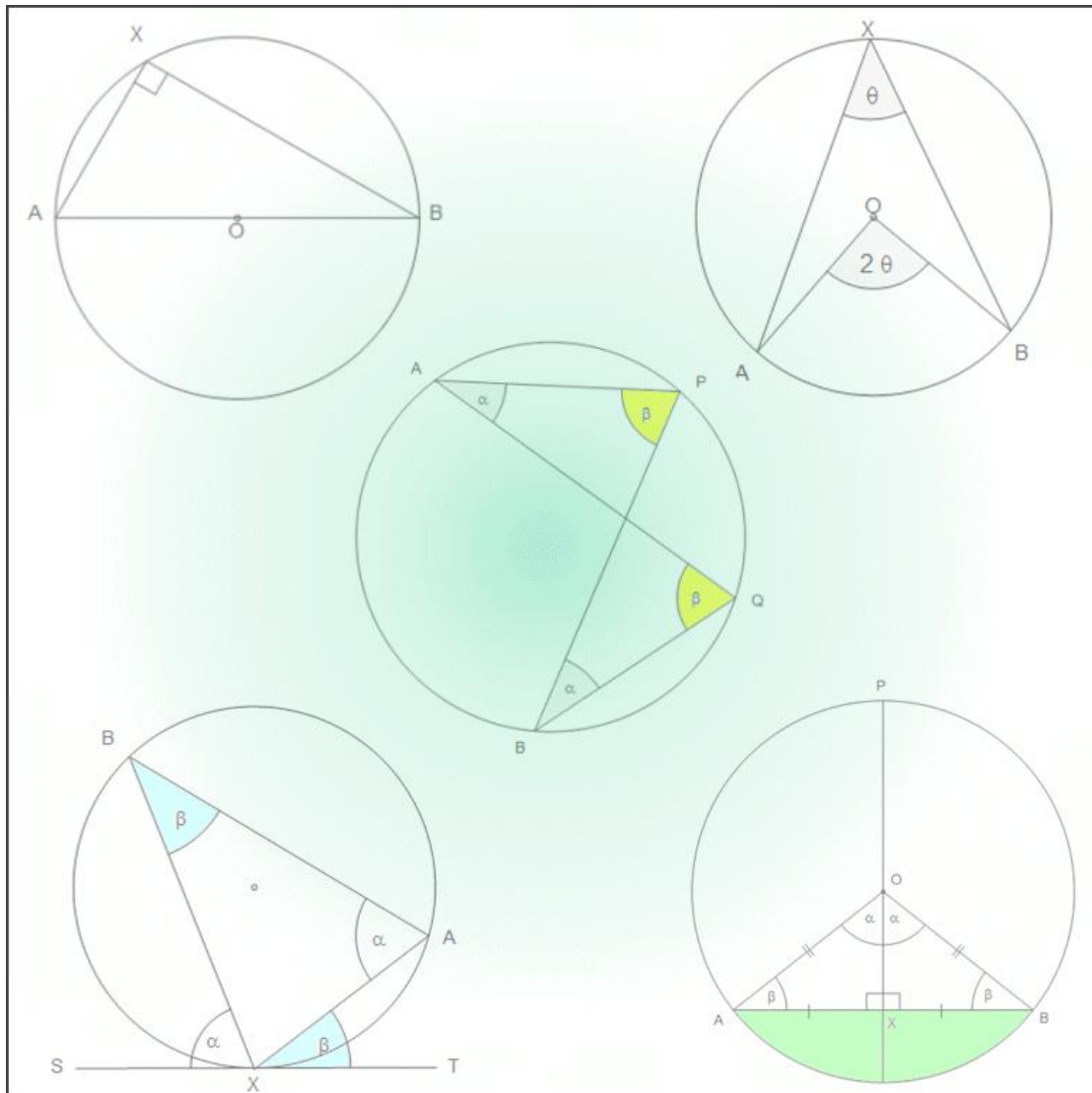


M.K. HOME TUITION

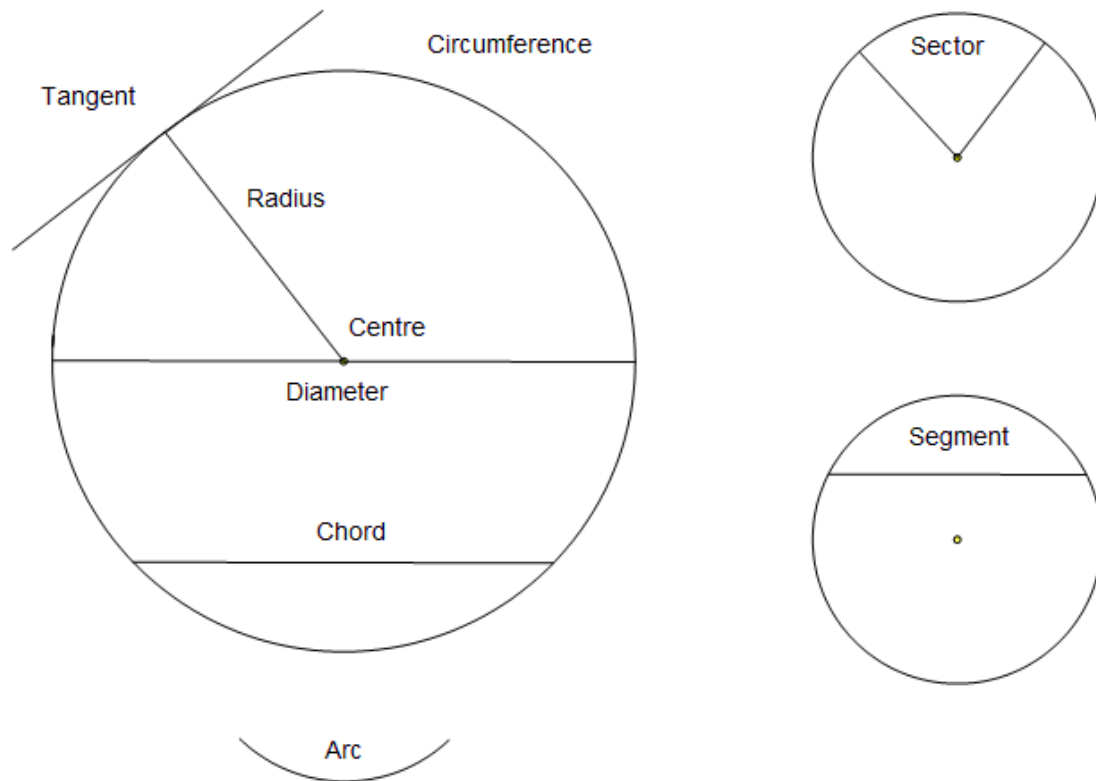
Mathematics Revision Guides
Level: GCSE Higher Tier

CIRCLE THEOREMS



CIRCLE THEOREMS

Recall the following definitions relating to circles:



A circle is the set of points at a fixed distance from the **centre**.
The perimeter of a circle is the **circumference**, and any section of it is an **arc**.
A line from the centre to the circumference is a **radius** (plural: **radii**).

A line dividing a circle into two parts is a **chord**.
If the chord passes through the centre, then it is a **diameter**.
A diameter divides a circle into two equal parts, and its length is twice that of a radius.

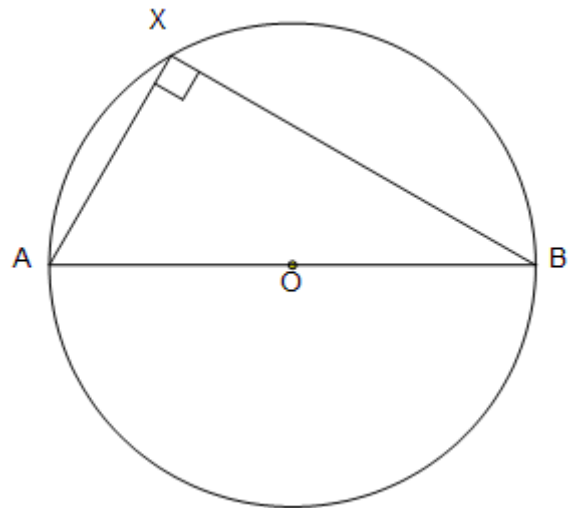
If the chord is not a diameter, the two resulting unequal regions of the circle are **segments**. The minor segment is the smaller; the major segment the larger.

A **sector** is a region of a circle bounded by two radii. The smaller one is the minor sector, the larger one the major sector.

A **tangent** is a line touching a circle at one point.

The angle at the circumference subtended by a diameter is a right angle, or more simply, the angle in a semicircle is a right angle.

The line AB is a diameter of the circle, passing through the centre, O . Angle AXB is therefore a right angle.



Proof.

The triangle AXB can be divided into two smaller triangles, AOX and BOX . Both triangles are isosceles because OA , OX and OB are all radii and therefore equal in length.

$$\angle AXB = \alpha + \beta.$$

$\angle OAX$ and $\angle OXA$ (marked α) are therefore equal, and $\angle AOX$ is thus $(180-2\alpha)^\circ$.

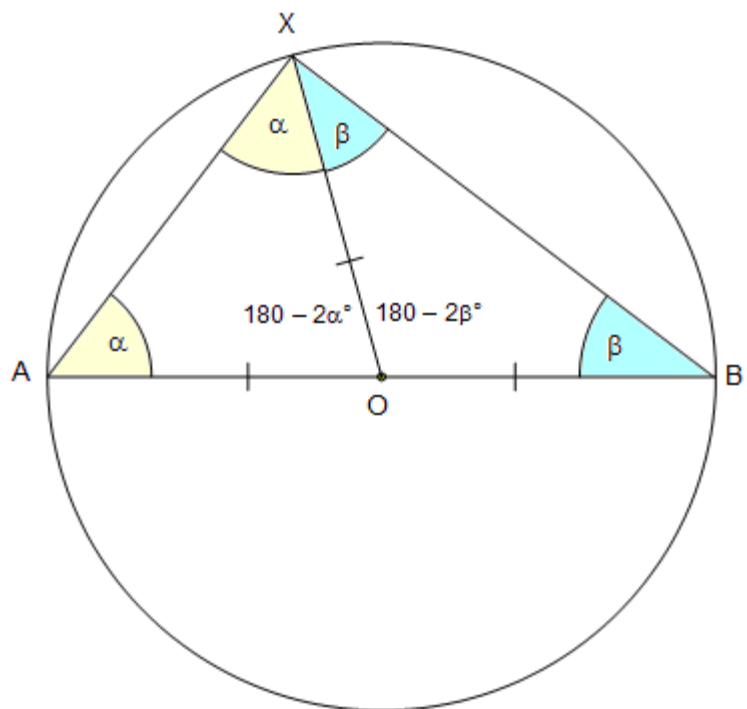
Similarly $\angle OBX$ and $\angle OXB$ (marked β) are equal,

$$\text{with } \angle BOX = (180-2\beta)^\circ.$$

Since $\angle AOX$ and $\angle BOX$ form a straight line, their angle sum is 180° .

$$\begin{aligned} \text{Hence } (180-2\alpha)^\circ + (180-2\beta)^\circ &= 180^\circ \\ \rightarrow (360-2\alpha-2\beta)^\circ &= 180^\circ \\ \rightarrow 2\alpha + 2\beta &= 180^\circ \\ \rightarrow \alpha + \beta &= 90^\circ. \end{aligned}$$

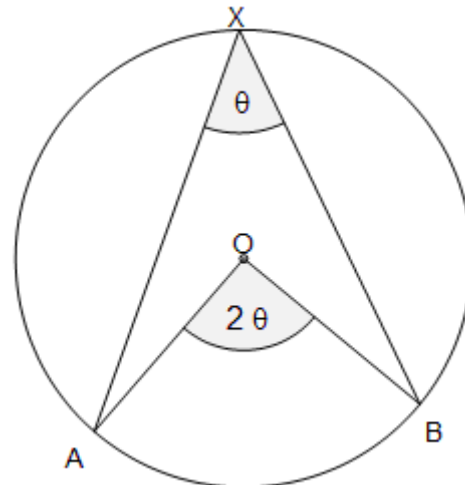
$\therefore \angle AXB$ is a right angle.



The angle subtended at the centre of a circle is double the angle subtended at the circumference.

Points A , X and B lie on the circumference of a circle centred on O .

The angle AOB at the centre is double the angle AXB at the circumference (labelled θ),



Proof.

The figure $OAXB$ can be divided into two smaller triangles, AOX and BOX . Both triangles are isosceles because OA , OX and OB are all radii and therefore equal in length.

$$\angle AXB = \alpha + \beta.$$

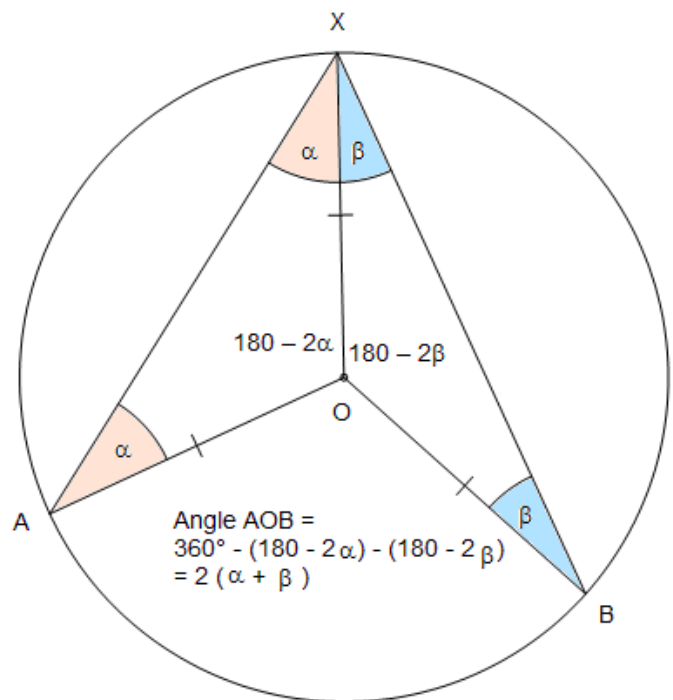
$\angle OAX$ and $\angle OXA$ (marked α) are therefore equal, and $\angle AOX$ is thus $180 - 2\alpha$ °.

Similarly $\angle OBX$ and $\angle OXB$ (marked β) are equal, with $\angle BOX = 180 - 2\beta$ °.

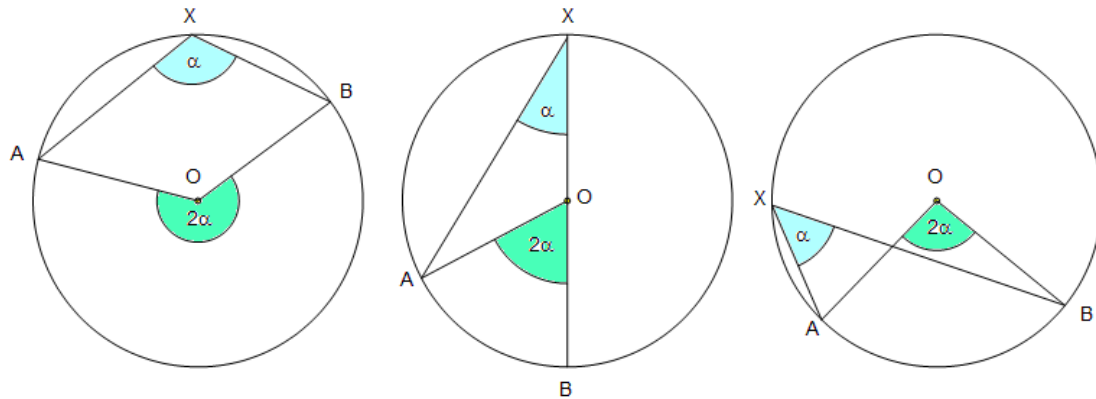
Since angles AOX , BOX and AOB add up to 360 ° around O , it follows that

$$\begin{aligned} \angle AOB &= 360^\circ - (180^\circ - 2\alpha) - (180^\circ - 2\beta) \\ \rightarrow \angle AOB &= 360^\circ - (360^\circ - 2\alpha - 2\beta) \\ \rightarrow \angle AOB &= 2(\alpha + \beta). \end{aligned}$$

\therefore The angle AOB at the centre is double the angle AXB at the circumference .



Although the ‘arrowhead’ configuration is the one most commonly shown in textbooks, there are other patterns where the rule can be applied – in each case below, angle AOB at the centre is double the angle AXB at the circumference .



In the left-hand diagram, $\angle AOB$ at the centre is reflex because $\angle AXB$ at the circumference is obtuse.

In the middle diagram, point B has moved such that XB is a diameter of the circle, and points X , O and B are collinear.

In the right-hand diagram, point X has moved such that the line XB now crosses line OA .

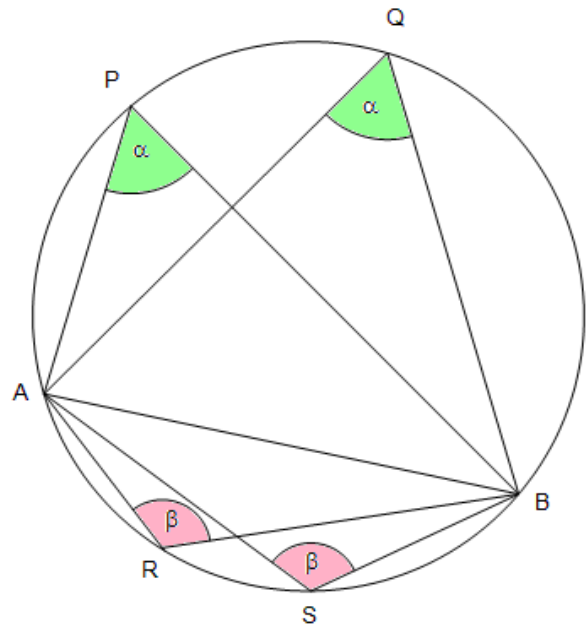
Angles subtended by the same chord are equal.

Let AB be a chord of a circle.

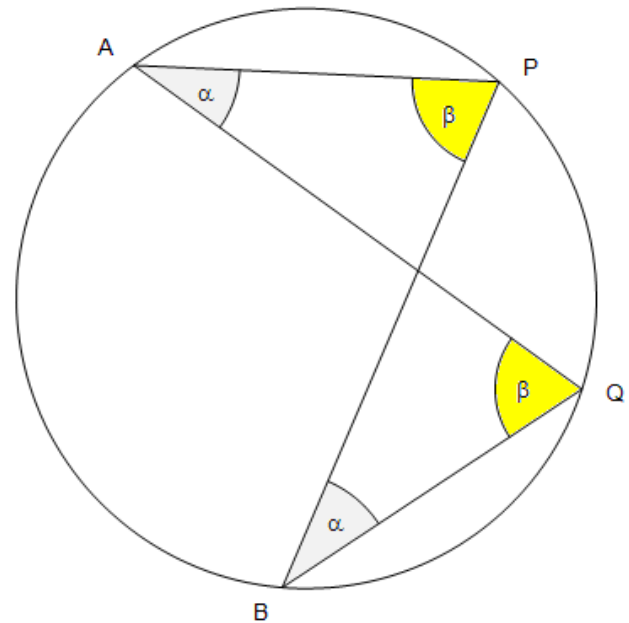
$\angle APB$ and $\angle AQB$ (marked α) are equal, as are $\angle ARB$ and $\angle ASB$ (marked β).

Also, the sum of the angles on opposite sides of the chord AB is equal to 180° .

(See the section on Cyclic Quadrilaterals for further details.)



This circle does not actually have the chords AB or PQ drawn on it, but the rule still holds: $\angle APB$ and $\angle AQB$ are equal, as are $\angle PAQ$ and $\angle PBQ$.



The bisector of a chord is a diameter.

Any line drawn across a circle is a **chord**, and the perpendicular bisector of the chord passes through the centre, and is therefore also a diameter of the circle.

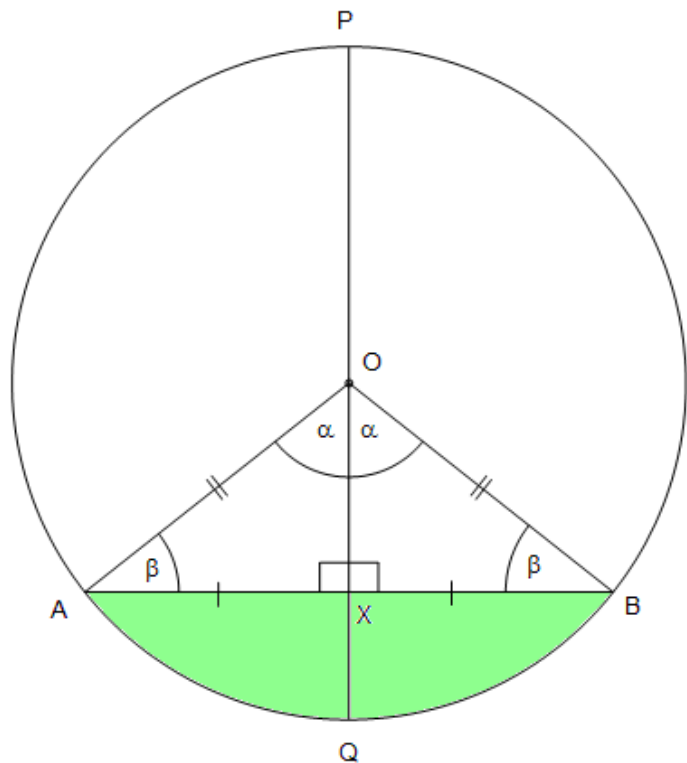
In the diagram, the perpendicular bisector of chord AB is the line PQ . Since PQ passes through the centre O , it is also a diameter.

Point X is the midpoint of the chord AB and also lies on the diameter PQ .

The triangle AOB is isosceles because OA and OB are radii, and hence equal.

Additionally triangles AOX and BOX are congruent.

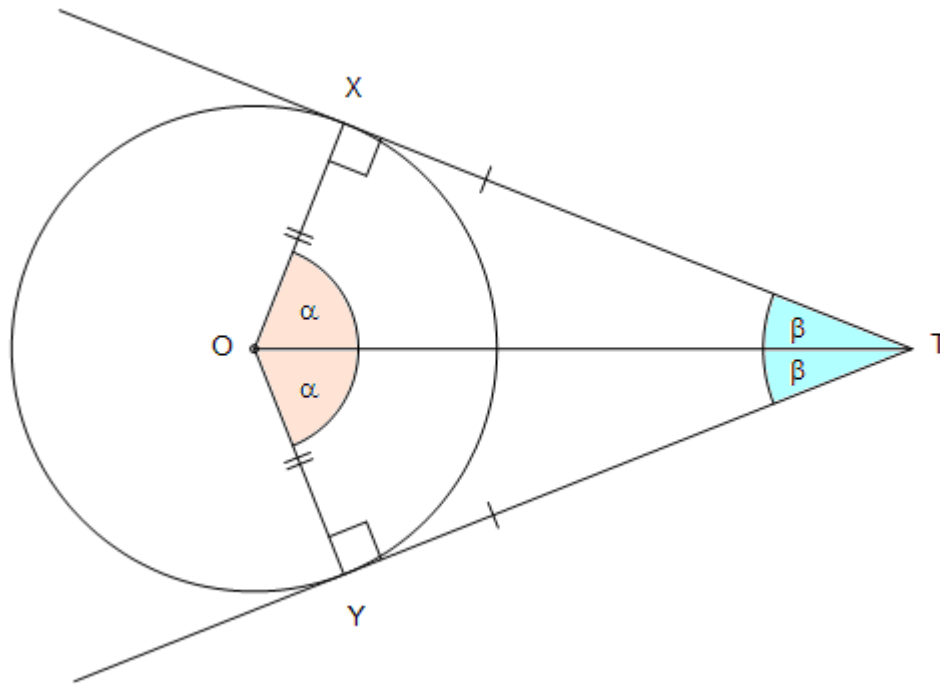
Notice also the symmetry of the triangle AOB – it is isosceles because AO and OB are both radii.



A tangent and a radius meet at right angles; also, two tangents drawn from a point to a circle describe two lines equal in length between the point and the circle.

The lines TX and TY both originate from the point T and form tangents with the circle at points X and Y . These tangents are perpendicular to the radii of the circle OX and OY , and are also equal in length.

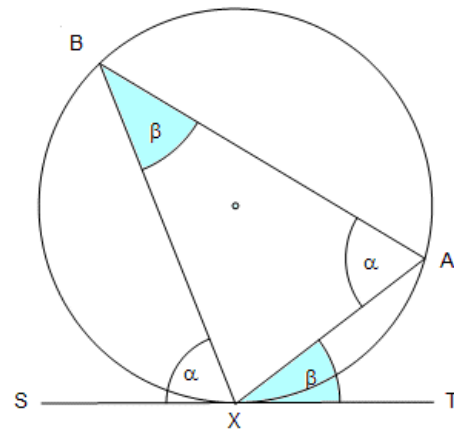
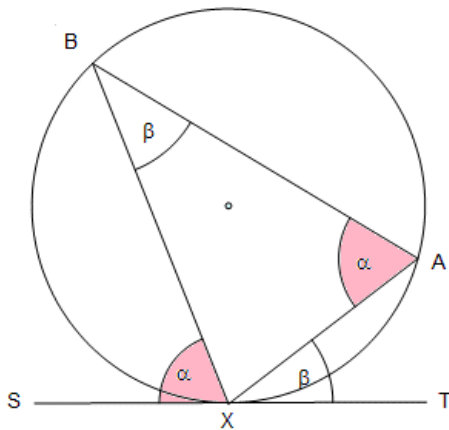
Because the triangles OXT and OYT also have side OT in common, both also have all three sides equal, and are therefore congruent. As a consequence, angles XOT , YOT (labelled α) and angles XTO , YTO (labelled β) are also equal.



Angles in the alternate (opposite) segment are equal.

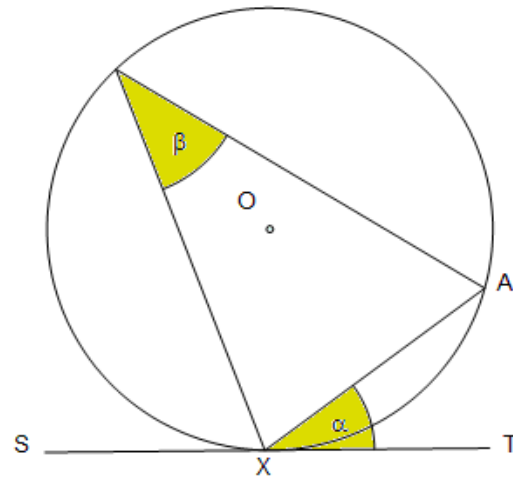
In the diagram below left, the chord BX and the tangent ST meet at X .
Let SXB be the angle between the chord and the tangent.
Let XAB be the angle in the alternate segment (bounded by chord BX).
angles SXB and XAB (shaded and labelled α) are equal.

There is another angle pair in the right-hand diagram.
This time, TXA is the angle between the chord AX and the tangent ST .
Let XBA be the angle in the alternate segment (bounded by chord AX).
Then angles TXA and XBA (shaded and labelled β) are equal.



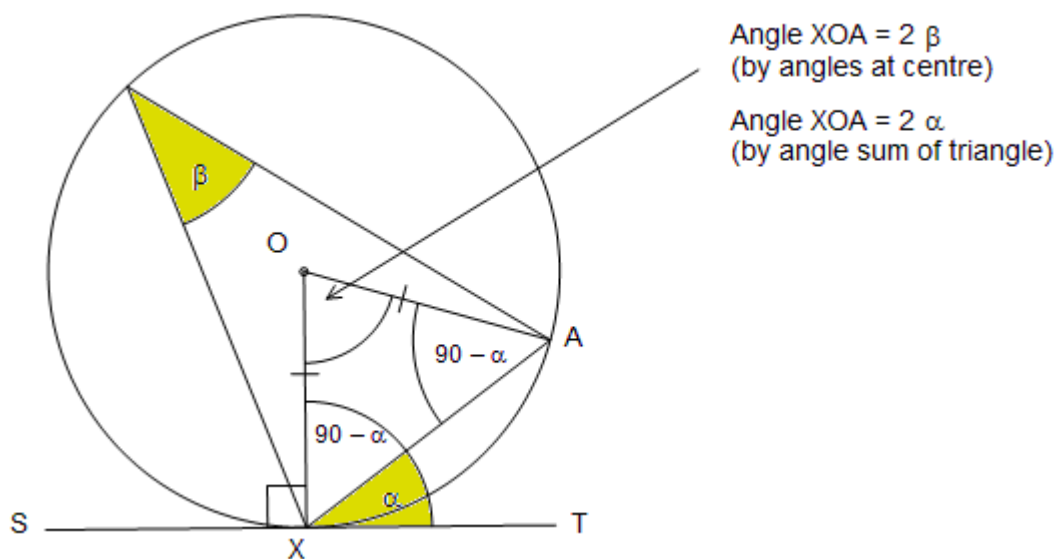
Proof. (Using angles at the centre)

The line ST is a tangent to the circle centred on O , and α is the angle between TX and the chord XA .



We begin by drawing radii at OX and OA .
 Because the tangent ST and the radius OX meet at right angles,

$$\angle OXS = \angle OXT = 90^\circ.$$



Hence $\angle OXA = (90 - \alpha)^\circ$.

Furthermore, $\angle OAX$ also $= (90 - \alpha)^\circ$, since $\triangle XOA$ is isosceles (OX and OA are radii).

The angles in a triangle add to 180° , and so $\angle XOA = (180 - (180 - 2\alpha))^\circ = 2\alpha$.

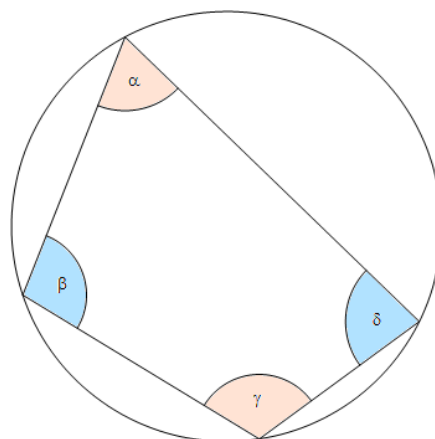
But, the angle at the centre ($\angle XOA$) is double the angle at the circumference (β),
 so $\angle XOA$ also $= 2\beta$.

\therefore Angles α and β are equal.

A **cyclic quadrilateral** is one that can be inscribed in a circle, in other words one whose vertices lie on a circle's circumference.

The opposite angles of a cyclic quadrilateral add up to 180° .

Thus, in the diagram shown right, angles α and γ have a sum of 180° , as do angles β and δ .



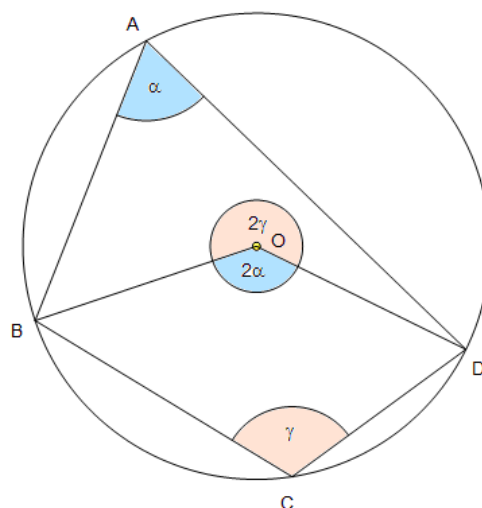
Proof (using angles at the centre)

Take the quadrilateral $ABCD$ inscribed in the circle centred on O .

By the rules of angles at the centre, if $\angle BAD = \alpha$, then the obtuse angle $BOD = 2\alpha$.

Similarly, if $\angle BCD = \gamma$, then the reflex angle $BOD = 2\gamma$.

Since $2\alpha + 2\gamma = 360^\circ$, $\alpha + \gamma = 180^\circ$.



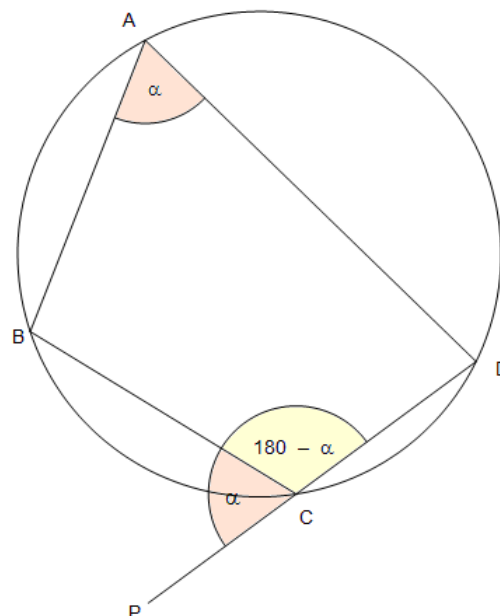
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Let $\angle BAD = \alpha$.

Since the opposite angles of a cyclic quadrilateral add to 180° , $\angle BCD = 180 - \alpha$.

Extending side DC to point P , $\angle BCP + \angle BCD = 180^\circ$ since DCP is a straight line.

Hence $\angle BCP = 180^\circ - (180 - \alpha) = \alpha$.



Examination questions usually feature more than just one of the theorems, and often include some algebra and Pythagoras as well.

Also, diagrams in exam questions are not usually drawn accurately. This is also true for the examples in this section.

Example (1): The points A , B , C and D lie on a circle with centre O .

i) Given $\angle DAC = 26^\circ$, find angles DBC and ACD .

ii) Explain why angle BCD is not a right angle.

i) Angle $DBC = 26^\circ$ because it is in the same segment as angle DAC .

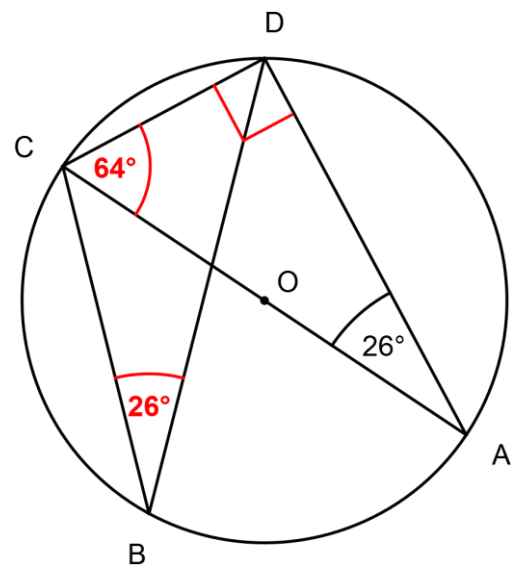
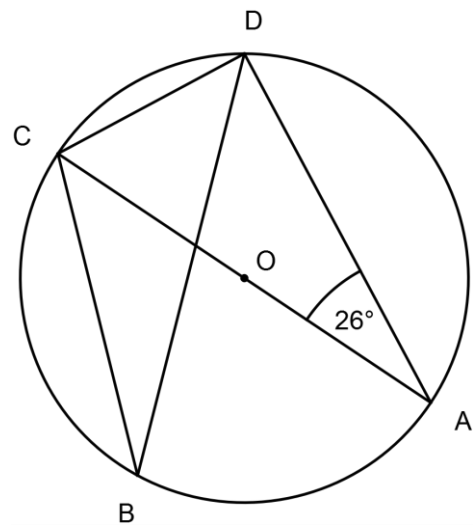
As COA is a diameter of the circle, angle $CDA = 90^\circ$ because the angle in a semicircle is a right angle.

\therefore Angle $ACD = 180^\circ - (90 + 26)^\circ = 64^\circ$.

More simply, the two acute angles in a right-angled triangle add to 90° , so angle $ACD = (90 - 26)^\circ = 64^\circ$.

ii) Angle BCD is not a right angle because BD is not a diameter of the circle.

This is the converse of the “angle in a semicircle” theorem.

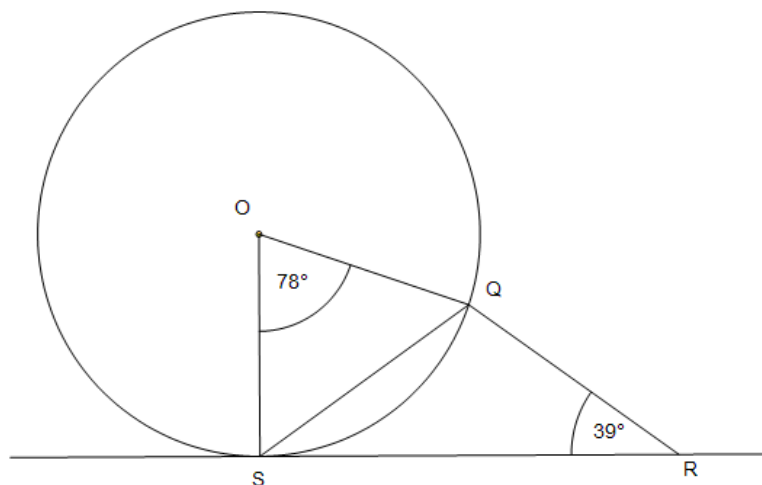


Example (2): Points Q and S lie on a circle centred on O .

SR is a tangent to the circle at S .

Angle $QRS = 39^\circ$ and angle $SOQ = 78^\circ$.

Prove that triangle SQR is isosceles.



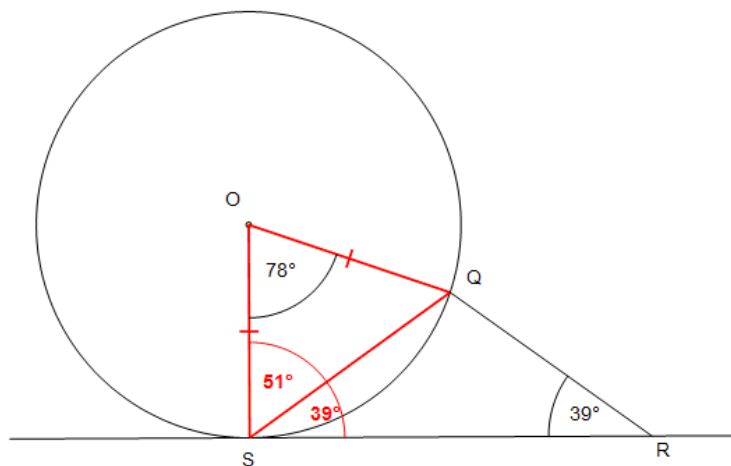
We know that $\triangle SOQ$ is isosceles because OQ and OS are radii, and therefore equal.

This makes $\angle OSQ = \angle OQS$, and as the angle sum of a triangle is 180° , $\angle OSQ = \frac{1}{2}(180-78)^\circ = 51^\circ$.

But we are also told that SR is a tangent, so $\angle OSR$ (the angle between the radius OS and the tangent SR) is a right angle.

But, $\angle OSR = \angle OSQ + \angle QSR = 90^\circ$,
 so $\angle QSR = (90 - 51)^\circ = 39^\circ$

$\therefore \angle QSR = \angle QRS$, and so $\triangle QSR$ is isosceles.



Example (3): Which of the following quadrilaterals are cyclic ?

- i) a rectangle; ii) a rhombus; iii) a square; iv) a kite with a pair of opposite right angles ?

A rectangle has all its angles equal to 90° , as does a square. Both are therefore cyclic, since opposite pairs of angles add to 180° .

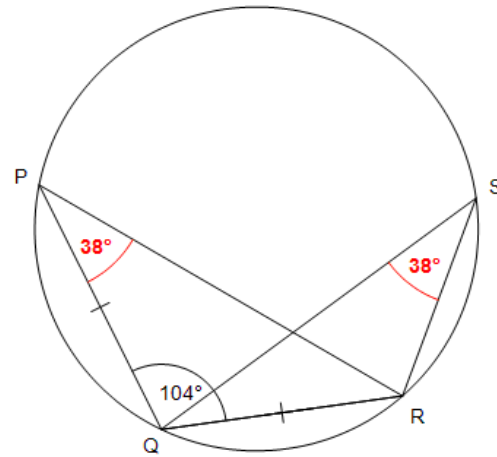
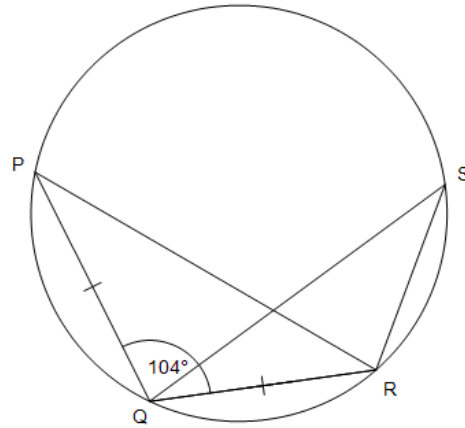
The kite with a pair of opposite right angles is also cyclic for the very same reason.

A rhombus has opposite pairs of angles equal, but because they are not generally right angles, a rhombus is not cyclic.

Example (4): Points P , Q , R and S lie on a circle.
 $PQ = QR$, and angle $PQR = 104^\circ$.
Explain why angle $QSR = 38^\circ$.

If $PQ = QR$, then $\triangle PQR$ is isosceles, and hence
 $\angle QPR = \frac{1}{2}(180 - 104)^\circ = 38^\circ$.

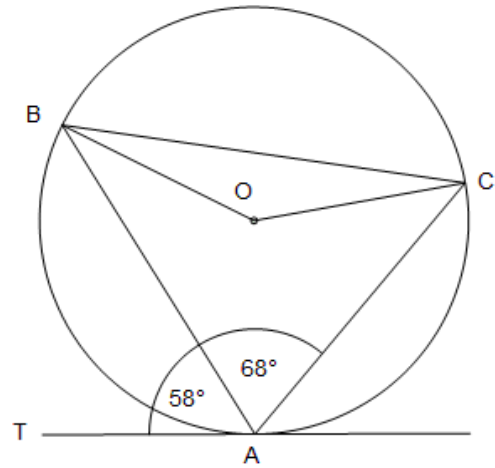
Also $\angle QPR$ and $\angle QSR$ are in the same segment,
so $\angle QSR$ is also $= 38^\circ$ because angles in the same
segment are equal.



Example (5):

Line TA is a tangent to the circle centred on O .
 Angles BAC and BAT are 68° and 58° respectively.

Calculate angles BOC and OCA .



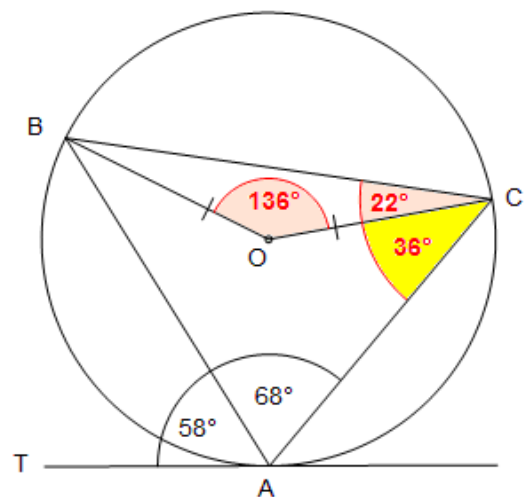
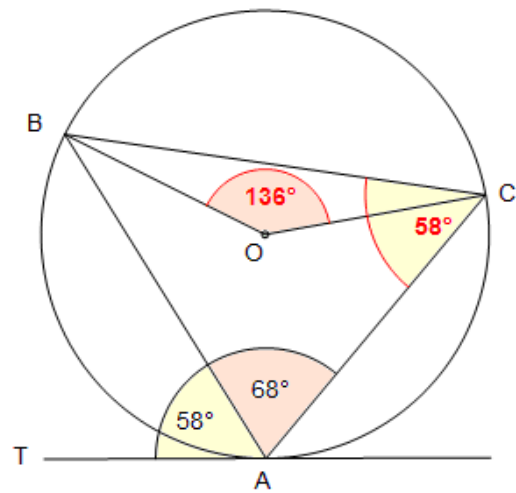
The angle at the centre is double the angle at the circumference, so $\angle BOC = 2 \times 68^\circ = 136^\circ$.

Then we use the alternate segment theorem to deduce that $\angle BCA = \angle BAT = 58^\circ$.

This angle BCA is then divided into two smaller angles, namely BCO and OCA .

$\triangle BOC$ is isosceles, so $\angle BCO = \frac{1}{2}(180-136)^\circ = 22^\circ$.

Finally, angle $OCA = \angle BCA - \angle BCO = (58 - 22)^\circ = 36^\circ$.



Example (6): The quadrilateral $ABCD$ is cyclic, and PAQ is a tangent to the circle. Also, $BC = CD$, $\angle QAB = 41^\circ$ and $\angle BAD = 82^\circ$.

Show that AD and BC are parallel, giving reasons for justification.

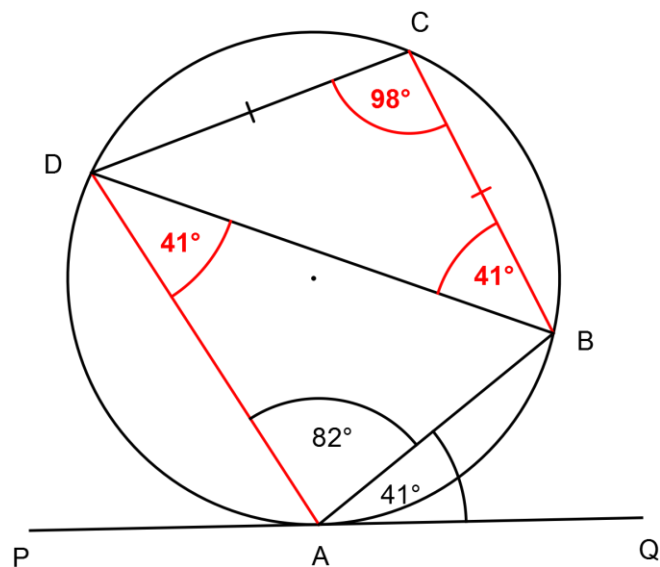
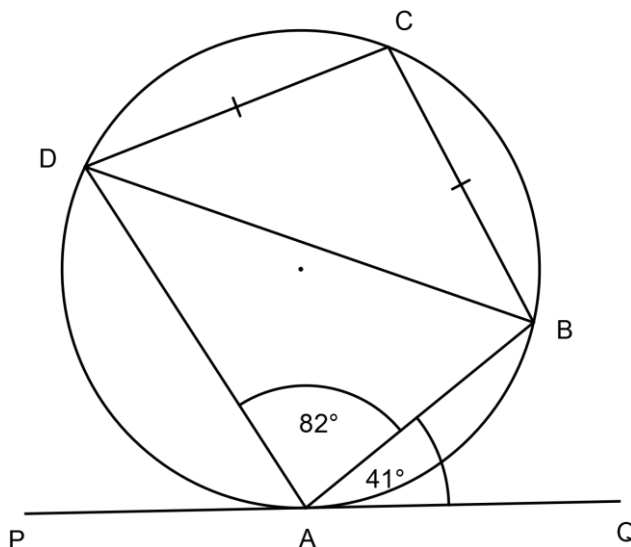
If lines AD and BC were parallel, angles CBD and ADB would form a pair of alternate angles and be equal to each other.

Angle $DCB = (180 - 82)^\circ = 98^\circ$ because opposite angles of a cyclic quadrilateral sum to 180° .

Triangle DCB is isosceles, so $\angle CBD = \frac{1}{2}(180 - 98)^\circ = 41^\circ$.

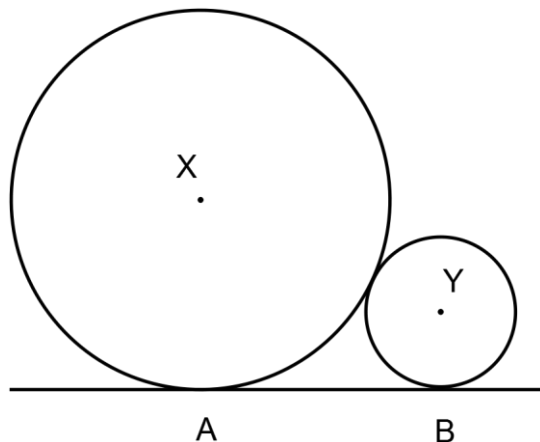
By the alternate segment theorem, $\angle ADB = 41^\circ$.

Angles CBD and ADB are indeed equal, so conversely lines AD and BC are parallel.



Example (7): The circles centred on X and Y have the tangent AB in common and have radii of 9 cm and 4 cm respectively.

- i) Explain why $ABYX$ is a trapezium.
- ii) Show that $AB = 12$ cm.



i) The tangents at A and B meet their respective radii AX and BY at right angles, so angles XAB and ABY are both equal to 90° , forming a pair of co-interior angles. Sides AX and BY are therefore parallel, i.e. $ABYX$ is a trapezium.

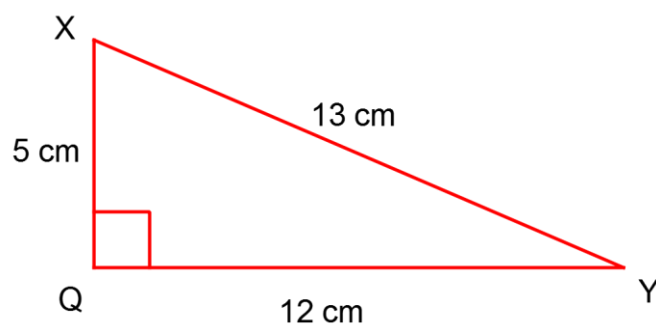
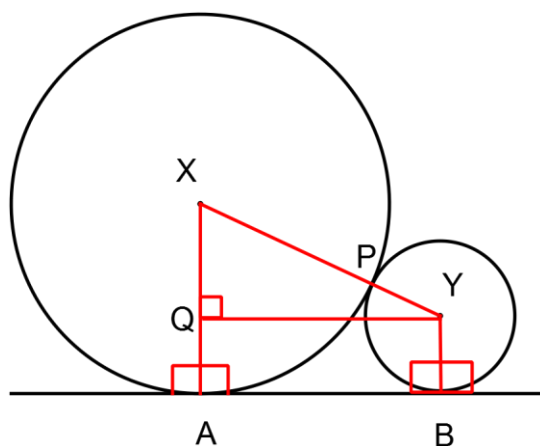
ii) We label a point Q on the radius AX such that $ABYQ$ is a rectangle. Now $XP = 9$ cm and $PY = 4$ cm, so $XY = 13$ cm. Also $AX = 9$ cm and $BY = 4$ cm, so $QX = 9 - 4 = 5$ cm.

The resulting triangle XQY has a hypotenuse of 13 cm and side $QX = 5$ cm. We use Pythagoras to find the distance QY .

$$QY = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12 \text{ cm.}$$

Because $ABYQ$ is a rectangle, $AB = QY$.

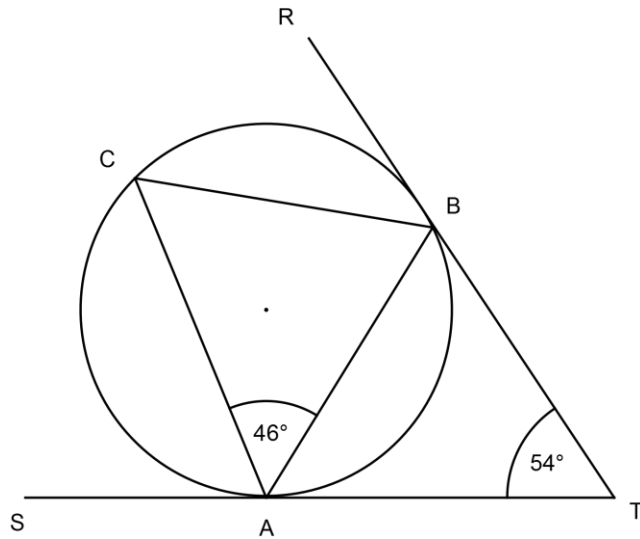
Hence the distance $AB = 12$ cm.



Example (8): The points A , B and C lie on the circumference of a circle. The lines SAT and RBT are tangents to the circle at points A and B respectively. These tangents meet at T .

Angle $CAB = 46^\circ$ and angle $BTA = 54^\circ$.

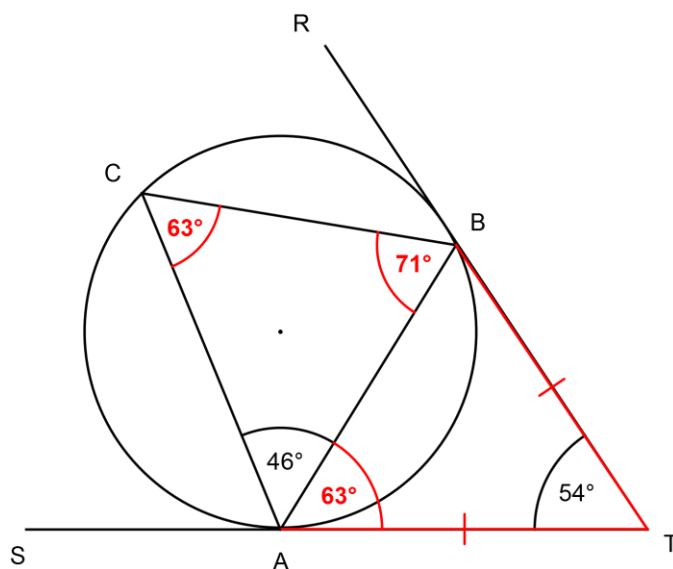
Find angles BAT and ABC , justifying your answers.



Tangents originating from the same point are equal in length so $TA = TB$. Triangle ATB is therefore isosceles, where angles BAT and $ABT = \frac{1}{2}(180-54) = 63^\circ$.

Additionally, $\angle ACB = \angle BAT = 63^\circ$ by the alternate segment theorem.

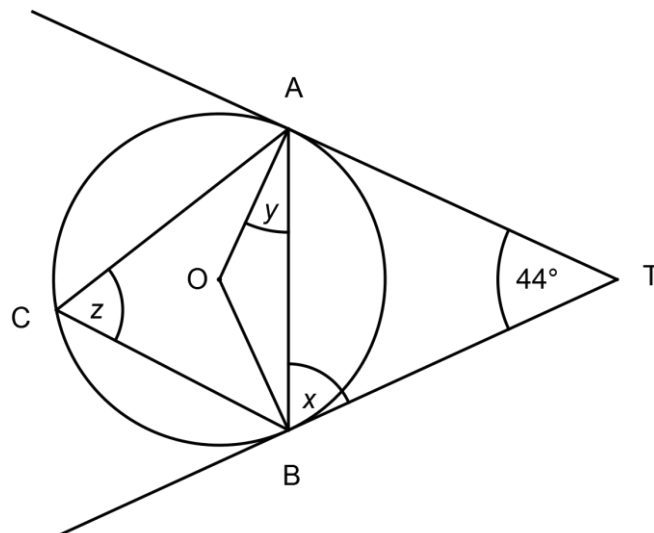
So $\angle ABC = (180 - (63 + 46))^\circ = 71^\circ$.



Example (9): A , B and C are points on the circumference of a circle whose centre is O .

TA and TB are tangents to the circle
 Angle $ATB = 44^\circ$.

Find angles x , y and z , giving reasons for your answers.



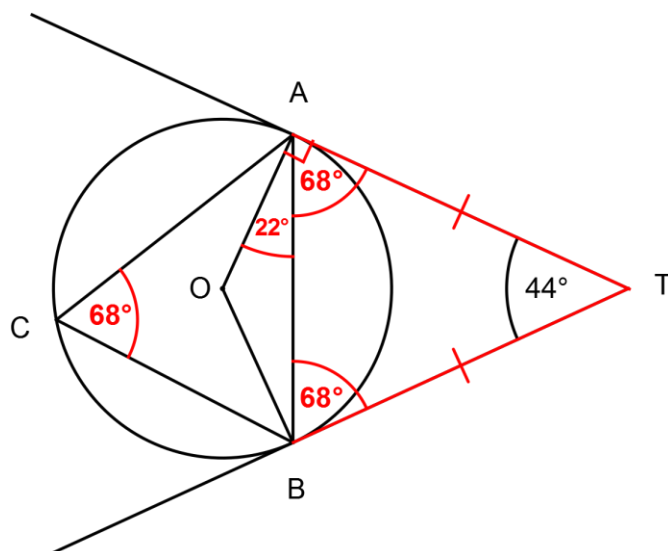
Tangents originating from the same point are equal in length so $TA = TB$.
 Triangle ATB is therefore isosceles, where angles BAT and ABT (labelled x) = $\frac{1}{2}(180-44) = 68^\circ$.

Also $\angle OBA = \angle OAB$ (labelled y) because ΔAOB is isosceles where OB and OA are radii. both radii),
 Additionally, $\angle OBT = 90^\circ$ because the tangent and radius at B meet at right angles.

We then find y by subtraction, i.e. $(90-68)^\circ = 22^\circ$.

The angle ACB (labelled z) is equal to angle x by the alternate segment theorem, i.e. it is also 68° .

We could have also calculated $\angle BOA = (180 - 44)^\circ = 136^\circ$. The size of angle ACB (labelled z) is half of that, or 68° , using the properties of angles at the centre.



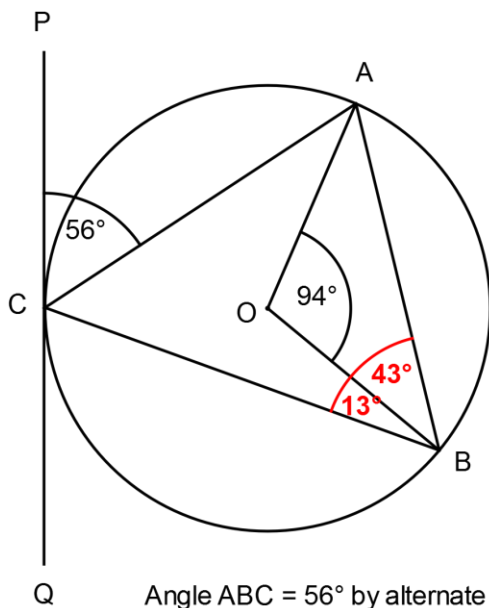
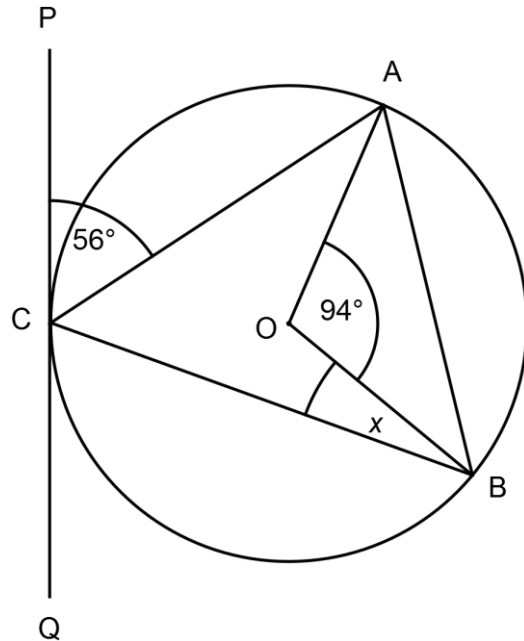
Example (10): Points A , B and C lie on the circumference of a circle centred on O .
 PQ is a tangent to this circle at P .
 $\angle PCA = 56^\circ$ and $\angle AOB = 94^\circ$.

Calculate angle OBC (labelled x), showing all working.

We use the alternate segment theorem to deduce that $\angle ABC = 56^\circ$. This angle is further divided up into angles ABO and OBC .

Since $\triangle AOB$ is isosceles,
 $\angle ABO = \frac{1}{2}(180-94) = 43^\circ$.

Hence $\angle OBC$ (labelled x) = $(56 - 43)^\circ = 13^\circ$.

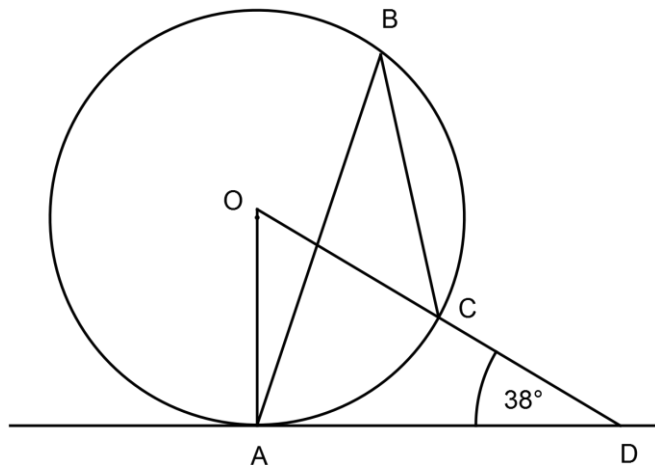


Angle $ABC = 56^\circ$ by alternate segment theorem

Example (11): The circle below is centred on O , and points A , B and C lie on its circumference.

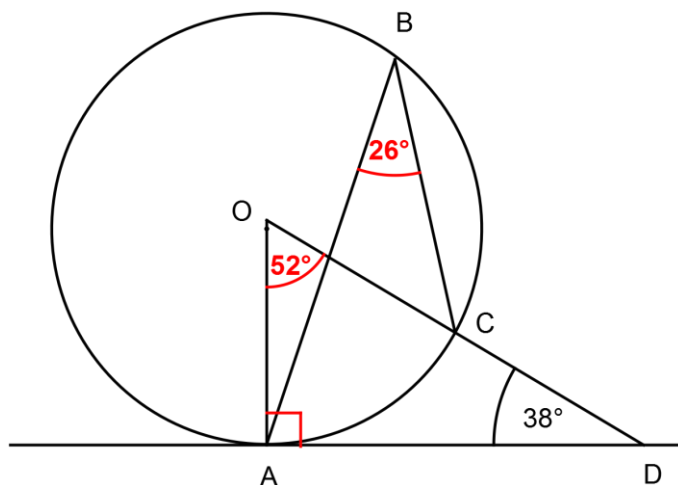
DCO is a straight line, DA is a tangent to the circle and $\angle ADO = 38^\circ$.

- i) Calculate angle AOD .
- ii) Calculate angle ABC , giving full reasons for your answer.

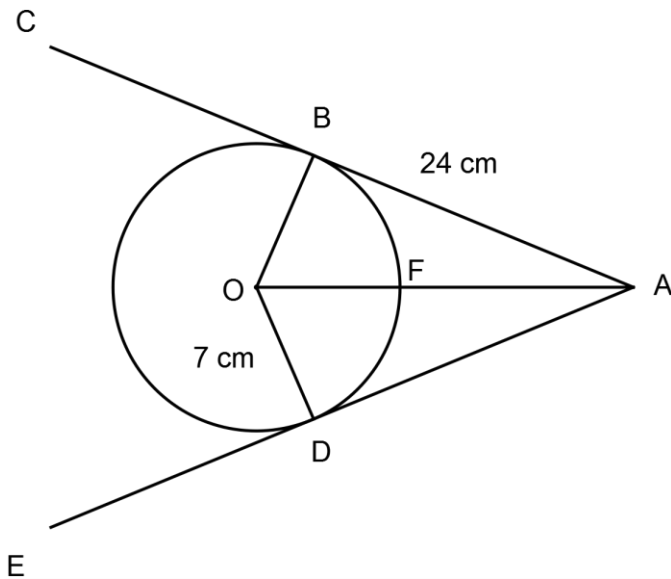


The tangent AD and the radius OA meet at 90° at point A , so $\angle OAD = 90^\circ$.
Hence $\angle AOD = (90 - 38)^\circ = 52^\circ$

We then use properties of angles at the centre to deduce that the angle ABC at the circumference is half the angle AOD at the circumference, so $\angle ABC$ is half of 52° or 26° .



Example (12): The lines ABC and ADE are tangents to the circle centred on O .
 $OD = 7$ cm and $AB = 24$ cm.
 F is the point where AO meets the circumference.
 Work out the distance AF .



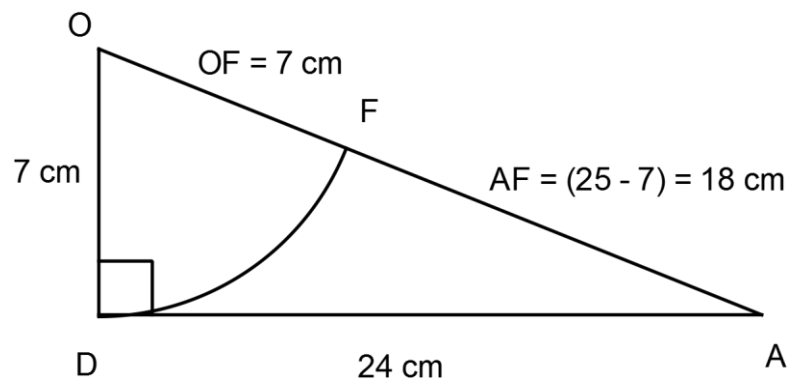
Tangents AB and AD are equal in length, as they originate from the same point A .
 Therefore $AD = 24$ cm.

Also, $\angle ODA = 90^\circ$ because the tangent and radius at D meet at right angles.
 Hence $\triangle ODA$ is right-angled.

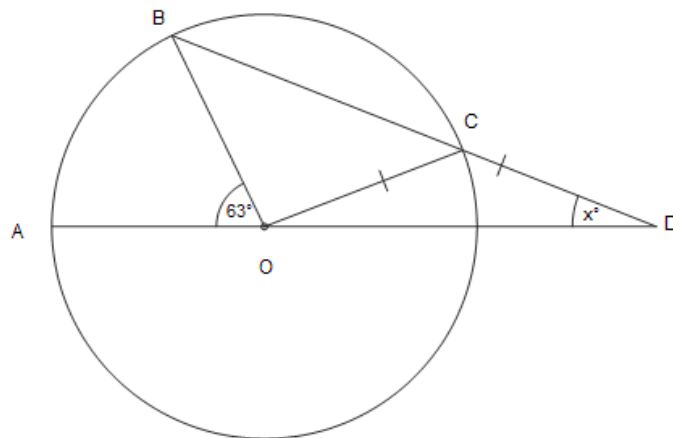
Using Pythagoras, we find that

$$OA = \sqrt{24^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25 \text{ cm.}$$

The distance OF is 7 cm, being a radius of the circle, so $AF = OA - OF = (25 - 7) \text{ cm} = 18 \text{ cm}$.



Example (13): In the circle below,
 $\angle AOB = 63^\circ$.
 The diameter through A and O is
 extended to point D such that $OC =$
 CD . Find angle x .



Because $\triangle OCD$ is isosceles,
 $\angle COD = x^\circ$ and $\angle OCD = (180-2x)^\circ$.

Hence $\angle BCO = 2x^\circ$ (180° in a straight
 line), and, because $\triangle OCB$ is isosceles
 (OB, OC are radii), it follows
 $\angle OBC = 2x^\circ$ as well.

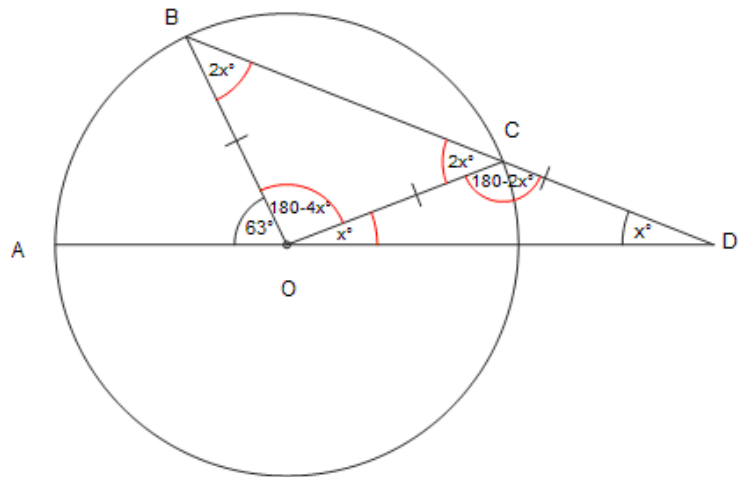
Hence $\angle BOC = 180-4x^\circ$.

Since AOD is a straight line, the sum of angles AOB, BOC and COD must equal 180° .
 This leads to the linear equation

$$63 + 180 - 4x + x = 180, \text{ so}$$

$$63 - 3x = 0, \therefore \text{angle } x = \mathbf{21^\circ}.$$

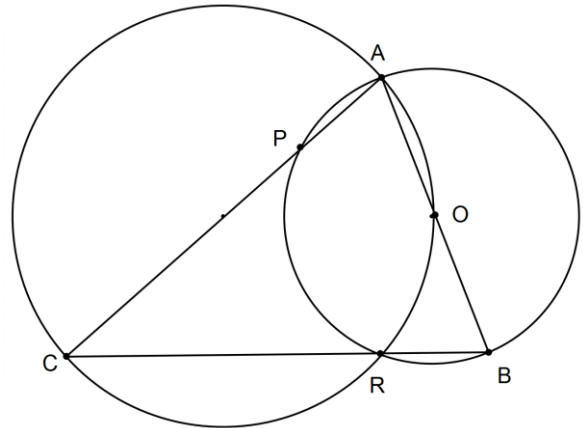
The angles of $\triangle OCD$ are thus 21° ,
 21° and 138° , while those of $\triangle BOC$
 are $42^\circ, 42^\circ$ and 96° .



Example (14): Tie-in: Congruent triangles

Points A, B, R and P lie on a circle with centre O .
 Points A, O, R and C lie on a different circle.
 The circles intersect at points A and R .
 CPA, CRB and AOB are straight lines.

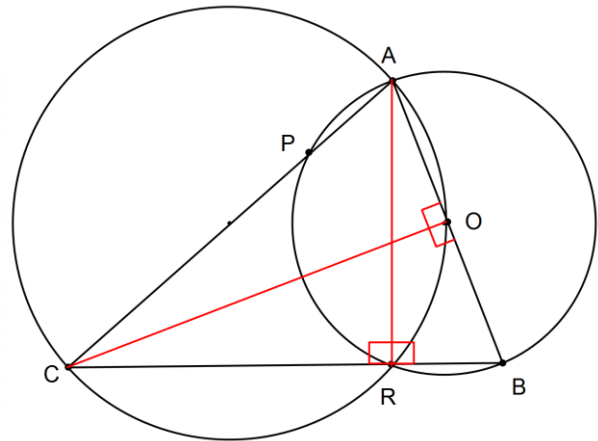
Prove that angle $CAB =$ angle ABC .



Because AB is a diameter of the smaller circle,
 $\angle ARB = 90^\circ$ (angle in a semicircle).

CRB is a straight line, so $\angle ARC$ in the larger circle
 $= 90^\circ$ (180° in a straight line)

Also, $\angle AOC = 90^\circ$ (angles in the same segment are equal) and $\angle BOC = 90^\circ$ (180° in a straight line).



Looking at triangles AOC and BOC we notice the following facts:

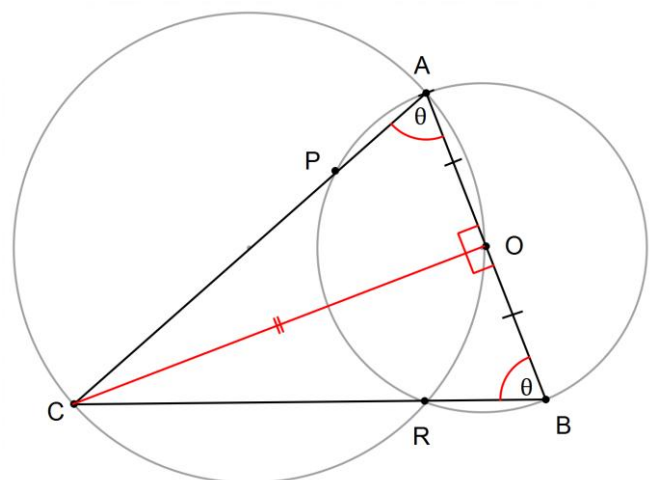
Sides AO and OB are equal in length since both are radii of the smaller circle.

Side OC is common to both triangles.

Angles AOC and BOC are equal.

The two triangles are therefore congruent (SAS – two sides and the included angle)

Hence angles CAB and ACB (labelled θ) are equal.



An important result (IGCSE).

AB and CD are two chords which intersect at point X , which is not necessarily the centre of the circle.

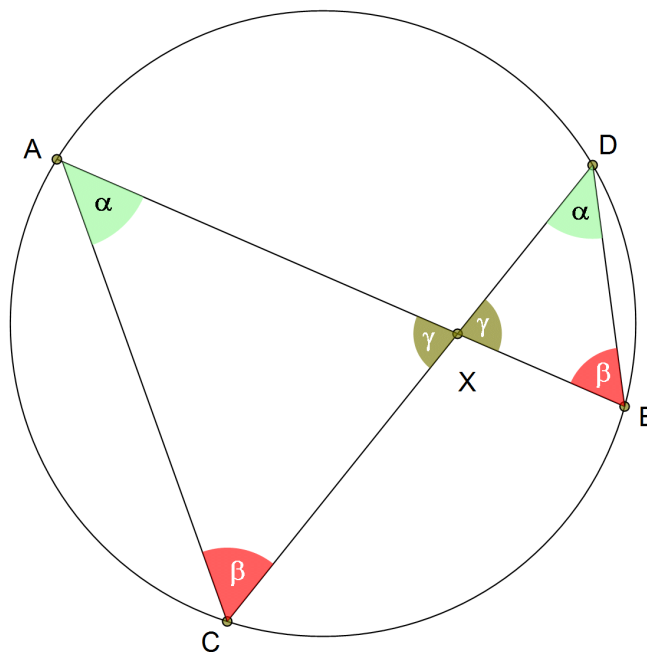
Prove that the products $(AX)(XB)$ and $(CX)(XD)$ are equal.

Proof.

Angles CAX and XDB (marked α) are equal because they are both subtended by the segment CB .

Angles ACX and XBD (marked β) are equal because they are both subtended by the segment AD .

Angles AXC and DXB (marked γ) are equal because they are vertically opposite.



From the three results above, the triangles AXC and DXB are similar.

Sides CX and XB correspond, as they are opposite the angles marked α .

The ratio of their lengths, $\frac{CX}{XB}$, is a constant value k .

Sides AX and XD correspond, as they are opposite the angles marked β .

The ratio of their lengths, $\frac{AX}{XD}$, takes the same value k , as triangles AXC and DXB are similar.

Hence $\frac{CX}{XB} = \frac{AX}{XD}$, and cross-multiplying, $(AX)(XB) = (CX)(XD)$.

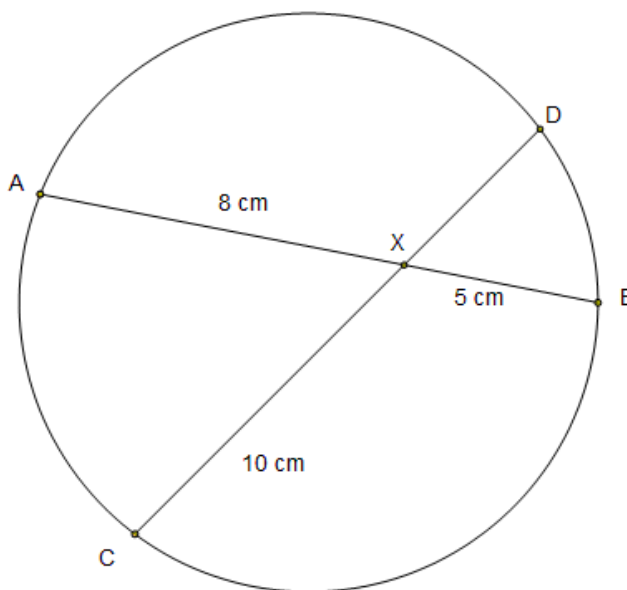
Example (15): In the circle shown, $AX = 8$ cm, $XB = 5$ cm and $CX = 10$ cm. Find the length of the chord CD .

Now, $(AX)(XB) = 8 \times 5 = 40$, so $(CX)(XD) = 40$.

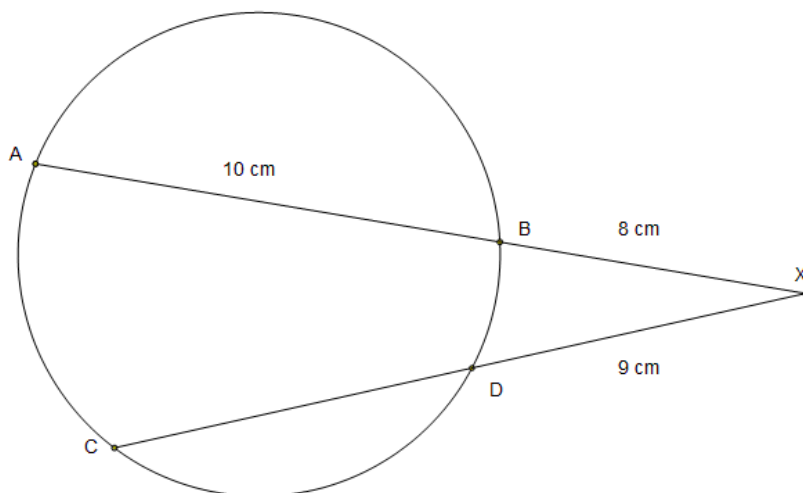
Therefore $XD = \frac{40}{10} = 4$ cm.

Thus the length of chord CD is $10 + 4 = 14$ cm.

This rule also holds when the chords intersect outside the circle, as the following examples will show.



Example (16): In the circle shown, $AB = 10$ cm, $XB = 5$ cm and $XD = 9$ cm.
Find the length of the chord CD .



Now, $AX = 10 + 5 = 15$, so $(AX)(XB) = 15 \times 5 = 75$. Since $(CX)(XD) = 75$, $CX = \frac{75}{9} = 8\frac{1}{3}$ cm.

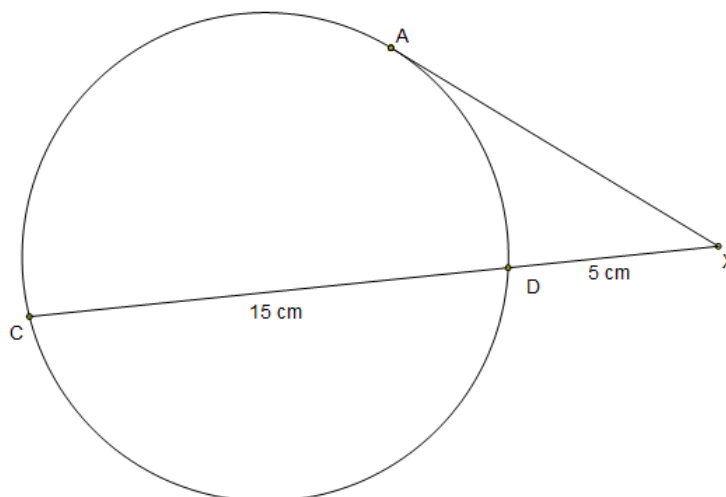
Hence the chord CD is $8\frac{1}{3} - 9$, or $-\frac{5}{3}$ cm, long.

Additionally, if the chord AB degenerates into a tangent at point A , the product $(AX)(XB)$ becomes $(AX)(XA)$ or $(AX)^2$. In other words, we have $(AX)^2 = (CX)(XD)$.

Example (17): In the circle shown, $CD = 15$ cm and $XD = 9$ cm.
Find the length of the tangent AX .

Here, $CX = 15 + 9 = 24$, and
 $(CX)(XD) = 24 \times 9 = 216$.

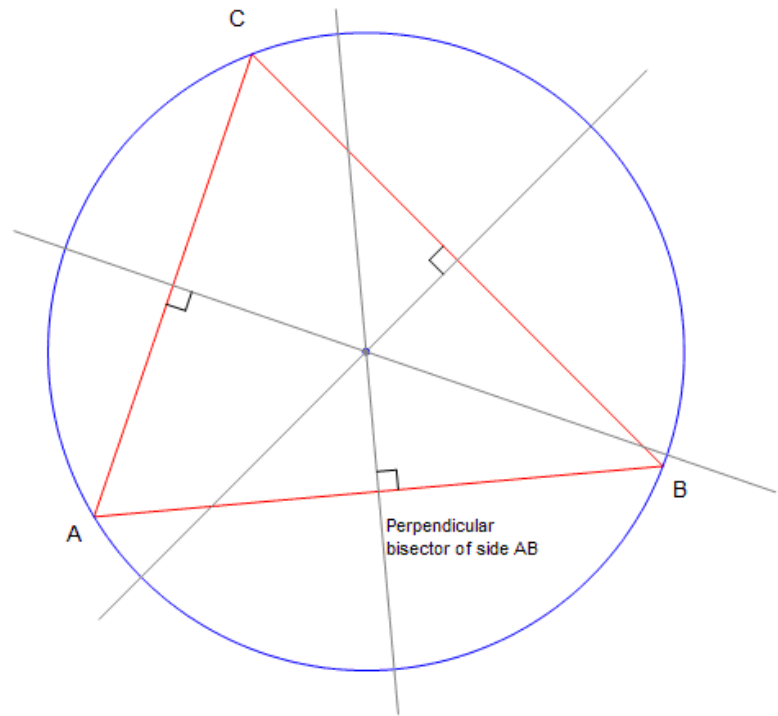
Since $(AX)^2 = 216$, $AX = \sqrt{216}$ or
 $6\sqrt{6}$ cm, i.e. the tangent AX is $6\sqrt{6}$
cm long.



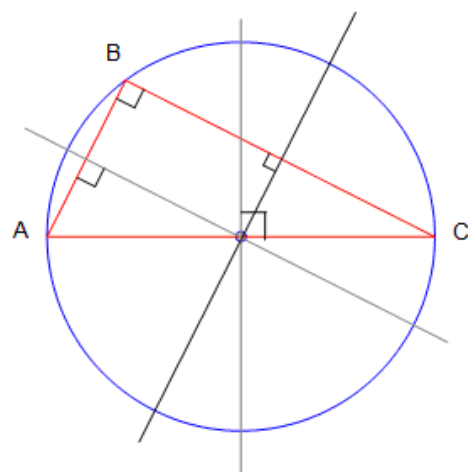
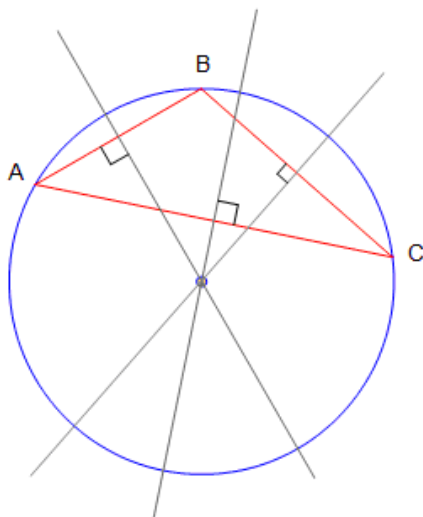
Circles and triangles.

The **circumcircle** of a triangle is the circle passing through all of its vertices. Its centre is the intersection of the perpendicular bisectors of its sides.

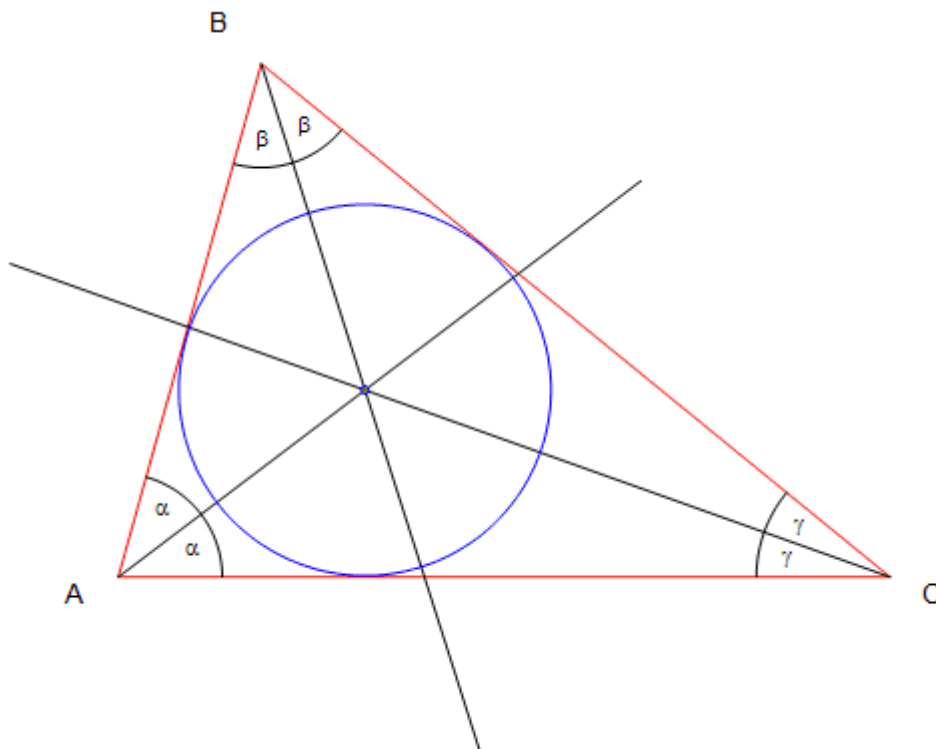
If the triangle is acute-angled, the circumcentre will be inside the triangle.



When we have an obtuse-angled triangle, the centre of the circumcircle lies outside the triangle, and with a right-angled triangle, the circumcentre lies on the midpoint of the hypotenuse.



The **incircle** of a triangle is the largest circle that can be inscribed within it. Its centre is the intersection of the bisectors of its angles, and the three sides of the triangle are tangents to it.



Note that the tangents to the incircle do not generally coincide with the angle bisectors.

To complete the discussion, a line joining the midpoint of a triangle's side to the opposite vertex is called a **median**. The three medians of the triangle meet at a point called the **centroid** of the triangle, which, incidentally, is also the triangle's centre of gravity.

The midpoints of sides AB, BC and CA are at points P, Q and R.

The medians are AQ, BR and CP and they intersect at the centroid X.

