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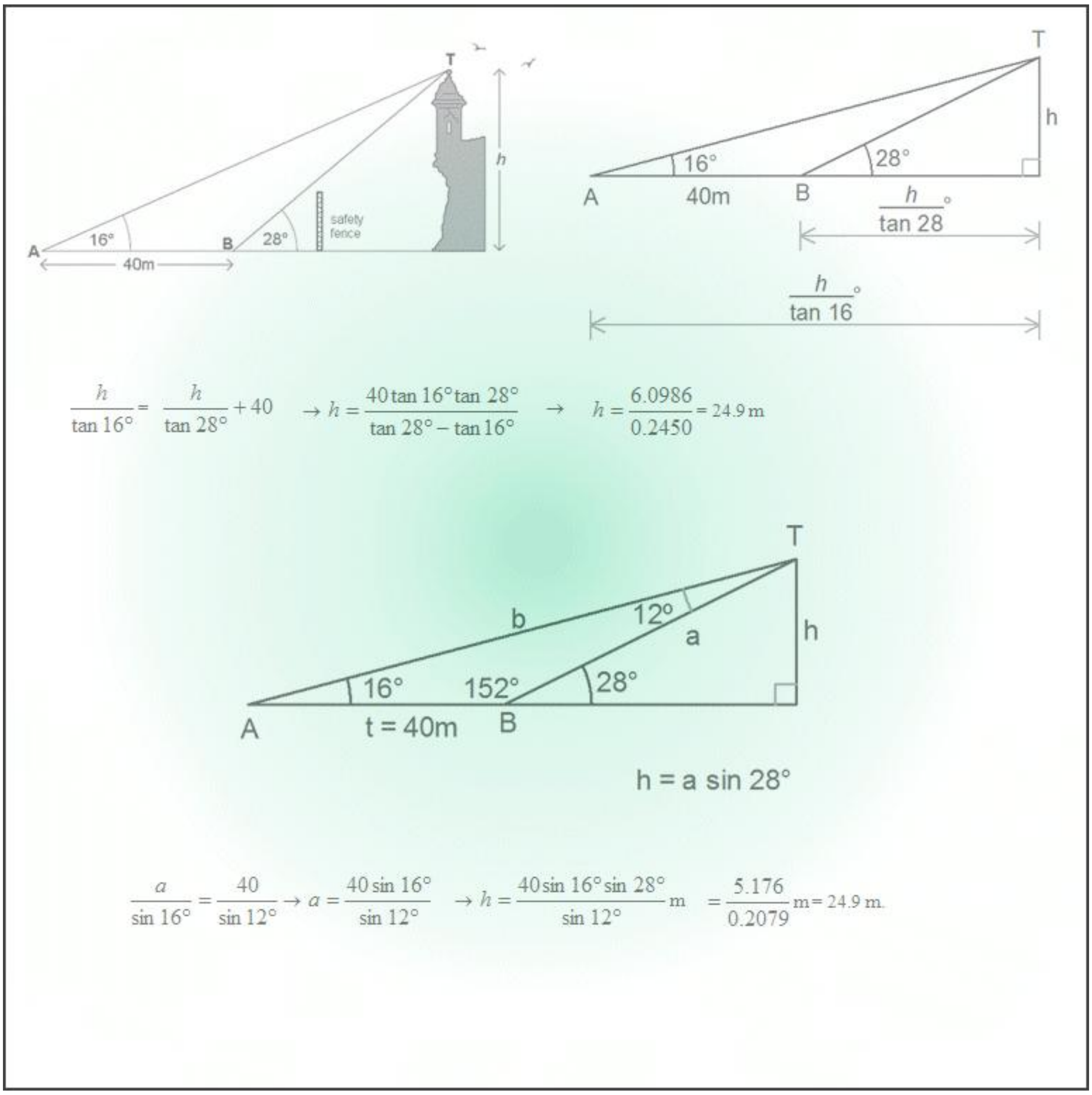
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

"REAL-LIFE" TRIG PROBLEMS



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"REAL-LIFE" TRIG PROBLEMS

Many everyday problems in trigonometry involve such terms as **bearings**, **angles of elevation**, and **angles of depression**.

Bearings.

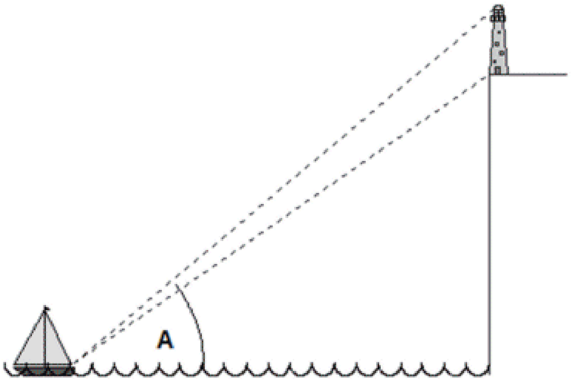
A **bearing** of a point B from point A is its compass direction generally quoted to the nearest degree, and stated as a number from 000 (North) to 359.

Bearings are measured **clockwise** from the **northline**.

Angle of elevation and depression.

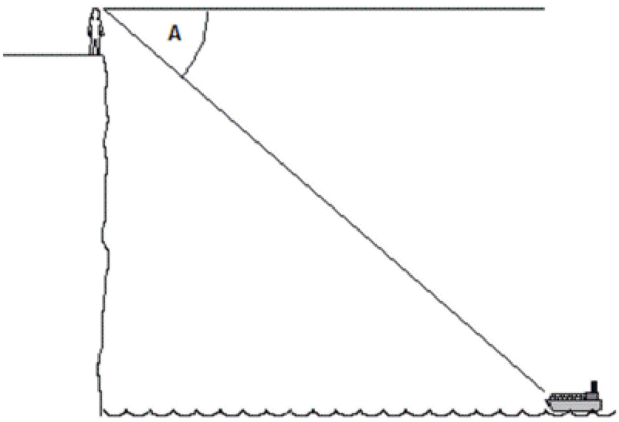
The angle of elevation of an object is its angular height (in degrees) above a reference line.

In the diagram on the right, the angle **A** is the angle of elevation of the base of the lighthouse from the yacht .



The angle of depression of an object is its angular 'depth' (in degrees) below a reference line.

In the diagram on the right, the boat makes an angle of depression **A** with the observer's horizon at the cliff top.



Comparison of "Bearing" and "Cartesian" notations of angles.

When given a trig problem, we might have to distinguish between different angle notations.

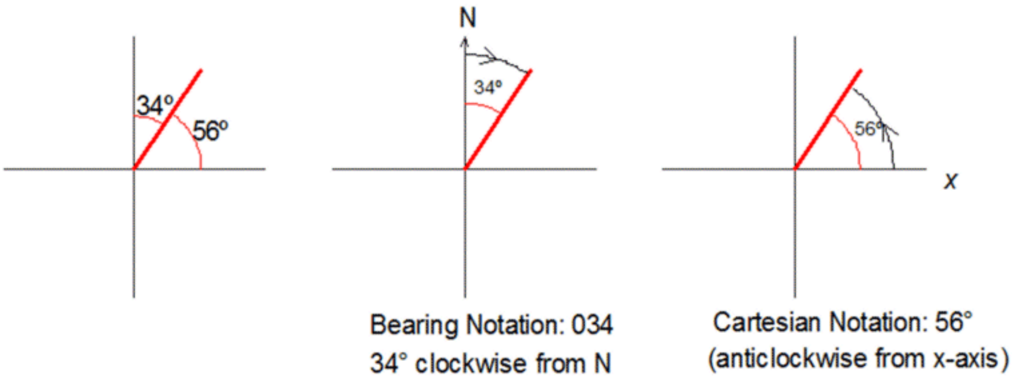
Bearings are measured **clockwise** from the **northline**.
Example(1): Express the eight points of the compass shown in the diagram as bearings from north.

N – 000;	NE – 045;	**E – 090**;	SE – 135
S – 180;	SW – 225;	**W – 270;**	NW -315

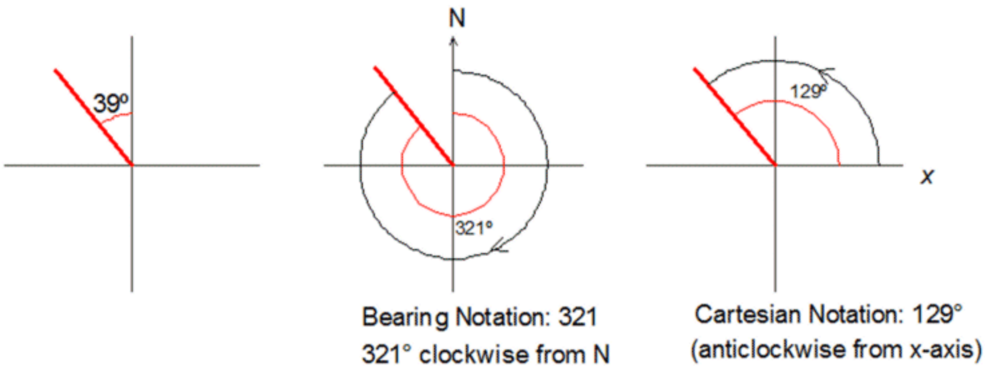
The angles made by the bold line and the axes are all identical, but different notations give seemingly different results.

In the leftmost diagram, we see that the bold line makes an angle of 34° with the *y*-axis and 56° with the *x*-axis.

When expressed as a bearing **clockwise** from **north**, the angle is given as 034.
When expressed in Cartesian form **anticlockwise** from the positive *x*-axis, the angle is 56°.



The last case was a first-quadrant example; here is an angle in the second quadrant.

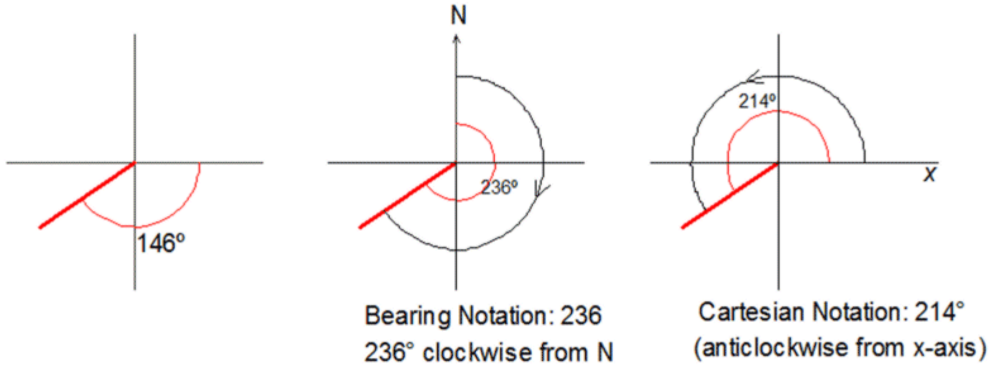


The left-hand diagram shows how the angle might appear in a question; here it is seen to be 39° anticlockwise from north.

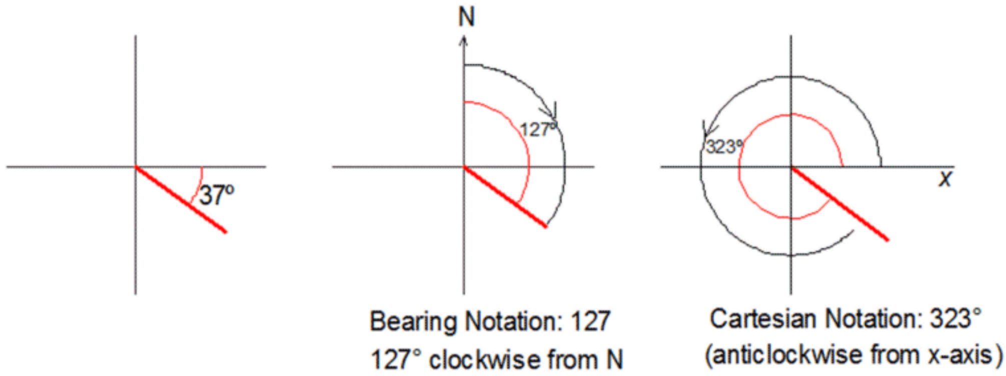
Bearings are measured clockwise from north, so we must subtract 39° from 360° to get the bearing, here 321.

The Cartesian measurement, taken anticlockwise from the *x*-axis,, can be found by adding 90° to 39°, giving 129°.

The next two examples are in the third and fourth quadrants.



In this third-quadrant example, we add 90° to obtain the bearing notation, namely 236; we subtract 146° from 360° to get the Cartesian value of 214°.



In the fourth-quadrant example above, we again add 90° to obtain the bearing notation, namely 127; we subtract 37° from 360° to get the Cartesian value of 323°.

Real-life problems can be solved by applying trigonometric rules, and often in differing ways. The following examples show a variety of such examples.

Example (1): Two ships leave port at 10:00 and each one continue on a straight-line course. Ship **A** travels on a bearing of 060 at a speed of 23 km/h, and ship **B** travels on a bearing of 115 at a speed of 28 km/h.
How far away are the ships from each other at 12:00, assuming no changes of course and speed ?

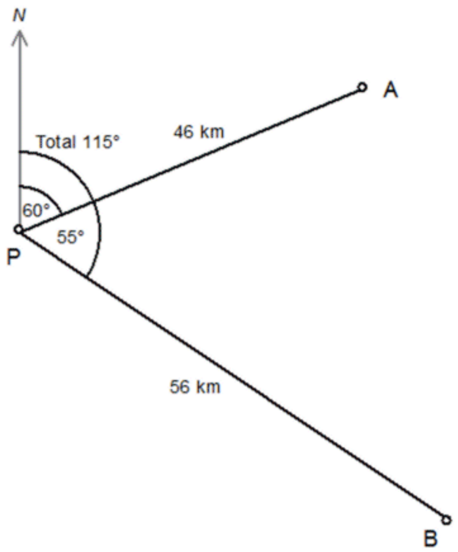
Method 1 – using right-angled triangles only.

Since two hours elapse between 10:00 and 12:00, ship **A** will have travelled 46 km and ship **B** will have travelled 56 km from port.

The situation at 12:00 is shown in the diagram on the right, where ship **A** is 46 km from port **P** at a bearing of 060 and ship **B** 56 km from port at a bearing of 115.

We therefore draw another diagram with the east line drawn in, and construct two right-angled triangles to find how far north and east of port the two ships are.

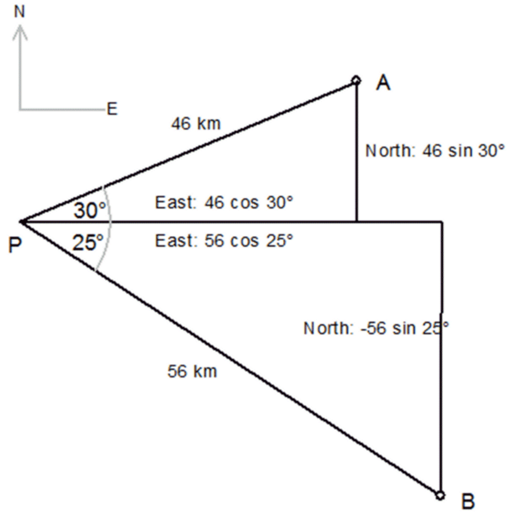
(Verify the angles of 30° and 25° in the diagram !)

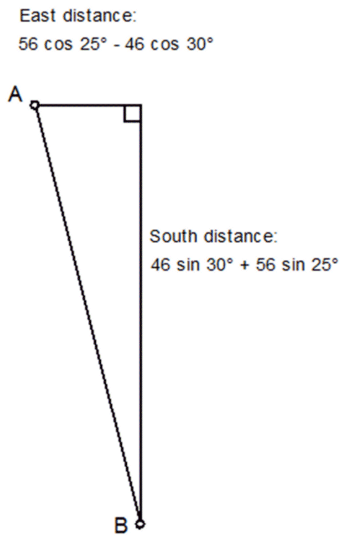


Ship **A** is 46 cos 30° km east of port and 46 sin 30° north of it.

Ship **B** is 56 cos 25° km east of port and -56 sin 25° north of it.
The negative value implies that the ship is 56 sin 25° **south** of port.

The separate diagram on the right is to find the distance between the ships.
The east distance of **B** from **A** is (56 cos 25° - 46 cos 30°) km, or 10.916km, and the corresponding south distance is (46 sin 30° + 56 sin 25°) km or 46.667 km.





The actual distance is therefore $\sqrt{10.916^2 + 46.667^2}$ km, or 47.9 km to 3 s.f.

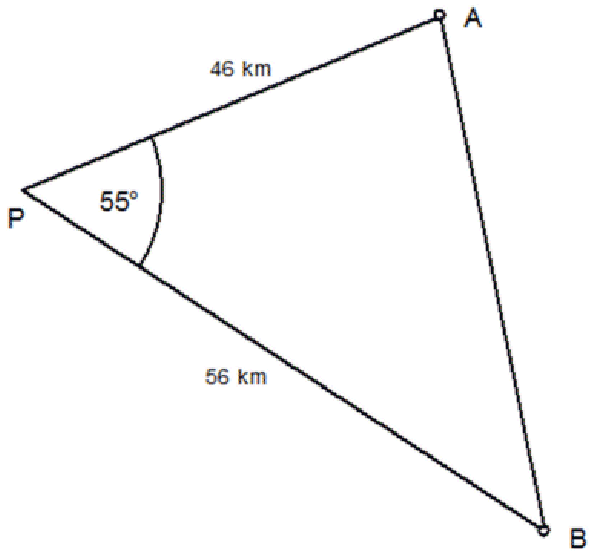
The ships are therefore 47.9 km apart.

Method 2 - Using the cosine rule.

Since two hours elapse between 10:00 and 12:00, ship **A** will have travelled 46 km and ship **B** will have travelled 56 km from port.

The situation at 12:00 is represented by a triangle where ship **A** is 46 km from port **P** at a bearing of 060 and ship **B** 56 km from port at a bearing of 115.

In other words, the angle between the ships' bearings is 55°.



We have two sides and the included angle given in the triangle PAB, and so we can find the distance **AB** using the cosine rule:

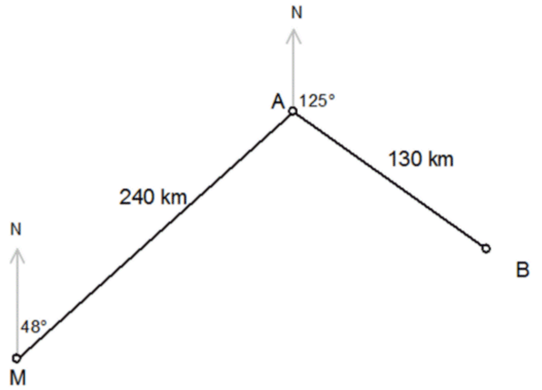
$(AB)^2 = (AP)^2 + (PB)^2 - 2(AP)(PB) \cos 55°$

This gives $(AB)^2 = 2116 + 3136 - 5152 \cos 55°$ or 2296.9, and hence **AB** = 47.9 km to 3 s.f.

Example (2a): A pilot in the sky above Manchester travels on a bearing of 048 for 240 km and then alters course and continues on a bearing of 125 for another 130 km. How far is he from Manchester after the second stage, to the nearest km ?

Method 1 – using right-angled triangles only.

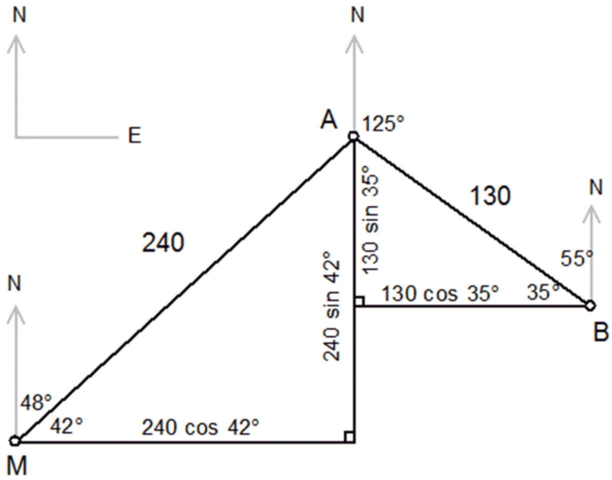
Let point **M** represent Manchester, **A** the pilot's position after the first stage of 240 km, and **B** his position after the second stage.



We next draw another diagram with both north and east lines drawn in, and construct two right-angled triangles to find out how far the pilot had flown north and east of Manchester in total.

On the first part of the flight, from **M** to **A,** the pilot had covered 240 cos 42° km in the eastbound direction and 240 sin 42° km in the northbound direction.

On the second part of the flight, from **A** to **B,** the pilot had covered 130 cos 35° km in the eastbound direction, and (-130 sin 35°) northbound. (The negative value implies a southbound path.)

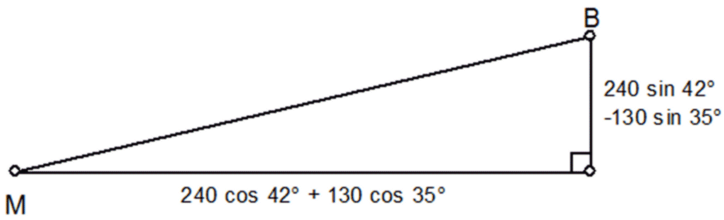


The total northbound distance travelled by the pilot is (240 sin 42° - 130 sin 35°) km, or 86.02 km.

The total eastbound distance travelled by the pilot is (240 cos 42° + 130 cos 35°) km, or 284.84 km.

The pilot's distance from Manchester can therefore be found by Pythagoras – it is given by

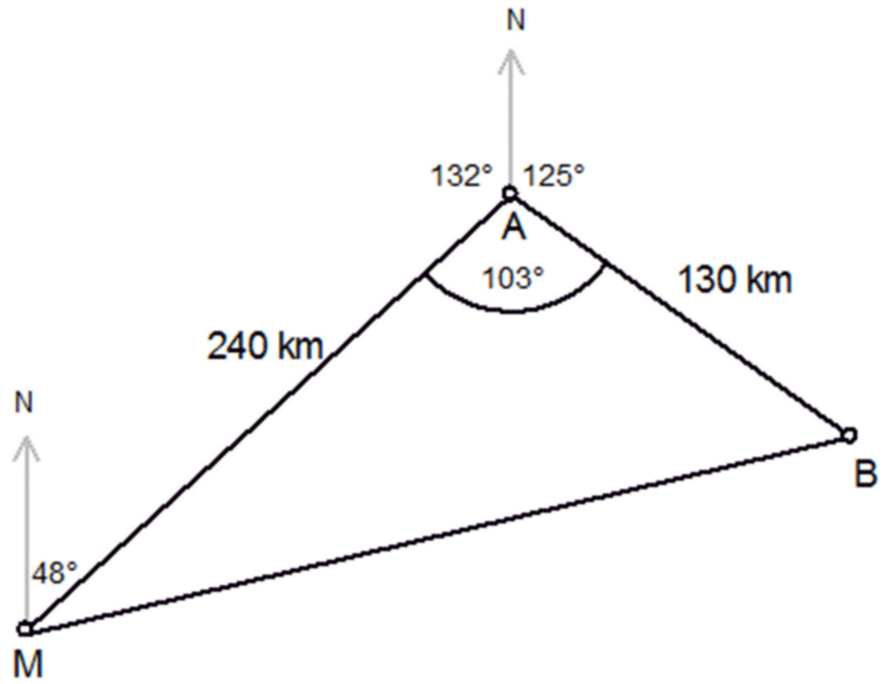
$\sqrt{86.02^2 + 284.84^2}$ km, or 298 km to the nearest km.



Method 2 - Using the cosine rule.

Let point **M** represent Manchester, **A** the pilot's position after the first stage of 240 km, and **B** his position after the second stage.

We can therefore construct triangle MAB using the original data.



The distance **MB** can be found using the cosine rule:

$(MB)^2 = (MA)^2 + (AB)^2 - 2(MA)(AB) \cos 103°$

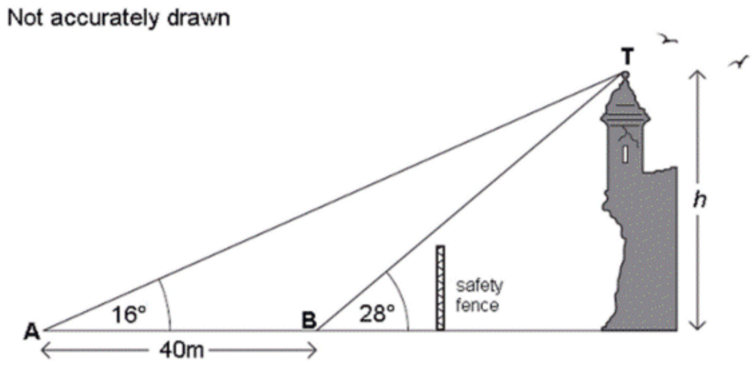
This gives $(MB)^2 = 57600 + 16900 - 62400 \cos 103°$ or 88537, and hence **MB** = 298 km to the nearest km.

In the last two examples, the cosine rule produced the result more quickly than using right-angled triangle methods alone.

The right-angled triangle methods introduced us to the idea of using vectors by splitting the distances into northbound (vertical) and eastbound (horizontal) components. This method is especially valuable for students who wish to study mechanics at higher levels.

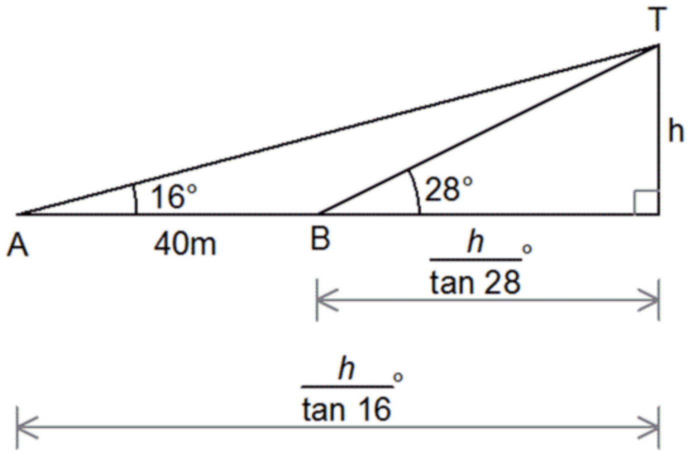
Example (3a): A ruined castle is fenced off for safety reasons, and a surveyor measures the angle of elevation of the tower at 16°.

He then walks another 40 metres in the direction of the tower to point **B**, where the angle of elevation is 28°. Find the height, *h*, of the castle tower.



Method 1 – using right-angled triangles only.

Using standard right-angled triangle methods, we know that the distance from the foot of the cliff to point **B** = $\frac{h}{\tan 28°}$ m and the distance from the cliff foot to point **A** = $\frac{h}{\tan 16°}$ m.



Since **AB** = 40m, we have an equation in *h* and two tangents:

$$\frac{h}{\tan 16°} = \frac{h}{\tan 28°} + 40 .$$

Multiplying both sides by tan 16° tan 28° we then have

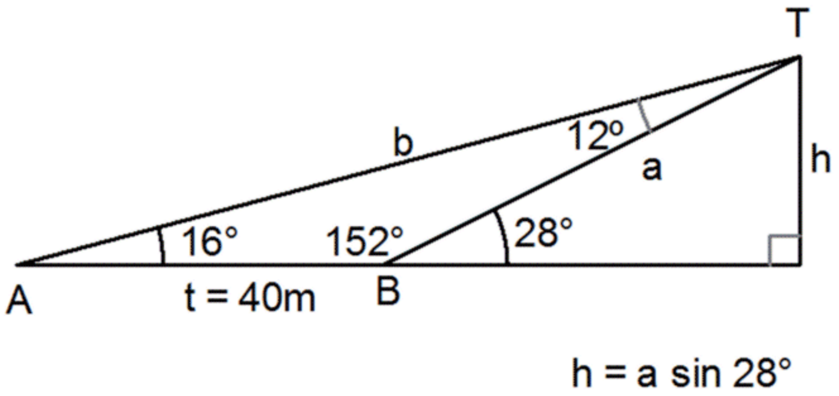
$h \tan 28° = h \tan 16° + 40 \tan 16° \tan 28° \rightarrow h \tan 28° - h \tan 16° = 40 \tan 16° \tan 28°$

$\rightarrow h (\tan 28° - \tan 16°) = 40 \tan 16° \tan 28° \rightarrow h = \dfrac{40 \tan 16° \tan 28°}{\tan 28° - \tan 16°}$,

and finally $h = \dfrac{6.0986}{0.2450}$ = 24.9 m.

$\therefore$ **the castle tower is 24.9 m high.**

Method 2 - Using the sine and cosine rules.



This time we concentrate on the triangle ABT and work out that angle ABT = 152° and therefore angle ATB = 12°.

We then use the sine rule to find the side labelled *a*:

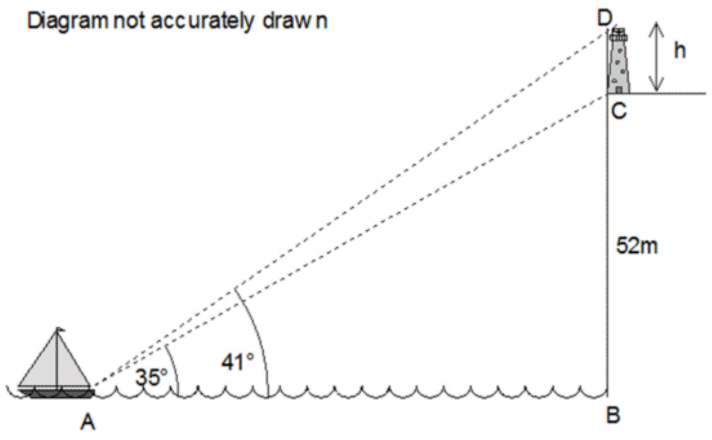
$$\frac{a}{\sin 16°} = \frac{40}{\sin 12°} \rightarrow a = \frac{40 \sin 16°}{\sin 12°}.$$

Then we can use right-angled methods to find $h = a \sin 28° \rightarrow h = \dfrac{40 \sin 16° \sin 28°}{\sin 12°}$ m

and finally $h = \dfrac{5.176}{0.2079}$ m = 24.9 m.

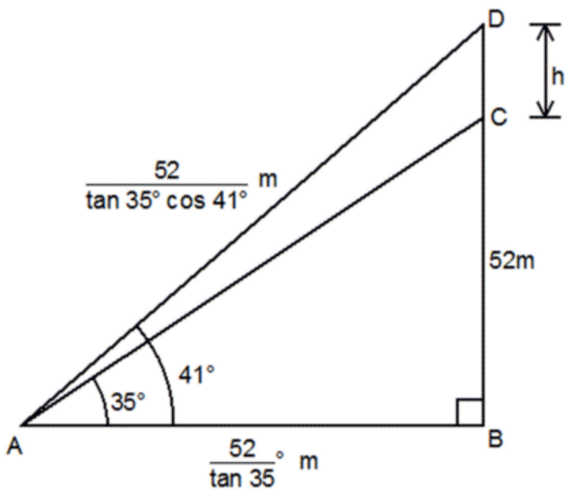
∴ the castle tower is 24.9 m high.

Example (4a): A yachtsman at **A** measures the angle of elevation of the base of a lighthouse at point **C** and finds it to be 35°. He then measures the angle of elevation of the top of the lighthouse at point **D** and finds it to be 41°. Given that the lighthouse is on the top of a vertical sea-wall 52m high meeting the sea at **B**, calculate the height *h* of the lighthouse.



Method 1 – using right-angled triangles only.

The first thing to notice is that triangles ABC and ABD have the base AB in common.



The length of the base, AB, is $\frac{52}{\tan 35°}$ m or 74.26m .

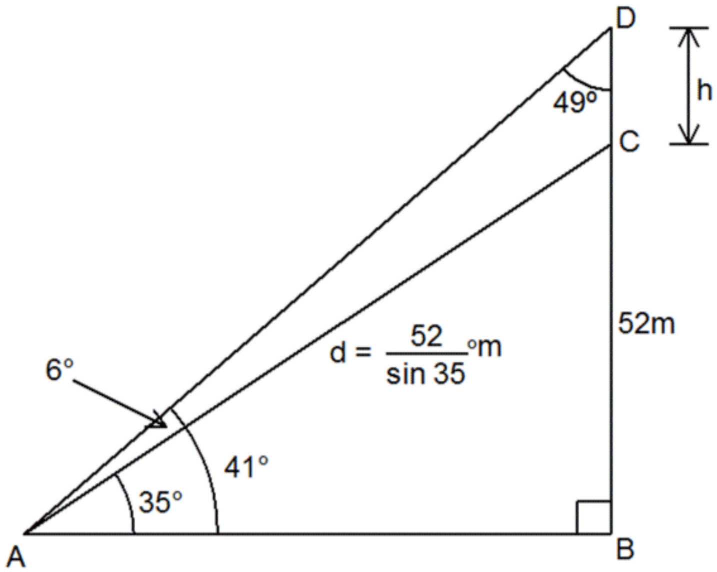
Because angle BAD = 41°, we can then find the hypotenuse of triangle BAD – it is the length AD divided by cos 41°, so AD = $\frac{52}{\tan 35° \cos 41°}$ m or 98.40 m.

We can then use Pythagoras to find the length BD:

$$BD = \sqrt{(AD)^2 - (AB)^2} = 64.56\text{m} .$$

The required height of the lighthouse, *h,* is therefore (64.56 – 52) metres, or **12.6 m.**

Method 2 - Using the sine and cosine rules.



We check out triangle ABD and work out that angle ADB = 49°.

Since angle BAC = 35°, the small angle DAC = 6°.

We can then use triangle ABC to find side AC (also labelled *d*).

Here $d = \dfrac{52}{\sin 35°}$ m.

We can then use the sine rule to find *h*:

$$\frac{h}{\sin 6°} = \frac{d}{\sin 49°} \rightarrow \; h = \frac{52 \sin 6°}{\sin 35° \sin 49°} \rightarrow h = \frac{5.435}{0.4329} \text{ m.}$$

Hence the height of the lighthouse, *h*, = 12.6 m.

The next two examples assume knowledge of the sine and cosine rules.

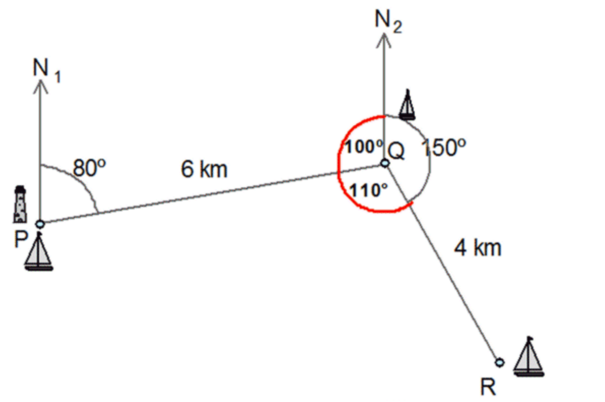
Example (5): A yachtsman passes a lighthouse at point **P** and sails for 6 km on a bearing of 080 until he reaches point **Q**. He then changes direction to sail for 4 km on a bearing of 150.

Work out the yachtsman's distance and bearing from the lighthouse at point **R**, after the second stage of his sailing.

(Although this is a fairly detailed diagram, only a sketch is required).

We can find $\angle$ PQR by realising that $\angle$ N_1PQ and $\angle$ PQN_2 are supplementary, i.e. their sum is 180°. Hence add $\angle$ PQN_2 = (180 – 80) = 100°.

Because angles at a point add to 360°, $\angle$ PQR = 360 – (100 + 150) = 110°



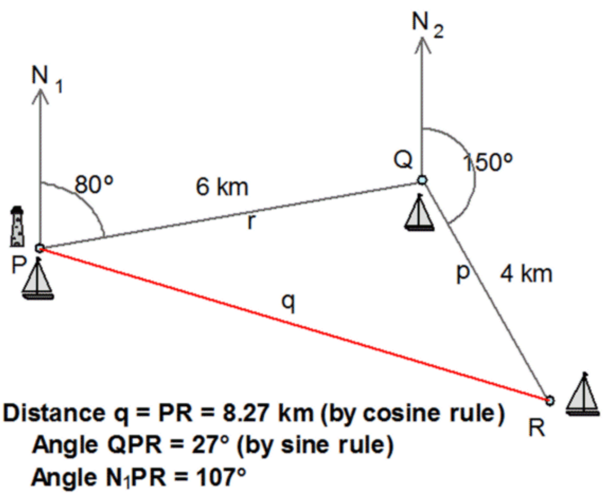
We can now find the distance PR as being the third side of triangle PQR – we have two sides (4 km and 6 km) and the included angle of 110°.

We label each side as opposite the angles, and use the cosine rule to find side q (PR) first :

$q^2 = r^2 + p^2 - 2pr \cos Q$

This gives $q^2 = 36 + 16 - 48 \cos 110°$ or 68.41, and hence q = 8.27 km (Keep higher accuracy for future calculation - 8.271).

We can then use the sine rule to find angle QPR (P) and hence the yacht's final bearing from northline N_1.



Applying the sine rule, we have $\dfrac{\sin \angle QPR}{4} = \dfrac{\sin 110°}{8.271}$, and thus

$$\sin \angle QPR = \frac{4 \sin 110°}{8.271}$$

or sin $\angle$QPR = 0.4545 to 4 d.p. Angle QPR is hence 27°

The yacht's final bearing from northline N_1 is (80 + 27) °, or 107, and its distance from the lighthouse at P is 8.27 km.

Example (6): Peel Tower is 12 km from Winter Hill, on a bearing of 083, whereas Jubilee Tower is 7 km from Winter Hill, on a bearing of 016.

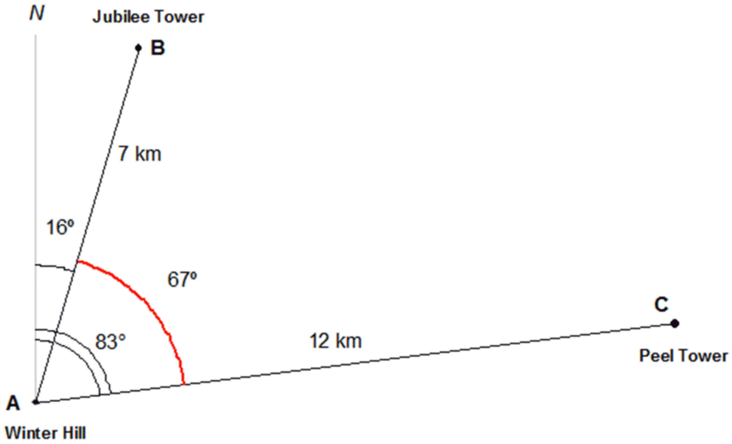
Find the distance and bearing of Jubilee Tower from Peel Tower.

First, we sketch the positions of the northline and the three landmarks in question.

We also label sides opposite corresponding angles with lower-case letters.
Note that angle A = (83 – 16) = 67°.

Firstly, we find the length of the side a of the triangle, and to do so, we use the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos A$



This gives $a^2 = 144 + 49 - 168 \cos 67°$ or 127.35, and hence a = 11.285 km (Keep higher accuracy for future calculations).

To find the bearing of Jubilee Tower from Peel Tower, we draw a southern continuation of the northline at S and use alternate angles to find ∠ACS = 83°

The bearing required is therefore 180 + 83 + angle C (to be determined).

Angle C can be found by the sine rule:

$$\frac{\sin C}{7} = \frac{\sin 67°}{11.285}$$, and thus $$\sin C = \frac{7\sin 67°}{11.285}$$ = 0.571 to 3 d.p.

Angle C is hence 35°, so the bearing of Jubilee Tower from Peel Tower is (180 + 83 + 35) = 298.

∴ Jubilee Tower is 11.3 km from Peel Tower on a bearing of 298.