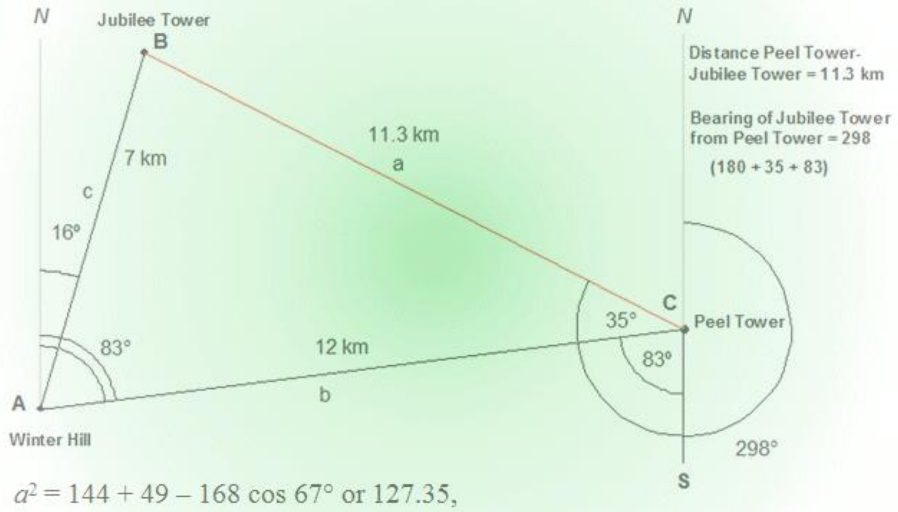
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SOLVING TRIANGLES USING THE SINE AND COSINE RULES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$a^2 = 144 + 49 - 168 \cos 67°$ or 127.35,
hence $a = 11.285$ km

$$\sin C = \frac{7\sin 67°}{11.285} \text{ or } 0.571\text{, hence } C = 35°$$

The bearing of Jubilee Tower from Peel Tower is $(180 + 83 + 35) = 298$.

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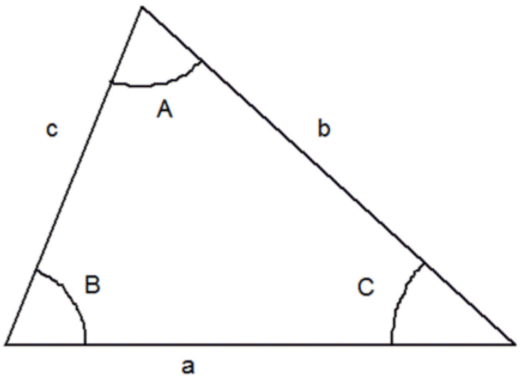
SOLUTION OF GENERAL TRIANGLES - THE SINE AND COSINE RULES.

The sine and cosine rules are used for finding missing sides or angles in all triangles, not just right-angled examples.

The labelling is important here; upper-case letters are used for angles and lower-case ones used for sides. Also, lettered sides are opposite the corresponding lettered angles.

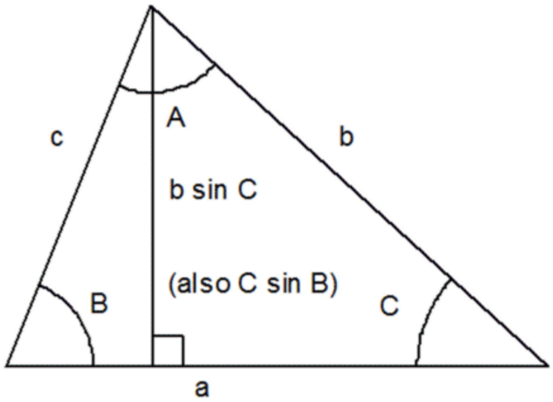


Area of a triangle.

One formula for finding the area of a triangle is ½ (base) × (height). This can be adapted as follows:

By drawing a perpendicular from A, its length can be deduced by realising that it is opposite to angle C, and that the hypotenuse is of length *b*. The length of the perpendicular, and thus the height of the triangle, is *b* sin C.

The area of the triangle is therefore ½*ab* sin C. Since any side can be used as the base, the formula can be rotated to give ½*ac* sin B or ½*bc* sin A.



This formula holds true for acute- and obtuse-angled triangles.

The sine rule.

The sides and angles of a triangle are related by this important formula:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The formula is normally used in the rearranged forms

$\sin A = \dfrac{a \sin B}{b}$ when finding an unknown angle, or

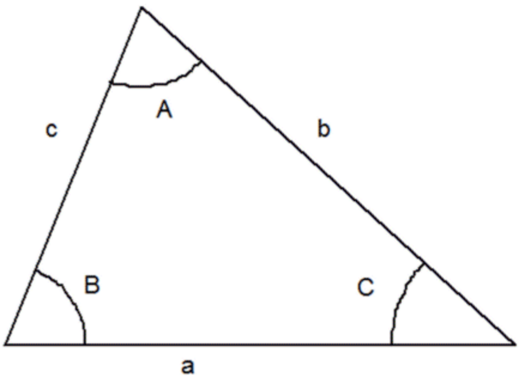
$a = \dfrac{b \sin A}{\sin B}$ when finding an unknown side.

(The corresponding letter-pairs are interchangeable, thus $\sin B = \dfrac{B \sin C}{c}$ and $c = \dfrac{a \sin C}{\sin A}$ are examples of other equally valid forms.)

Note that an equation of the form sin *A* = *x* has *two* solutions in the range 0° to 180°.
Thus 30° is not the only angle with a sine of 0.5 - 150° is another one. Any angle *A* will have the same sine as (180°-*A*). This is important when solving certain cases, but this is outside the scope of GCSE exams.

The cosine rule.

This is another formula relating the sides and angles of a triangle, slightly harder to apply than the sine rule.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Here, A is the included angle between the sides c and b.

It is used in this form when finding an unknown side a, but rearranged as

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

when used to find an unknown angle.

This formula can also be rotated between different sides and angles: thus

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

all have the same effect.

The formulae for missing angles can be similarly rotated:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Which rules should we use ?

This depends on the information given.

i) Given **two angles and one side** - use the **sine rule**. (If the side is not opposite one of the angles, you can work out the third angle simply by subtracting the sum of the other two from 180°).

ii) Given **two sides and an angle opposite one of them** - use the **sine rule**. (Some cases can give rise to two possible solutions, but there is no need to know this at GCSE).

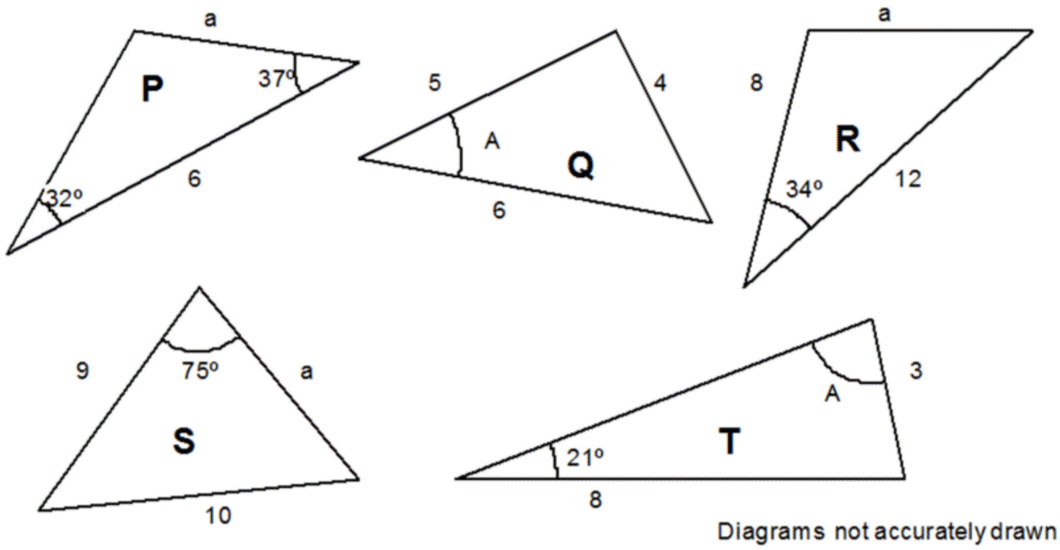
This is the "angle *not* included and two sides" case – questions will usually specify that the triangle is acute-angled. The section on 'Congruent Triangles' also discusses this case.

iii) Given **two sides and the included angle** - use the **cosine rule** to find the third side and then continue with the **three sides and one angle** case below.

iv) Given **three sides and no angles** - use the **cosine rule** to find the angle opposite the longest side, followed by the sine rule for either of the others. The third angle can be found by subtraction.

Example (1): Find the angles marked A and the sides marked *a* in the triangles below.

Note that triangles **S** and **T** are **acute-angled.**



Triangle P.

Two angles and a side are known. The known side is not opposite either of the known angles, but the opposite angle (call it *B*) can easily be worked out by subtracting the other two angles from 180°. This makes *B* = 111° and *b* = 6 units. We will also label the 32° angle *A* as it is opposite side *a*.

We therefore use the sine rule in the form $a = \frac{b \sin A}{\sin B}$, giving $a = \frac{6 \sin 32°}{\sin 111°}$, or 3.41 units to 2 d.p.

Triangle Q.

All three sides are known here but we are required to find angle *A*. Labelling side *a* as the opposite side (length 4 units), we will call the side of length 5 side *b* and the side of length 6 side *c*.

This time we use the cosine rule in the form $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

Substituting for *a*, *b* and *c* gives

$\cos A = \frac{25 + 36 - 16}{60}$ and hence *A* = 41.4° to 1 d.p.

Triangle R.

Here we have two sides plus the included angle given. Label the angle of 34° as *A*, the side of length 8 as *b*, and the side of length 12 as *c*.

We must therefore substitute the values of *A*, *b* and *c* into the cosine formula

$a^2 = b^2 + c^2 - 2bc \cos A$

This gives $a^2 = 64 + 144 - 192 \cos 34°$, and hence $a = \sqrt{208 - 159.2}$ or 6.99 units to 2 d.p.

Triangle S.

Here we have two sides given, plus an angle *not* included. Label the angle opposite *a* as *A*, the 75° angle as *B*, the side of length 10 as *b*, the side of length 9 as *c*, and the angle opposite *c* as *C*. To find *a* we need to apply the sine rule twice.

First we find angle *C* using $\sin C = \dfrac{c \sin B}{b}$, hence $\sin C = \dfrac{9 \sin 75°}{10}$

The value of sin *C* is 0.8693 to 4 dp, so in this acute-angled case angle *C* is therefore 60.4°.

To find side *a*, we must find angle *A*. The angle can be worked out as 180 - (75 + 60.4) degrees, or 44.6°.

Then we use the sine rule again: $a = \dfrac{b \sin A}{\sin B}$ or $a = \dfrac{10 \sin 44.6°}{\sin 75°}$, giving a = 7.27 units to 2 d.p.

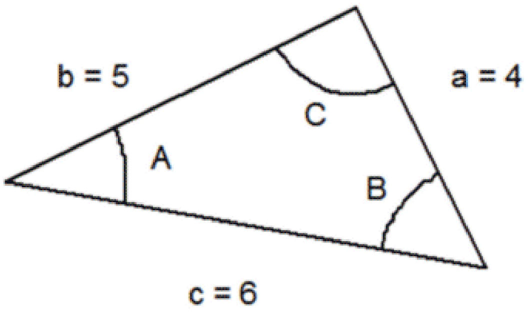
Triangle T.

Again we have two sides given, plus an angle *not* included. We use the sine rule again , this time to find angle *A*. Label the side of length 8 as *a*, the angle of 21° as *B*, and the side of length 3 as *b*.

Applying the sine formula in the form $\sin A = \dfrac{a \sin B}{b}$ we get $\sin A = \dfrac{8 \sin 21°}{3}$,

or sin *A* = 0.9556 to 4 d.p. This gives angle *A* = 72.9° and, by subtraction, angle *C* = 86.1°.

Example (2): Solve triangle **Q** from example 1 by finding all three missing angles, as well as its area.



After labelling as above, the first step would be to find angle *C*, opposite the longest side. This uses the cosine formula.

We use the form

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Substituting for *a*, *b* and *c* gives

$\cos C = \frac{16 + 25 - 36}{40}$ and hence *C* = 82.8° to 1 d.p. (keep more accuracy, 82.82°, for future working)

We now have enough information to work out the area of the triangle, as we have found the included angle *C*.

The area of the triangle is thus ½*ab* sin *C*, or 10 sin 82.8° = 9.85 sq.units.

To find the other two angles, we use the sine rule to find one of them and then subtract the sum of the other two angles from 180° to find the third.

The reason for using the longest side first is to prevent ambiguous results when using the sine rule. No triangle can have more than one obtuse angle, and the longest side is always opposite the largest angle. The cosine rule would take care of the obtuse angle if there was one, leaving no possibility of confusion when using the sine rule to work out the other two. In fact, angle *C* is acute in this case, so we have an acute-angled triangle.

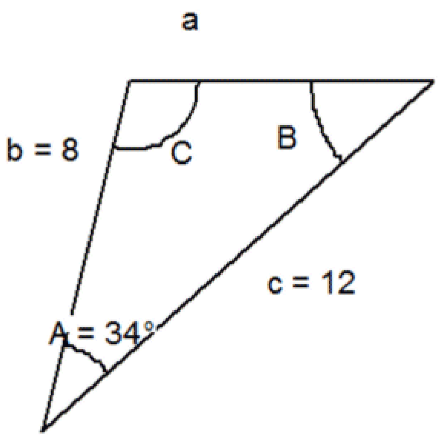
We can choose either remaining side to work out the other angles - here we'll find *B* first using the sine rule.

$\sin B = \frac{b \sin C}{c}$, giving $\sin B = \frac{5 \sin 82.82°}{6}$ and sin *B* = 0.8268.

This gives *B* = 55.8° to 1 d.p. (only the acute angle is valid here)

To find *C*, we subtract the sum of *A* and *B* from 180°, hence *C* = 41.4° to 1 d.p.

Example (3): Solve triangle **R** from example 1 by finding the two missing angles and the missing side. Find out its area as well.



We can work out the area at once as ½*bc* sin *A*. This gives 48 sin 34° or 26.84 sq.units.

First, we find the missing side *a* by substituting the values of *A*, *b* and *c* into the cosine formula
$a^2 = b^2 + c^2 - 2bc\cos A$

This gives $a^2 = 64 + 144 - 192\cos 34°$ or 48.8, and hence $a = \sqrt{208 - 159.2}$ or 6.99 units to 2 d.p. (Keep greater accuracy for future calculation - 6.987).

After finding *a* , the next step is to find one of the two missing angles. Both methods are shown here for illustrative purposes - choose the one you're happier with.

Using Cosine Rule.
Choose angle *C* as the next angle**,** since it is opposite the longer side, here *c*. (This will take care of a potential obtuse angle solution.)

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, or $\cos C = \dfrac{48.8 + 64 - 144}{2 \times 6.987 \times 8}$, or -0.2791.

This gives *C* = 106.2° to 1 d.p. (note that obtuse angles have a negative cosine).

Angle *B* can be found simply by subtracting the sum of *A* and *C* from 180°. It is thus 180 - (34 + 106.2)° or 39.8°.

Using Sine Rule.
We have one known angle, *A*, of 34°, so we know that one of the remaining ones must be acute since all triangles have at least two acute angles. We therefore use the sine rule to find the angle opposite the shorter of the remaining sides, namely side *b*.

Applying the sine formula in the form $\sin B = \dfrac{b\sin A}{a}$ we get $\sin B = \dfrac{8\sin 34°}{6.987}$,
or sin *B* = 0.6403 to 4 d.p. Angle *B* is hence 39.8°

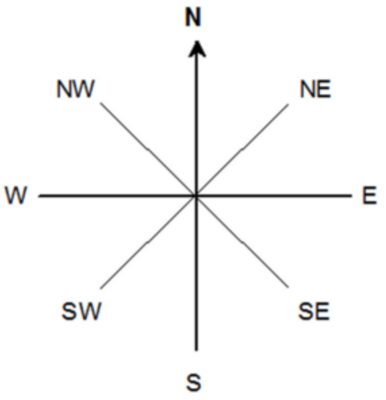
Angle *C* can be found by subtraction, being equal to (180 - (34 +39.8))°, or 106.2°.

Real-life applications of Sine and Cosine Rules.

The sine and cosine rules can be used to solve real-life trigonometry problems.

Many everyday problems in trigonometry involve such terms as **bearings**, **angles of elevation**, and **angles of depression**.

Bearings (revision).



A **bearing** of a point B from point A is its compass direction generally quoted to the nearest degree, and stated as a number from 000° (North) to 359°.

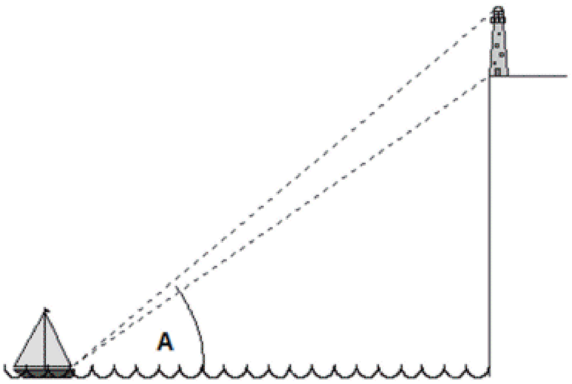
Bearings are measured **clockwise** from the **northline**.

Example(4): Express the eight points of the compass shown in the diagram as bearings from north.

N – 000°;	NE – 045°;	**E – 090°;**	SE – 135°
S – 180°;	SW – 225°;	**W – 270°;**	NW – 315°

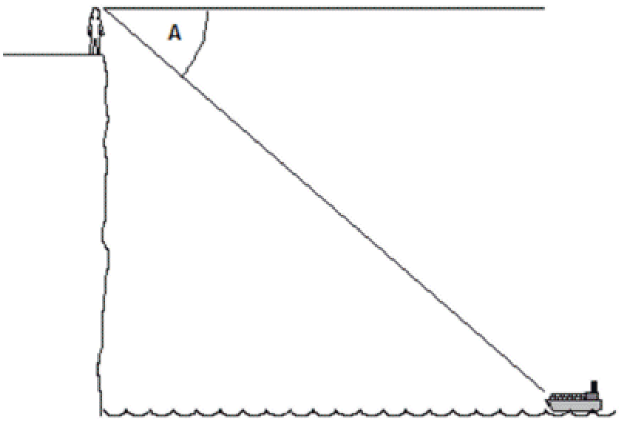
Real-life problems can be solved by applying trigonometric rules, and often in differing ways. See the following examples.

Angle of elevation and depression.



The angle of elevation of an object is its angular height (in degrees) above a reference line.

In the diagram on the right, the angle **A** is the angle of elevation of the base of the lighthouse from the yacht .



The angle of depression of an object is its angular 'depth' (in degrees) below a reference line.

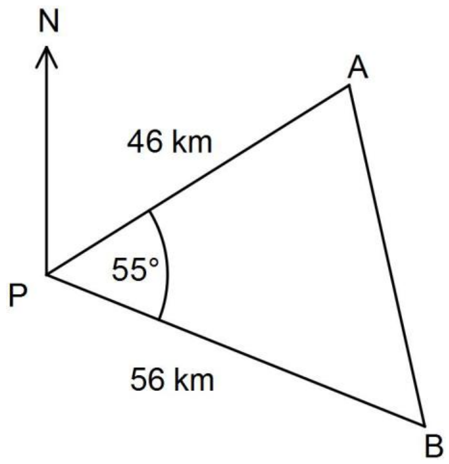
In the diagram on the right, the boat makes an angle of depression **A** with the observer's horizon at the cliff top.

Example (5): Two ships leave port at 10:00 and each one continue on a straight-line course. Ship **A** travels on a bearing of 060° at a speed of 23 km/h, and ship **B** travels on a bearing of 115° at a speed of 28 km/h.
How far away are the ships from each other at 12:00, assuming no changes of course and speed ?

Since two hours elapse between 10:00 and 12:00, ship **A** will have travelled 46 km and ship **B** will have travelled 56 km from port.

The situation at 12:00 is represented by a triangle where ship **A** is 46 km from port **P** at a bearing of 060° and ship **B** 56 km from port at a bearing of 115°. We have also included the northline at **N.**

The angle between the ships' bearings is 55° because angle NPA = 60° and angle NPB = 115°.



We have two sides and the included angle given in the triangle PAB, and so we can find the distance **AB** using the cosine rule:

$(AB)^2 = (AP)^2 + (PB)^2 - 2(AP)(PB) \cos 55°$

This gives $(AB)^2 = 2116 + 3136 - 5152 \cos 55°$ or 2296.9, and hence **AB** = 47.9 km to 3 s.f.

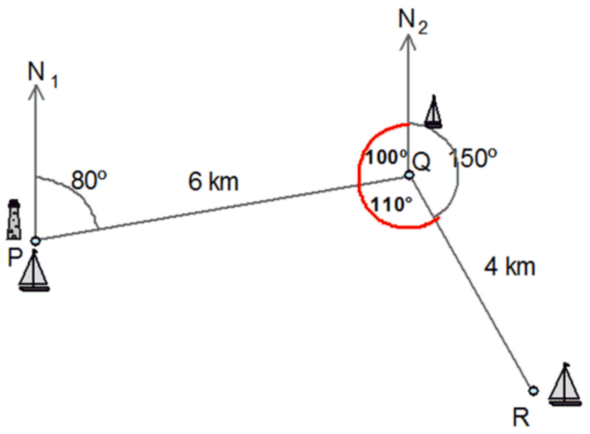
Example (6): A yachtsman passes a lighthouse at point **P** and sails for 6 km on a bearing of 080° until he reaches point **Q**. He then changes direction to sail for 4 km on a bearing of 150°.

Work out the yachtsman's distance and bearing from the lighthouse at point **R**, after the second stage of his sailing.

(Although this is an accurate diagram, only a sketch is required).

We can find $\angle$ PQR by realising that $\angle$ N_1PQ and $\angle$ PQN_2 are supplementary, i.e. their sum is 180°. Hence add $\angle$ PQN_2 = (180 – 80) = 100°.

Because angles at a point add to 360°, $\angle$ PQR = 360 – (100 + 150) = 110°



We can now find the distance PR as being the third side of triangle PQR – we have two sides (4 km and 6 km) and the included angle of 110°.

We label each side as opposite the angles, and use the cosine rule to find side q (PR) first :

$q^2 = r^2 + p^2 - 2pr \cos Q$

This gives $q^2 = 36 + 16 - 48 \cos 110°$ or 68.41, and hence q = 8.27 km (Keep higher accuracy for future calculation - 8.271).

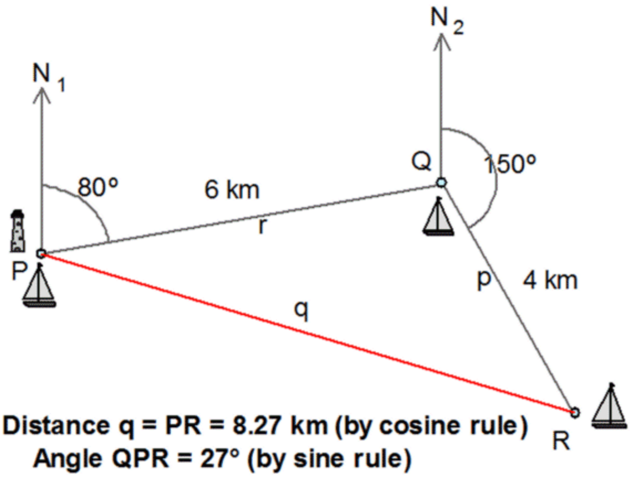
We can then use the sine rule to find angle QPR (P) and hence the yacht's final bearing from northline N_1.

Applying the sine rule, we have

$$\frac{\sin \angle QPR}{4} = \frac{\sin 110°}{8.271}$$, and thus

$$\sin \angle QPR = \frac{4 \sin 110°}{8.271}$$

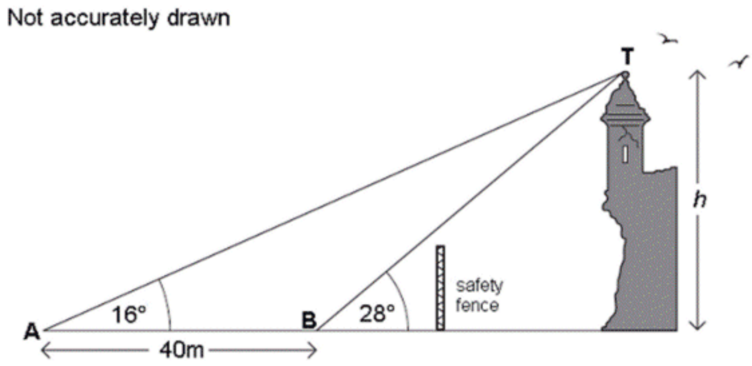
or sin $\angle$QPR = 0.4545 to 4 d.p. Angle QPR is hence 27°

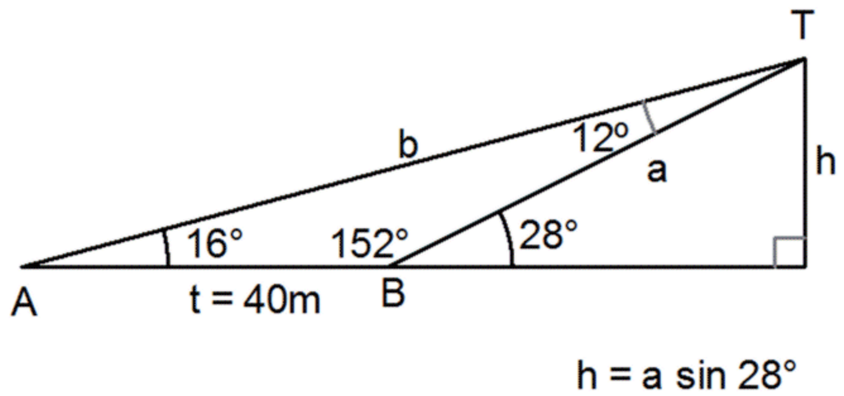


The yacht's final bearing from northline N_1 is (80 + 27) °, or 107° , and its distance from the lighthouse at P is 8.27 km.

Example (7): A ruined castle is fenced off for safety reasons, and a surveyor measures the angle of elevation of the tower at 16°.

He then walks another 40 metres in the direction of the tower to point **B**, where the angle of elevation is 28°. Find the height, *h*, of the castle tower.





We start by looking at the triangle ABT, because we can work out its angles ; ABT = 152° (180° in a straight line) and therefore angle ATB = 12°.

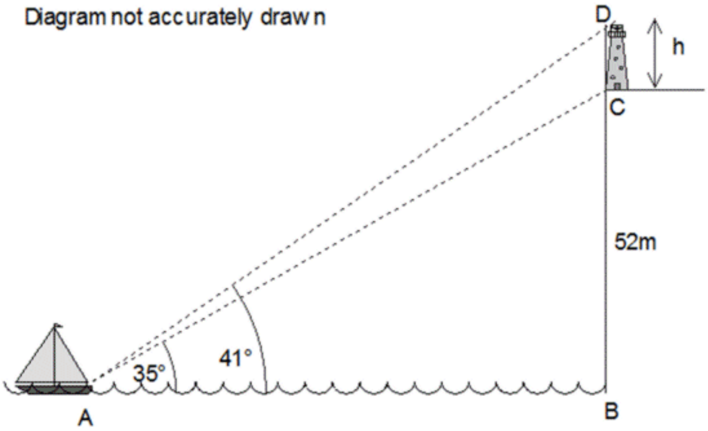
We then use the sine rule to find the side labelled *a*:

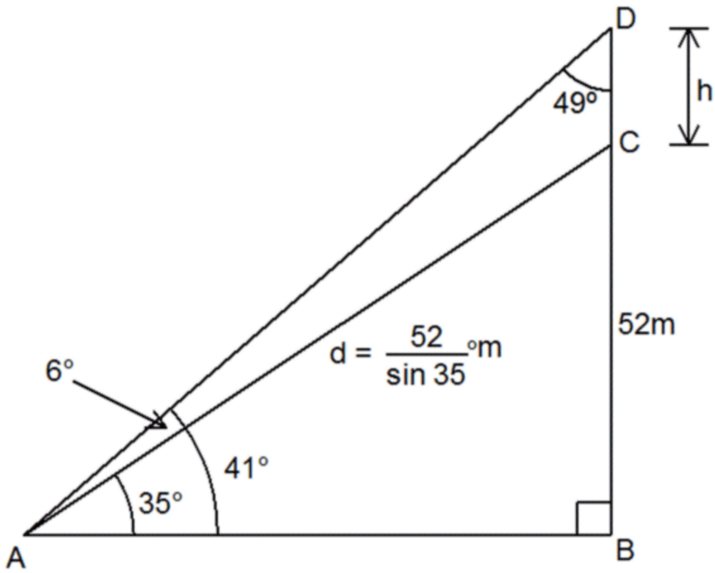
$$\frac{a}{\sin 16°} = \frac{40}{\sin 12°} \rightarrow a = \frac{40 \sin 16°}{\sin 12°} = 53.03 \text{ m.}$$

Then we can use right-angled methods to find h = 53.03 sin 28° m = 24.9 m

$\therefore$ **the castle tower is 24.9 m high.**

Example (8): A yachtsman at **A** measures the angle of elevation of the base of a lighthouse at point **C** and finds it to be 35°. He then measures the angle of elevation of the top of the lighthouse at point **D** and finds it to be 41°. Given that the lighthouse is on the top of a vertical sea-wall 52m high meeting the sea at **B**, calculate the height *h* of the lighthouse.





Since the triangle ABD is right-angled, angle ADB = 49°.

Since angle BAC = 35°, the small angle DAC = 6°.

We can then use triangle ABC to find side AC (also labelled *d*).

Here $d = \dfrac{52}{\sin 35°}$ m = 90.66m.

We can then use the sine rule to find *h*:

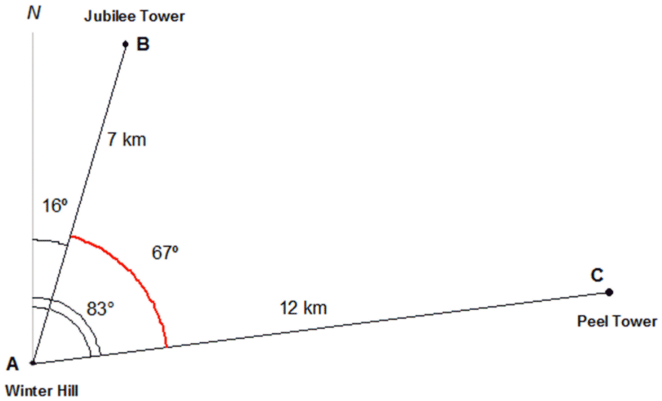
$$\frac{h}{\sin 6°} = \frac{d}{\sin 49°} \rightarrow h = \frac{90.66 \sin 6°}{\sin 49°} \rightarrow h = 12.6 \text{ m.}$$

Hence the height of the lighthouse, *h*, = 12.6 m.

Example (9): Peel Tower is 12 km from Winter Hill, on a bearing of 083°, whereas Jubilee Tower is 7 km from Winter Hill, on a bearing of 016°.

Find the distance and bearing of Jubilee Tower from Peel Tower.

First, we sketch the positions of the northline and the three landmarks in question.



We also label sides opposite corresponding angles with lower-case letters.
Note that angle A = (83 – 16) = 67°.

Next we find the length of the side a of the triangle, and to do so, we use the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos A$

This gives
$a^2 = 144 + 49 - 168 \cos 67°$ or 127.35,
and hence a = 11.285 km (Keep higher accuracy for future calculations).

To find the bearing of Jubilee Tower from Peel Tower, we draw a southern continuation of the northline at S and use alternate angles to find ∠ACS = 83°

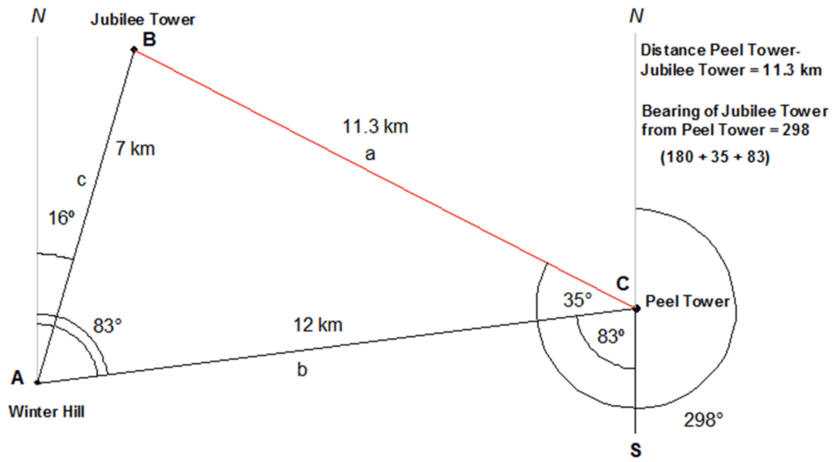
The bearing required is therefore (180 + 83)° + angle C (to be determined).

Angle C can be found by the sine rule:

$$\frac{\sin C}{7} = \frac{\sin 67°}{11.285}$$, and thus

$$\sin C = \frac{7 \sin 67°}{11.285}$$

or sin C = 0.571 to 3 d.p.
Angle C is hence 35°, so the bearing of Jubilee Tower from Peel Tower is
(180 + 83 + 35) = 298°.



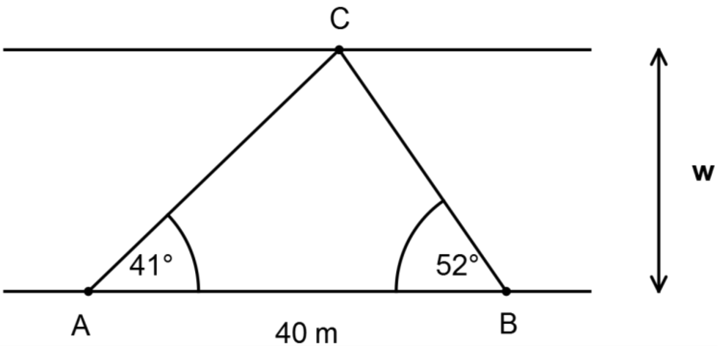
∴ Jubilee Tower is 11.3 km from Peel Tower on a bearing of 298°.

Example (10) :

Two lamp-posts A and B are 40 metres apart on the same side of a straight road.

The points A and B make angles of 41° and 52° respectively with a lamp-post C on the opposite side of the road.

Calculate the width of the road, *w*, to 3 significant figures. .

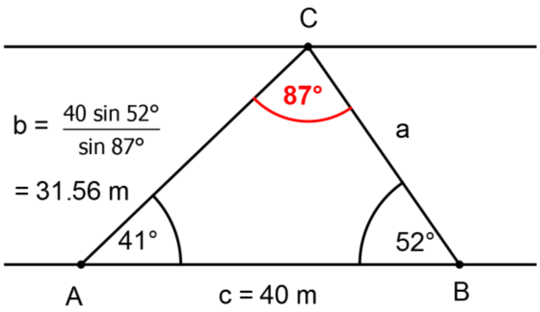


The first step is to find angle ACB, which works out as 87° (sum of angles of a triangle = 180°), and from there we use the sine rule to find the distance AC.

Using the convention of labelling opposite sides and angles, AC = *b*.

By the sine rule, $\dfrac{b}{\sin 52°} = \dfrac{40}{\sin 87°}$

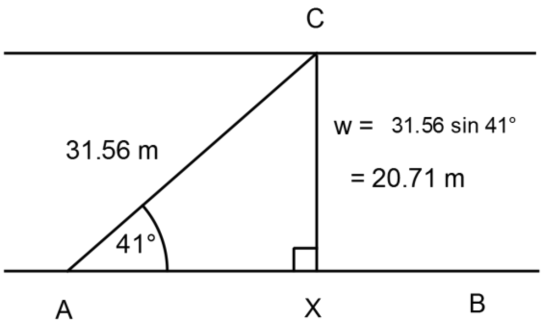
hence $b = \dfrac{40 \sin 52°}{\sin 87°}$ = 31.56 m.



Finally we draw a perpendicular across the road from C at the point X, and use right-angled triangle methods to find the width of the road, CX.

Side AC is the hypotenuse of the triangle AXC, and CX is the opposite, so CX = 31.56 sin 41° = 20.71m.

The width of the road is **20.7m** to 3 significant figures.



(We could equally well have chosen to find the length of BC, or *a*.

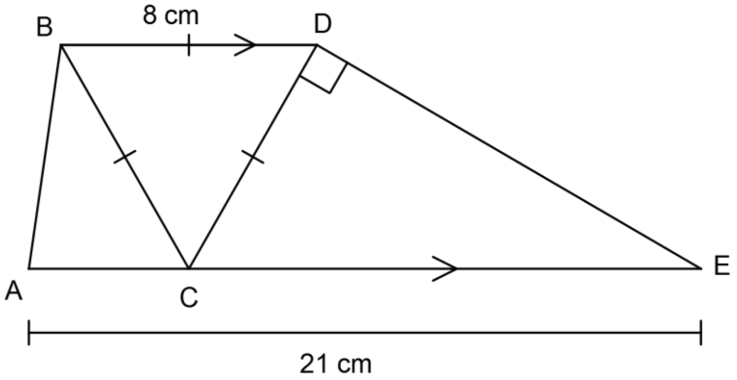
$\dfrac{a}{\sin 41°} = \dfrac{40}{\sin 87°}$, hence $a = \dfrac{40 \sin 41°}{\sin 87°}$ = 26.28 m.

From there the width of the road would be calculated as 26.28 sin 52°, or 20.7m)

Example (11): (Non-calculator)

ABDE is a trapezium whose base length *AE* is 21 cm, and additionally *BC* = *CD* = *BD* = 8 cm.
In addition, angle *CDE* = 90°.

Calculate the perimeter of the trapezium, giving your result in the form $a + b\sqrt{c}$ where a,b and c are integers.



From the given data, the triangle *BCD* is equilateral, so $\angle CBD = \angle CDB = 60°$, and because *AE* and *BD* are parallel, angles *ACB* and *DCE* equal 60° by alternate angles.

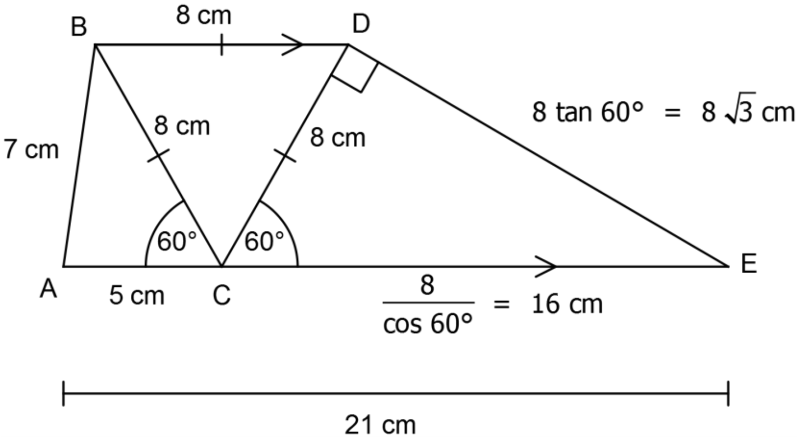
Triangle *CDE* is right-angled, so $DE = 8 \tan 60° = 8\sqrt{3}$ cm, and $CE = \dfrac{8}{\cos 60°} = 16$ cm.

(Note that cos 60° = ½ and tan 60° = √3).

By subtraction, *AC* = (21 – 16) cm = 5 cm, which leaves us with side *AB*.
The length of that side can be worked out using the cosine rule.

$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC)\cos 60°$

This gives $(AB)^2 = 25 + 64 - 80 \cos 60°$ or 89 – 40, or 49. Hence *AB* = 7cm.



∴ The perimeter of the trapezium is $(8 + 7 + 21 + 8\sqrt{3})$ cm, or $\mathbf{36 + 8\sqrt{3}}$ **cm**.