diagonal of the square *ABCD*.

The distance *AC* is therefore $\sqrt{6^2+6^2} = \sqrt{72} = 8.485$ cm, or 8.49cm to 3 s.f.

ii) The height of the pyramid can be worked out by using Pythagoras on the triangle *VMB* where angle *VMB* is the right angle.

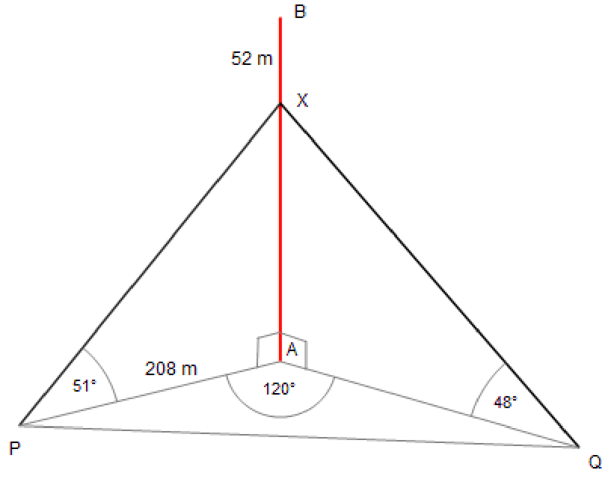
The hypotenuse of triangle *VMB*, namely *VB*, is the slant height of 10cm, and the length of side *MB* is half that of the diagonal *AC* by symmetry, or $\frac{\sqrt{72}}{2} = \sqrt{18}$ cm.
(Notice how we kept the working in surd form to keep accuracy.)

The vertical height, *VM*, is therefore given as $\sqrt{100-18} = \sqrt{82} = 9.055$ cm, or 9.06cm to 3 s.f.

iii) The angle $x$ ° between *VB* and the base of the pyramid can be worked out using any of the trigonometric relationships:

$\sin x = \frac{\sqrt{82}}{10}$ or $\cos x = \frac{\sqrt{18}}{10}$ or $\tan x = \frac{\sqrt{82}}{\sqrt{18}}$ . In each case, $x$ = 64.9°.
(Again, surd expressions have been used for accuracy).

**Example (11):** A TV mast $AB$ is anchored at $X$ by two cables $XP$ and $XQ$.
The ground angle at $PAQ$ = 120°.
The distance $PA$ from the foot of the mast $A$ to the ground anchor at $P$ = 208 m.
The angle of elevation of $X$ from the ground anchor at $P$ = 51°.
The angle of elevation of $X$ from the ground anchor at $Q$ = 48°.
The height $XB$ from the anchor at $X$ to the top of the mast at $B$ = 52 m.

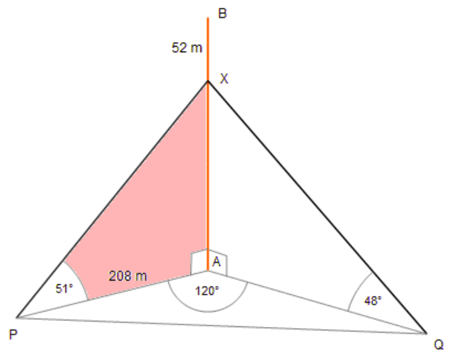


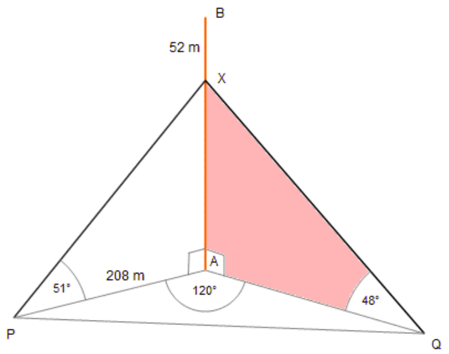

i) Find the length of the cable $PX$.

ii) Find the height $AX$, and hence the total height of the mast, $AB$.

iii) Using the results from ii), find the length of the cable $QX$.

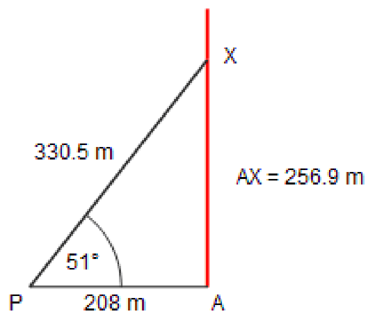
iv) Find the ground distance $PQ$ between the cable anchors, and hence the angle $PXQ$ between the cables at $X$.




Firstly we spot the two right-angled triangles $PAX$ and $QAX$.

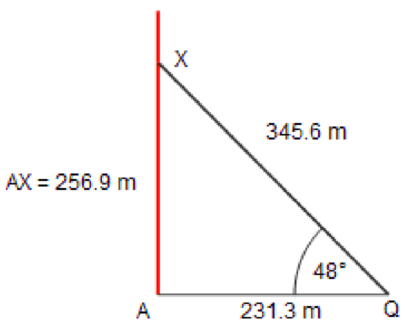
i) $PX = \dfrac{208}{\cos\ 51°} = 330.5$ m.




ii) $AX = 208 \tan 51° = 256.9$ m.

Hence the total height of the mast $AB$ = 52 + 256.9 = **308.9 m.**

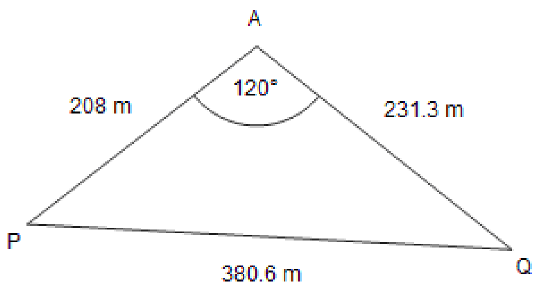
iii) $QX = \dfrac{256.9}{\sin\ 48°} = 345.6$m.




iv) $AQ = \dfrac{256.9}{\tan\ 48°} = 231.3$m.

Unlike triangles *PAX* and *QAX*, *PAQ* is not right-angled, so we need to use the cosine rule to find the ground distance *PQ* between the anchors.

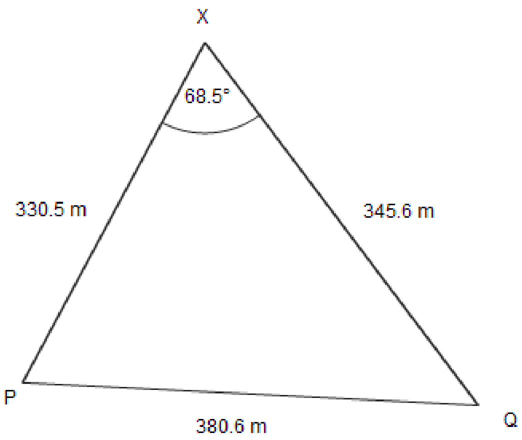
$(PQ)^2 = 208^2 + 231.3^2 - 2(208)(231.3)\cos 120°$
and hence *PQ* = 380.6 m.



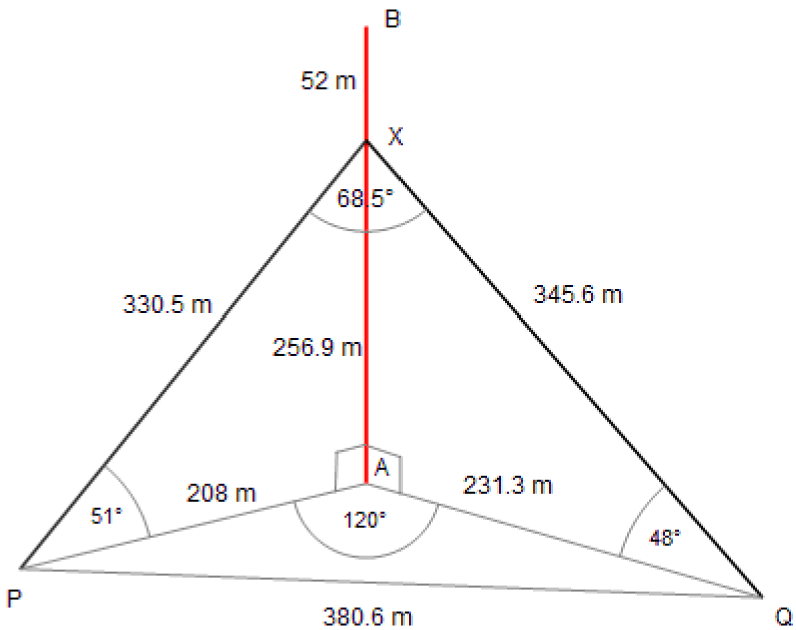

We apply the cosine rule again to find the angle *PXQ* between the cables at *X* :

$$\cos PXQ = \frac{330.5^2 + 345.6^2 - 380.6^2}{2 \times 330.5 \times 345.6}$$

and therefore *PXQ* = 68.5°.




Completed diagram:

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