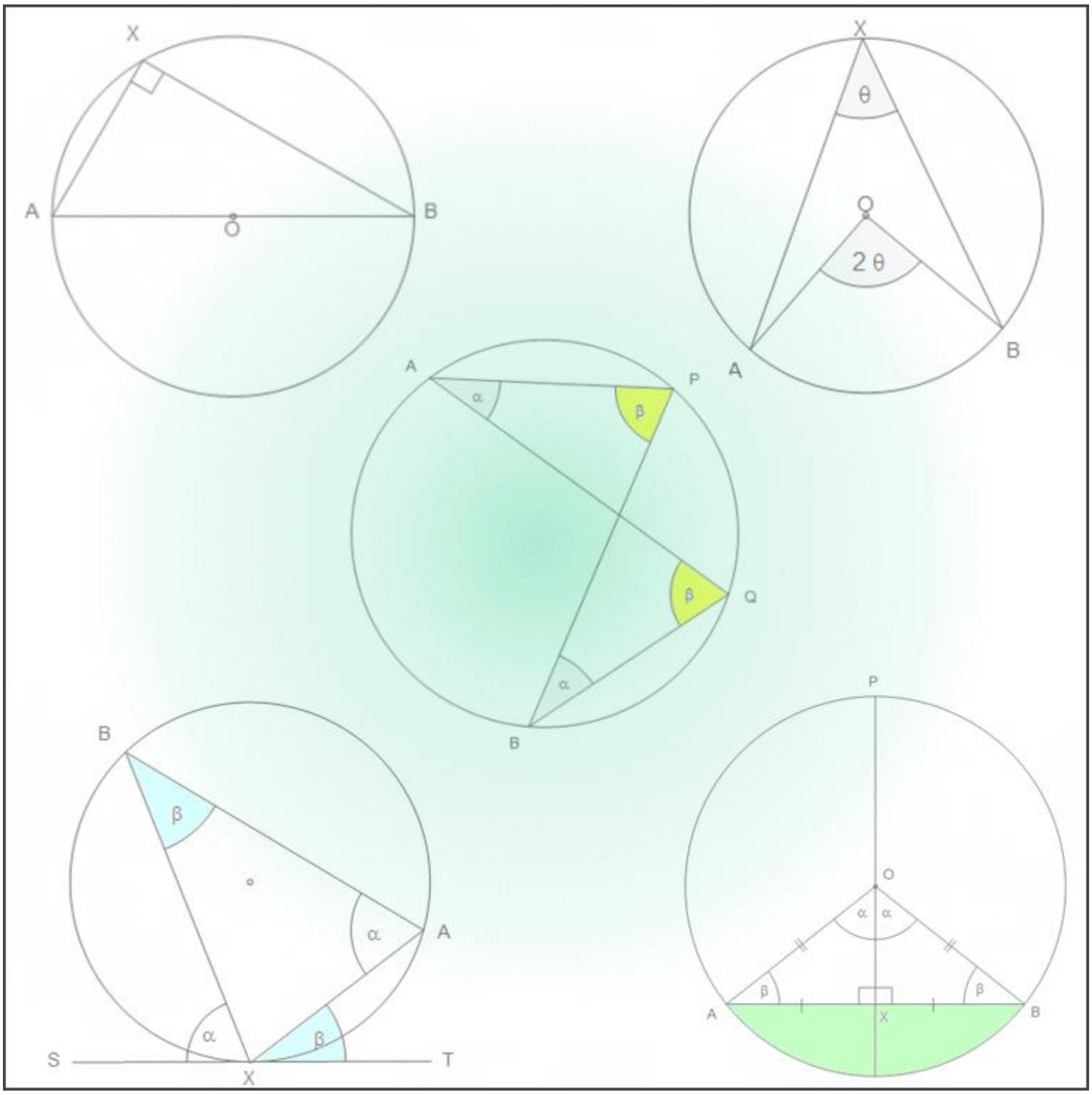
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

CIRCLE THEOREMS



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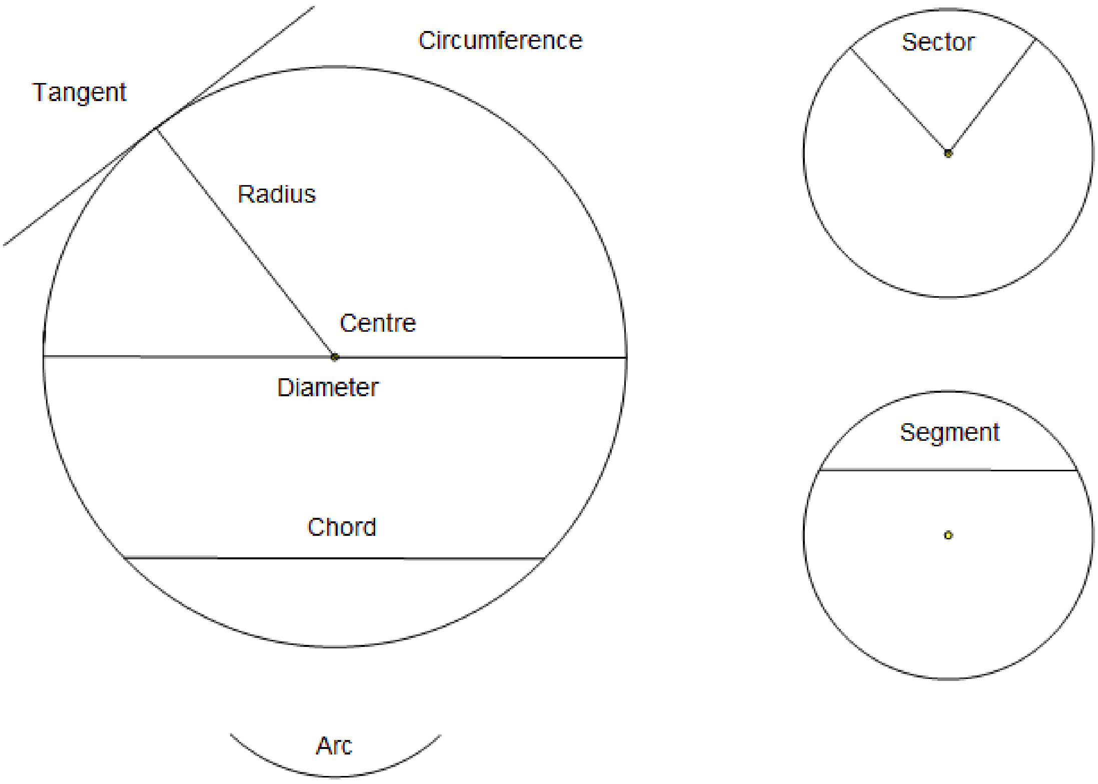
CIRCLE THEOREMS

Recall the following definitions relating to circles:



A circle is the set of points at a fixed distance from the **centre**.
The perimeter of a circle is the **circumference**, and any section of it is an **arc**.
A line from the centre to the circumference is a **radius** (plural: **radii**).

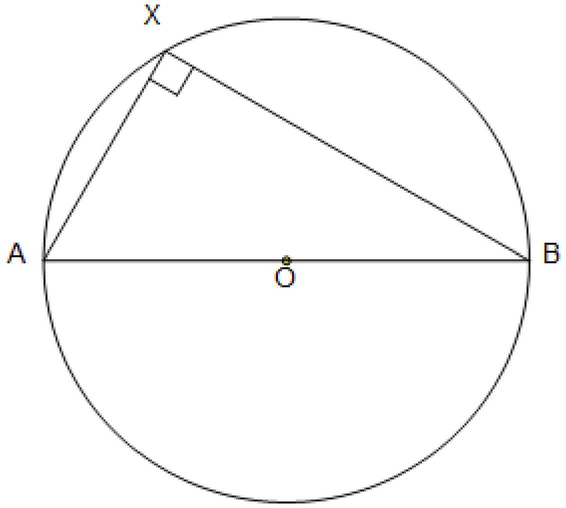
A line dividing a circle into two parts is a **chord**.
If the chord passes through the centre, then it is a **diameter**.
A diameter divides a circle into two equal parts, and its length is twice that of a radius.

If the chord is not a diameter, the two resulting unequal regions of the circle are **segments**. The minor segment is the smaller; the major segment the larger.

A **sector** is a region of a circle bounded by two radii. The smaller one is the minor sector, the larger one the major sector.

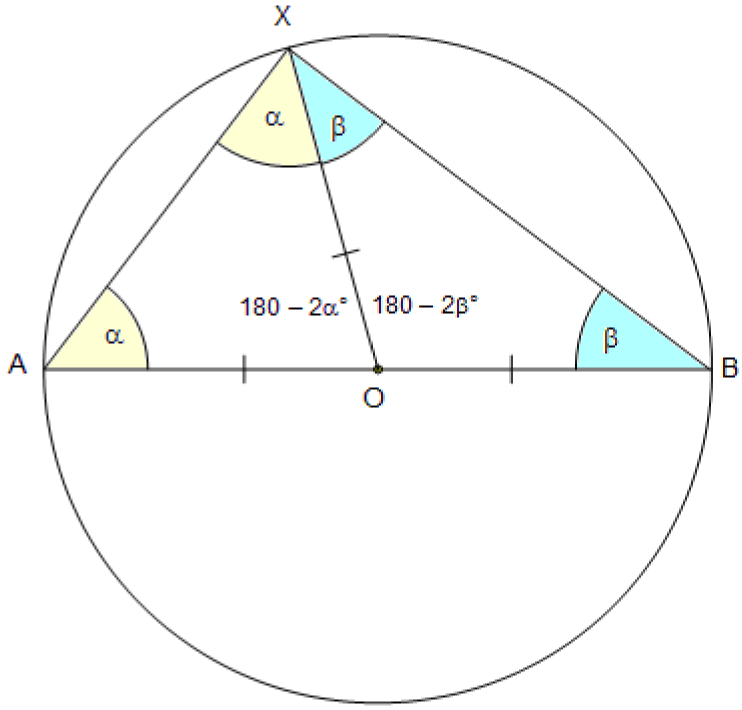
A **tangent** is a line touching a circle at one point.

The angle at the circumference subtended by a diameter is a right angle, or more simply, the angle in a semicircle is a right angle.



The line *AB* is a diameter of the circle, passing through the centre, *O*. Angle *AXB* is therefore a right angle.

Proof.



The triangle *AXB* can be divided into two smaller triangles, *AOX* and *BOX*. Both triangles are isosceles because *OA*, *OX* and *OB* are all radii and therefore equal in length.

$\angle AXB = \alpha + \beta$.

$\angle OAX$ and $\angle OXA$ (marked α) are therefore equal,
and $\angle AOX$ is thus $(180-2\alpha)°$.

Similarly $\angle OBX$ and $\angle OXB$ (marked β) are equal,

with $\angle BOX = (180-2\beta)°$.

Since $\angle AOX$ and $\angle BOX$ form a straight line, their angle sum is 180°.

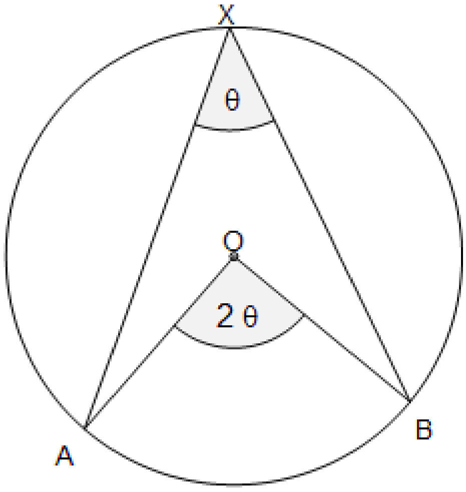
Hence $(180-2\alpha)° + (180-2\beta)° = 180°$
$\rightarrow (360-2\alpha-2\beta)° = 180°$
$\rightarrow 2\alpha + 2\beta = 180°$
$\rightarrow \alpha + \beta = 90°$.

$\therefore \angle AXB$ is a right angle.

The angle subtended at the centre of a circle is double the angle subtended at the circumference.

Points *A*, *X* and *B* lie on the circumference of a circle centred on *O*.

The angle *AOB* at the centre is double the angle *AXB* at the circumference (labelled θ),



Proof.

The figure *OAXB* can be divided into two smaller triangles, *AOX* and *BOX*. Both triangles are isosceles because *OA*, *OX* and *OB* are all radii and therefore equal in length.

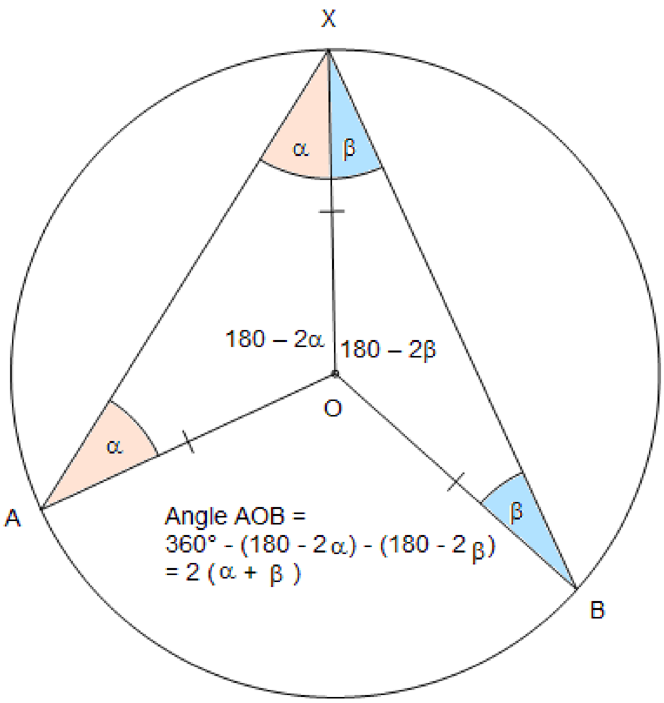
$\angle AXB = \alpha + \beta$.

$\angle OAX$ and $\angle OXA$ (marked α) are therefore equal, and $\angle AOX$ is thus $180-2\alpha$ °.

Similarly $\angle OBX$ and $\angle OXB$ (marked β) are equal, with $\angle BOX = 180-2\beta$ °.

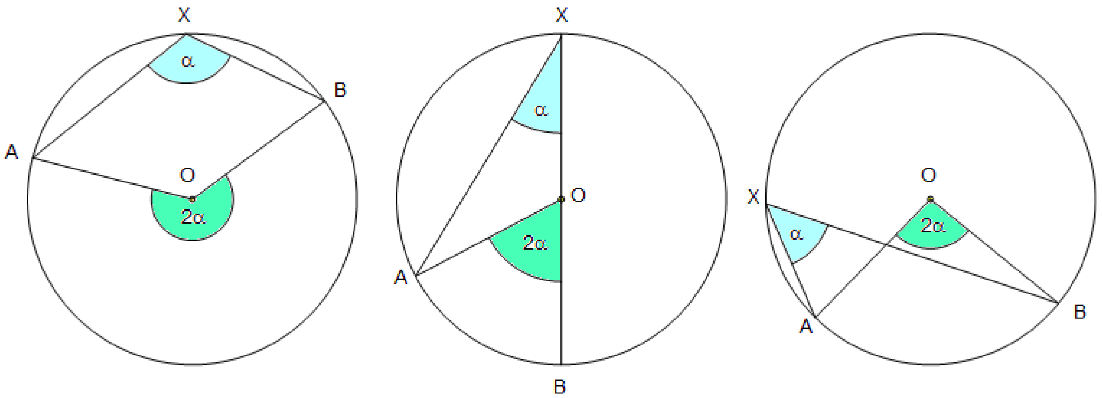
Since angles *AOX*, *BOX* and *AOB* add up to 360° around *O*, it follows that

$\angle AOB = 360° - (180° - 2\alpha) - (180° - 2\beta)$
$\rightarrow \angle AOB = 360° - (360° - 2\alpha - 2\beta)$
$\rightarrow \angle AOB = 2(\alpha + \beta)$.



$\therefore$ The angle *AOB* at the centre is double the angle *AXB* at the circumference .

Although the ‘arrowhead’ configuration is the one most commonly shown in textbooks, there are other patterns where the rule can be applied – in each case below, angle *AOB* at the centre is double the angle *AXB* at the circumference .



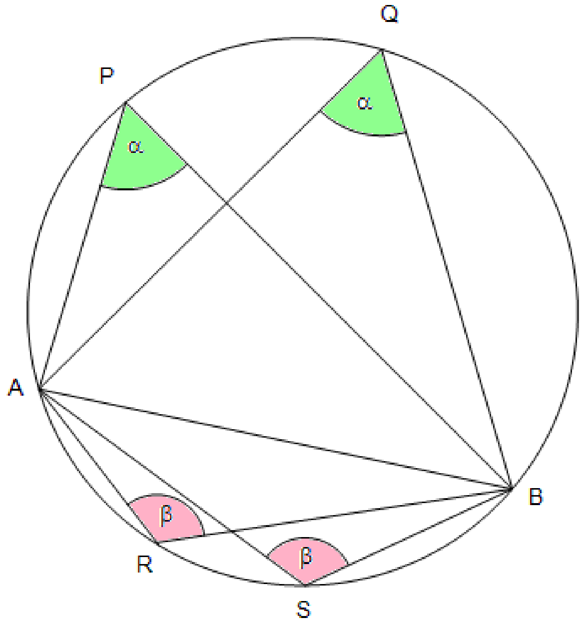
In the left-hand diagram, $\angle AOB$ at the centre is reflex because $\angle AXB$ at the circumference is obtuse.

In the middle diagram, point *B* has moved such that *XB* is a diameter of the circle, and points *X*, *O* and *B* are collinear.

In the right-hand diagram, point *X* has moved such that the line *XB* now crosses line *OA*.

Angles subtended by the same chord are equal.

Let *AB* be a chord of a circle.

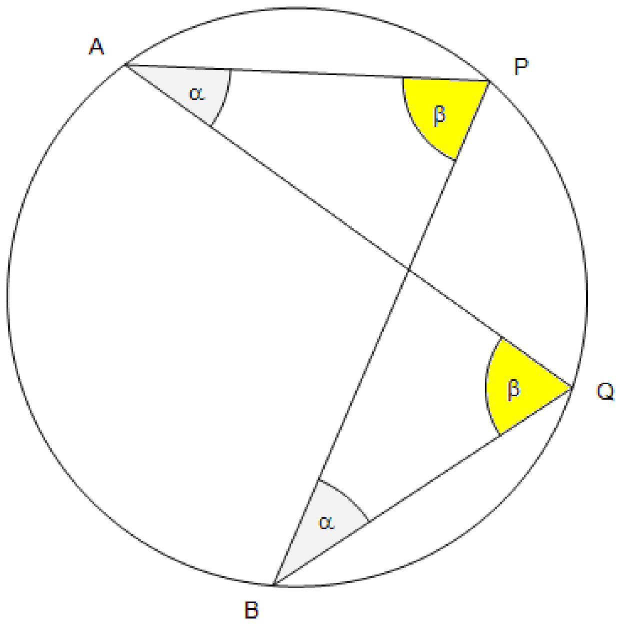


∠*APB* and ∠*AQB* (marked α) are equal, as are ∠*ARB* and ∠*ASB* (marked β) .

Also, the sum of the angles on opposite sides of the chord *AB* is equal to 180°.

(See the section on Cyclic Quadrilaterals for further details.)

This circle does not actually have the chords *AB* or *PQ* drawn on it, but the rule still holds: ∠ *APB* and ∠*AQB* are equal, as are ∠*PAQ* and ∠*PBQ*.



The bisector of a chord is a diameter.

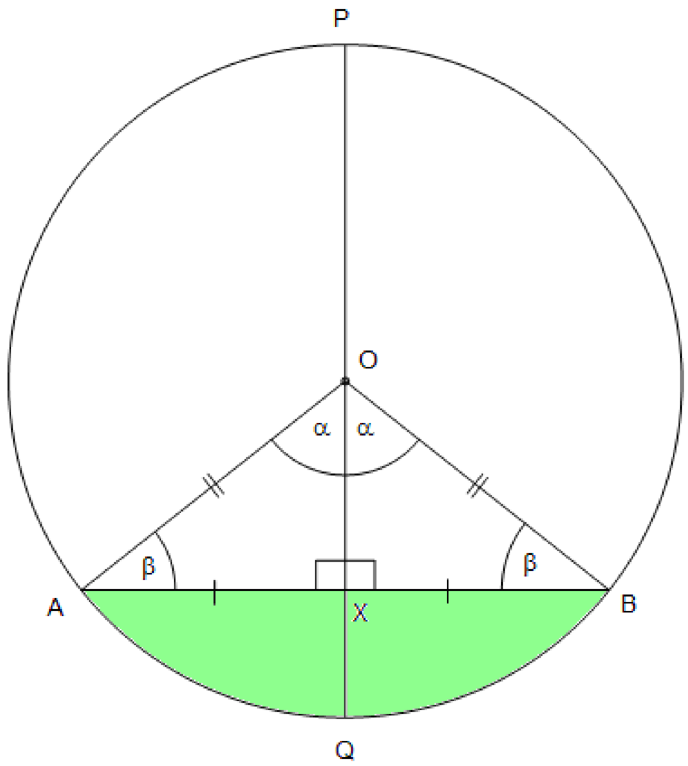
Any line drawn across a circle is a **chord**, and the perpendicular bisector of the chord passes through the centre, and is therefore also a diameter of the circle.

In the diagram, the perpendicular bisector of chord *AB* is the line *PQ*. Since *PQ* passes through the centre *O*, it is also a diameter.

Point *X* is the midpoint of the chord *AB* and also lies on the diameter *PQ*.

The triangle *AOB* is isosceles because *OA* and *OB* are radii, and hence equal.

Additionally triangles *AOX* and *BOX* are congruent.

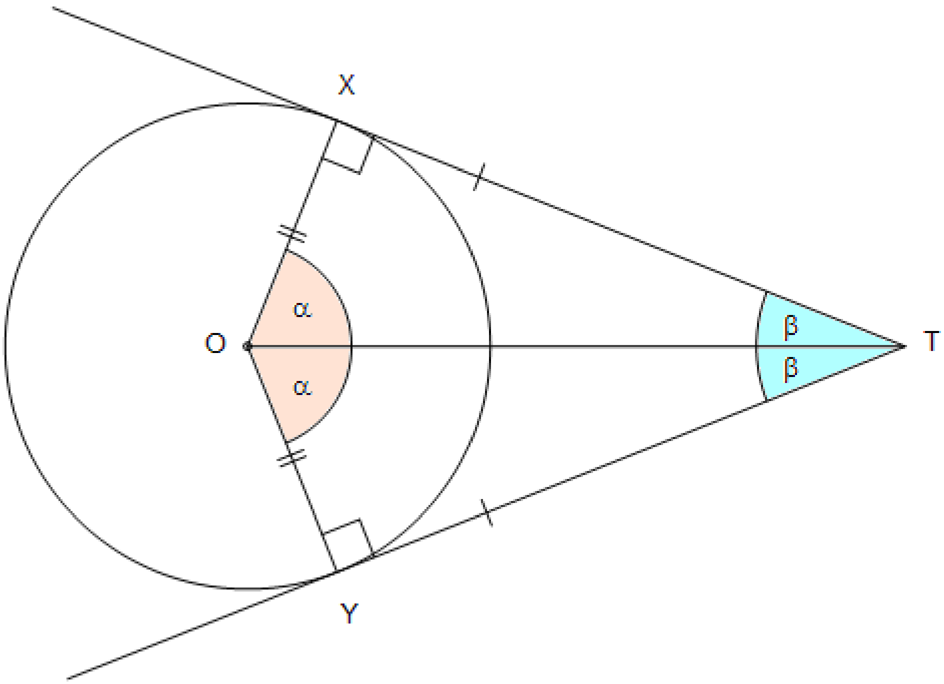


Notice also the symmetry of the triangle *AOB* – it is isosceles because *AO* and *OB* are both radii.

A tangent and a radius meet at right angles; also, two tangents drawn from a point to a circle describe two lines equal in length between the point and the circle.

The lines *TX* and *TY* both originate from the point *T* and form tangents with the circle at points *X* and *Y*. These tangents are perpendicular to the radii of the circle *OX* and *OY*, and are also equal in length.

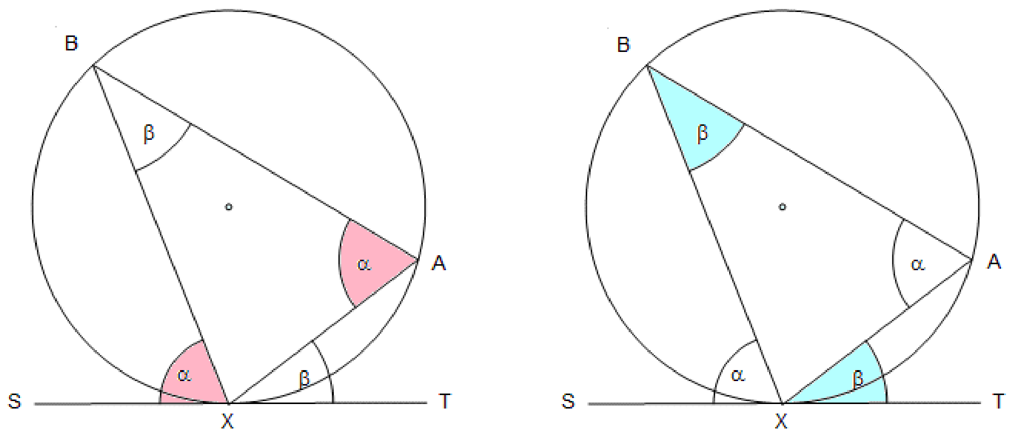
Because the triangles *OXT* and *OYT* also have side *OT* in common, both also have all three sides equal, and are therefore congruent. As a consequence, angles *XOT*, *YOT* (labelled α) and angles *XTO*, *YTO* (labelled β) are also equal.



Angles in the alternate (opposite) segment are equal.

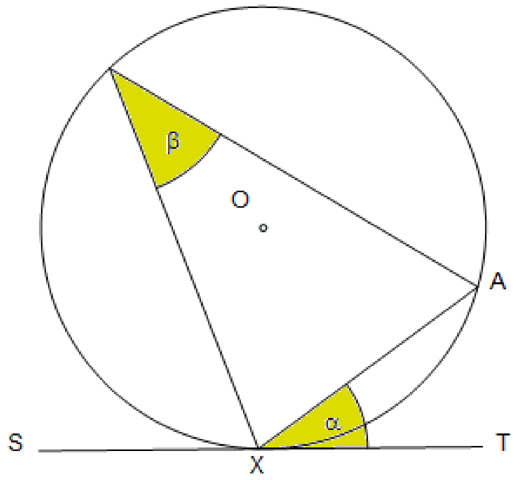
In the diagram below left, the chord *BX* and the tangent *ST* meet at *X*.
Let *SXB* be the angle between the chord and the tangent.
Let *XAB* be the angle in the alternate segment (bounded by chord *BX*).
angles *SXB* and *XAB* (shaded and labelled α) are equal.

There is another angle pair in the right-hand diagram.
This time, *TXA* is the angle between the chord *AX* and the tangent *ST*.
Let *XBA* be the angle in the alternate segment (bounded by chord *AX*)
Then angles *TXA* and *XBA* (shaded and labelled β) are equal.



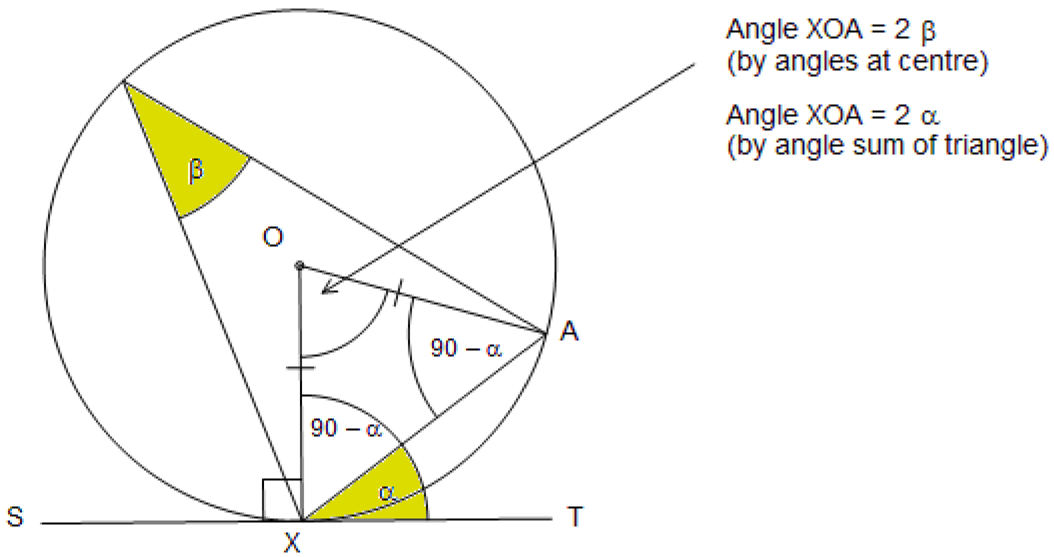
Proof. (Using angles at the centre)

The line *ST* is a tangent to the circle centred on *O*, and α is the angle between *TX* and the chord *XA*.



We begin by drawing radii at *OX* and *OA*.
Because the tangent *ST* and the radius *OX* meet at right angles,

$\angle OXS = \angle OXT = 90°$.



Hence $\angle OXA = (90 - \alpha)°$.

Furthermore, $\angle OAX$ also = $(90 - \alpha)$ °, since Δ *XOA* is isosceles (*OX* and *OA* are radii).

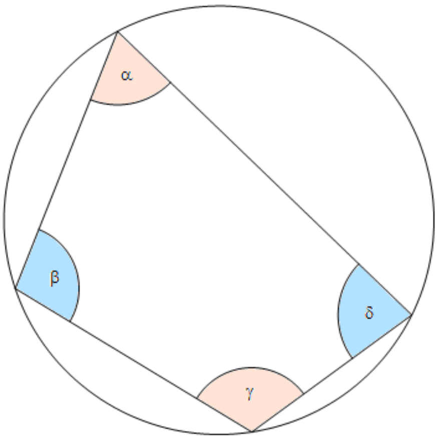
The angles in a triangle add to 180°, and so $\angle XOA = (180 – (180 – 2\alpha))° = 2\alpha$.

But, the angle at the centre ($\angle XOA$) is double the angle at the circumference (β) ,
so $\angle XOA$ also = 2β.

$\therefore$ Angles α and β are equal.

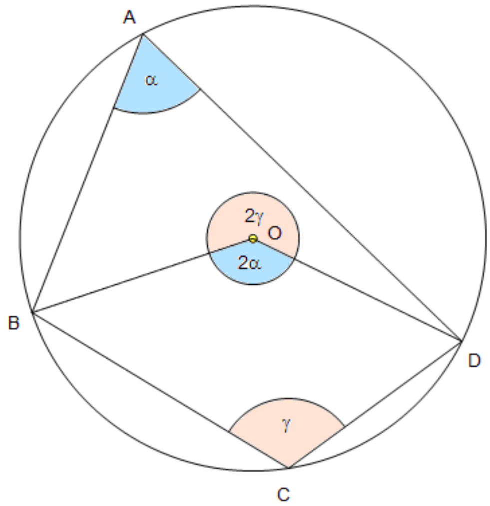
A **cyclic quadrilateral** is one that can be inscribed in a circle, in other words one whose vertices lie on a circle's circumference.

The opposite angles of a cyclic quadrilateral add up to 180°.



Thus, in the diagram shown right, angles α and γ have a sum of 180°, as do angles β and δ.

Proof (using angles at the centre)



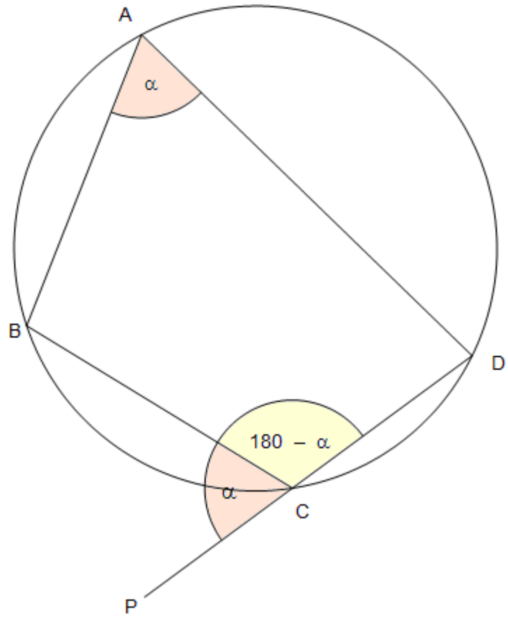
Take the quadrilateral *ABCD* inscribed in the circle centred on *O*.

By the rules of angles at the centre, if ∠ *BAD* = α, then the obtuse angle *BOD* = 2α.

Similarly, if ∠ *BCD* = γ, then the reflex angle *BOD* = 2γ.

Since 2α + 2γ = 360°, α + γ = 180°.

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Let ∠ *BAD* = α.

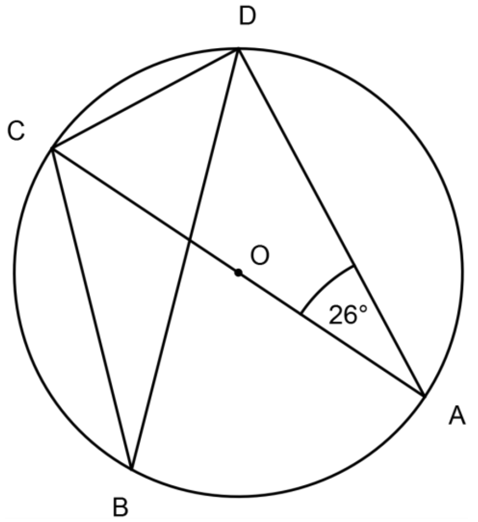
Since the opposite angles of a cyclic quadrilateral add to 180°, ∠ *BCD* = 180-α.

Extending side *DC* to point *P*, ∠ *BCP* + ∠ *BCD* = 180° since *DCP* is a straight line.

Hence ∠ *BCP* = 180° - (180-α) = α.

Examination questions usually feature more than just one of the theorems, and often include some algebra and Pythagoras as well.

Also, diagrams in exam questions are not usually drawn accurately. This is also true for the examples in this section.



Example (1): The points *A* , *B*, *C* and *D* lie on a circle with centre *O*.

i) Given $\angle DAC$ = 26°, find angles *DBC* and *ACD*.

ii) Explain why angle *BCD* is not a right angle.

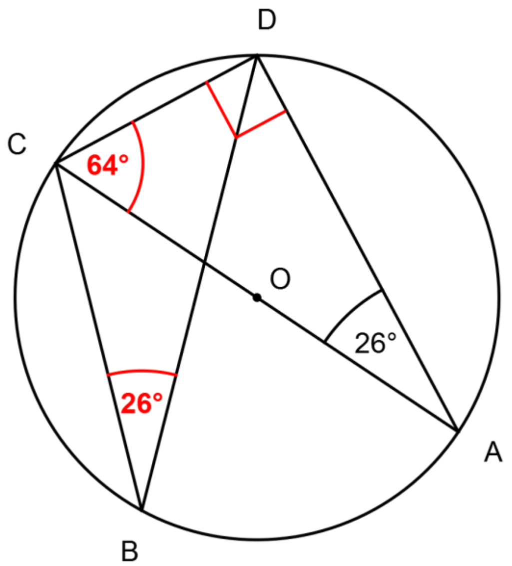
i) Angle *DBC* = 26° because it is in the same segment as angle *DAC*.

As *COA* is a diameter of the circle, angle *CDA* = 90° because the angle in a semicircle is a right angle.

$\therefore$ Angle *ACD* = 180°- (90 + 26)° = 64°.

More simply, the two acute angles in a right-angled triangle add to 90°, so angle *ACD* = (90 – 26)° = 64°.

ii) Angle *BCD* is not a right angle because *BD* is not a diameter of the circle.
This is the converse of the "angle in a semicircle" theorem.

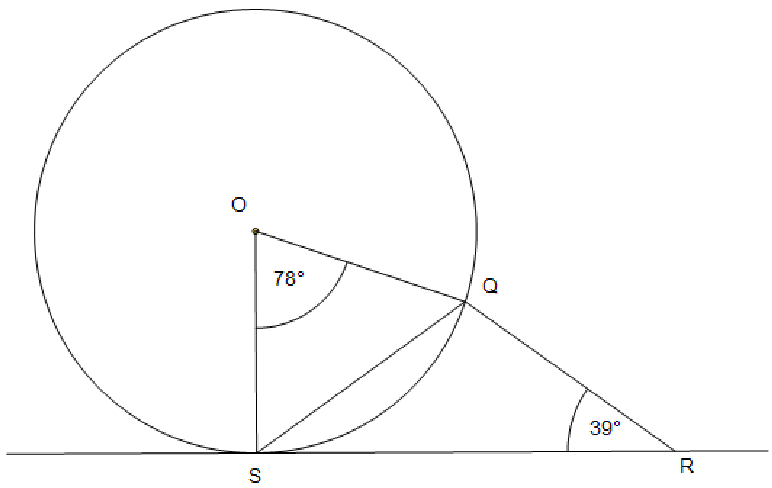


Example (2): Points *Q* and *S* lie on a circle centred on *O*.

SR is a tangent to the circle at *S*.

Angle *QRS* = 39° and angle *SOQ* = 78°.

Prove that triangle *SQR* is isosceles.

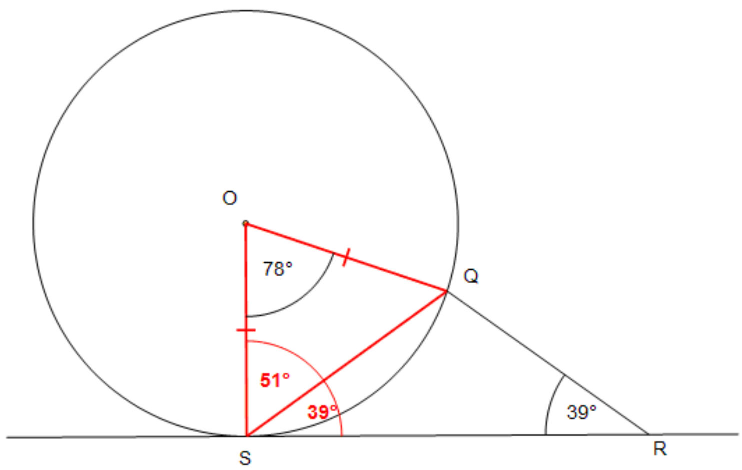


We know that Δ *SOQ* is isosceles because *OQ* and *OS* are radii, and therefore equal.

This makes ∠*OSQ* = ∠*OQS*, and as the angle sum of a triangle is 180°, ∠*OSQ* = ½(180-78)° = 51°.

But we are also told that *SR* is a tangent, so ∠*OSR* (the angle between the radius *OS* and the tangent *SR*) is a right angle.

But, ∠*OSR* =
∠*OSQ* + ∠*QSR* = 90°,
so ∠*QSR* = (90 – 51)° = 39°



∴ ∠*QSR* = ∠*QRS*, and so Δ *QSR* is isosceles.

Example (3): Which of the following quadrilaterals are cyclic ?

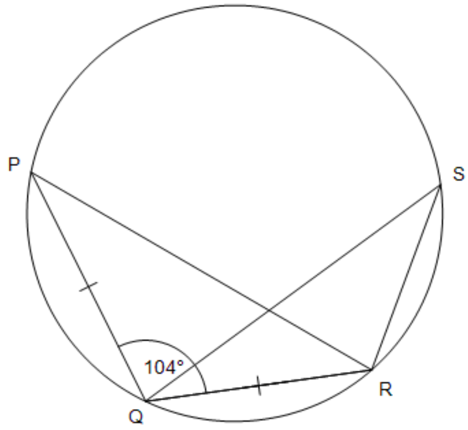
i) a rectangle; ii) a rhombus; iii) a square; iv) a kite with a pair of opposite right angles ?

A rectangle has all its angles equal to 90°, as does a square. Both are therefore cyclic, since opposite pairs of angles add to 180°.

The kite with a pair of opposite right angles is also cyclic for the very same reason.

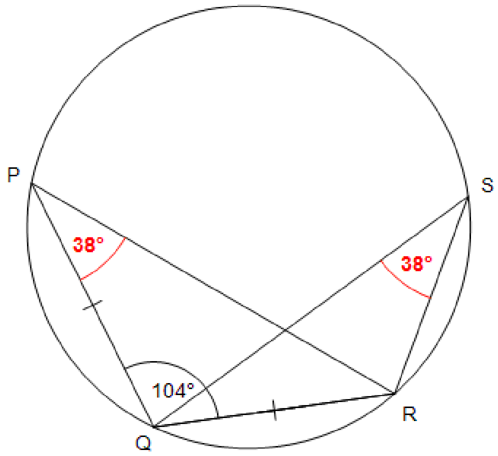
A rhombus has opposite pairs of angles equal, but because they are not generally right angles, a rhombus is not cyclic.

Example (4): Points P, Q, R and S lie on a circle.
$PQ = QR$, and angle $PQR = 104°$.
Explain why angle $QSR = 38°$.



If $PQ = QR$, then $\Delta\ PQR$ is isosceles, and hence $\angle\ QPR = ½(180 – 104)° = 38°$.

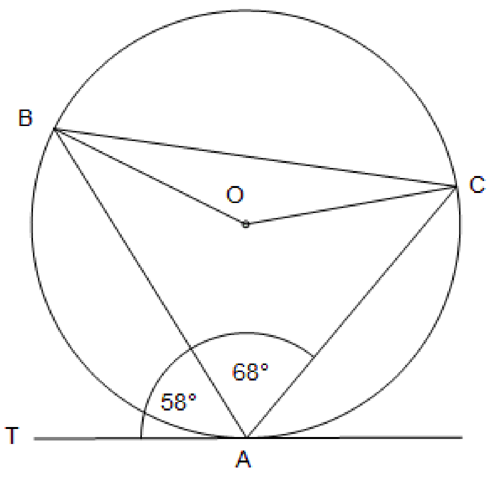
Also $\angle\ QPR$ and $\angle\ QSR$ are in the same segment, so $\angle\ QSR$ is also = 38° because angles in the same segment are equal.



Example (5):

Line *TA* is a tangent to the circle centred on *O*.
Angles *BAC* and *BAT* are 68° and 58° respectively.

Calculate angles *BOC* and *OCA*.



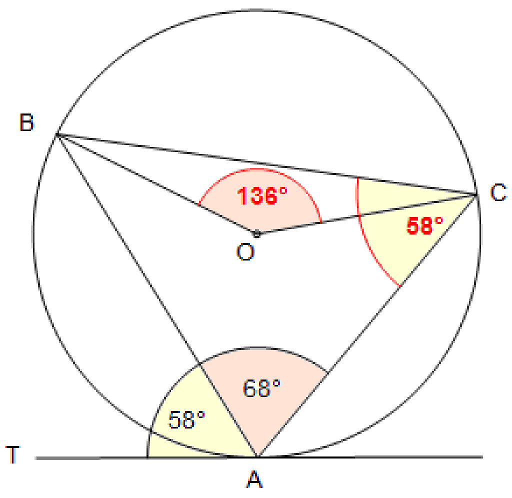
The angle at the centre is double the angle at the circumference, so ∠ *BOC* = 2 × 68° = 136°.

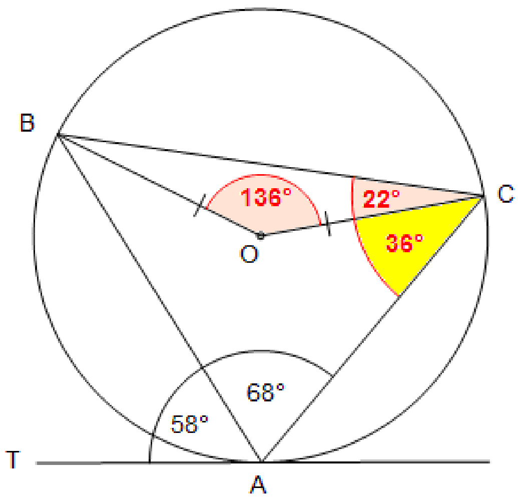
Then we use the alternate segment theorem to deduce that ∠ *BCA* = ∠ *BAT* = 58°.

This angle *BCA* is then divided into two smaller angles, namely *BCO* and *OCA*..

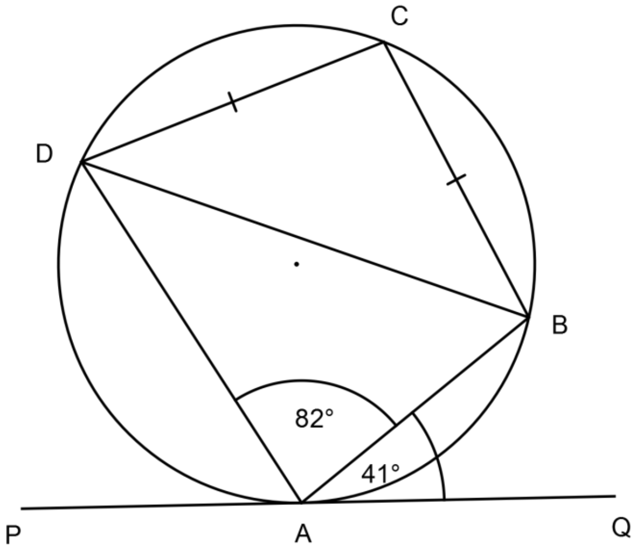
Δ *BOC* is isosceles, so ∠ *BCO* = ½(180-136)° = 22°.

Finally, angle *OCA* = ∠ *BCA* - ∠ *BCO*
= (58 – 22)° = 36°.





Example (6): The quadrilateral *ABCD* is cyclic, and *PAQ* is a tangent to the circle. Also, *BC* = *CD*, ∠ *QAB* = 41° and ∠ *BAD* = 82°.



Show that *AD* and *BC* are parallel, giving reasons for justification.

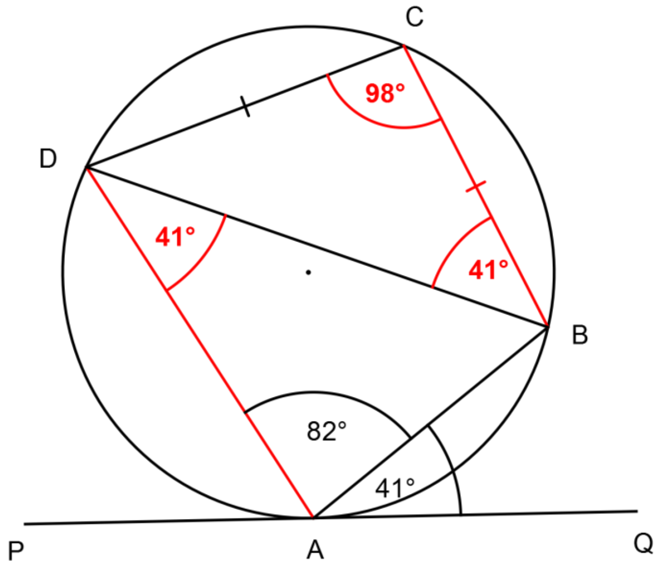
If lines *AD* and *BC* were parallel, angles *CBD* and *ADB* would form a pair of alternate angles and be equal to each other.

Angle *DCB* = (180-82)° = 98° because opposite angles of a cyclic quadrilateral sum to 180°.

Triangle *DCB* is isosceles, so ∠ *CBD* = ½(180-98)° = 41°.

By the alternate segment theorem, ∠ *ADB* = 41°.

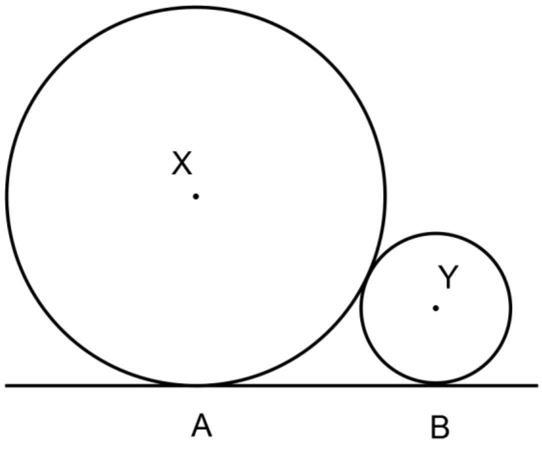
Angles *CBD* and *ADB* are indeed equal, so conversely lines *AD* and *BC* are parallel.



Example (7): The circles centred on X and Y have the tangent AB in common and have radii of 9 cm and 4 cm respectively.

i) Explain why $ABYX$ is a trapezium.

ii) Show that AB = 12 cm.



i) The tangents at A and B meet their respective radii AX and BY at right angles, so angles XAB and ABY are both equal to 90°, forming a pair of co-interior angles
Sides AX and BY are therefore parallel, i.e. $ABYX$ is a trapezium.

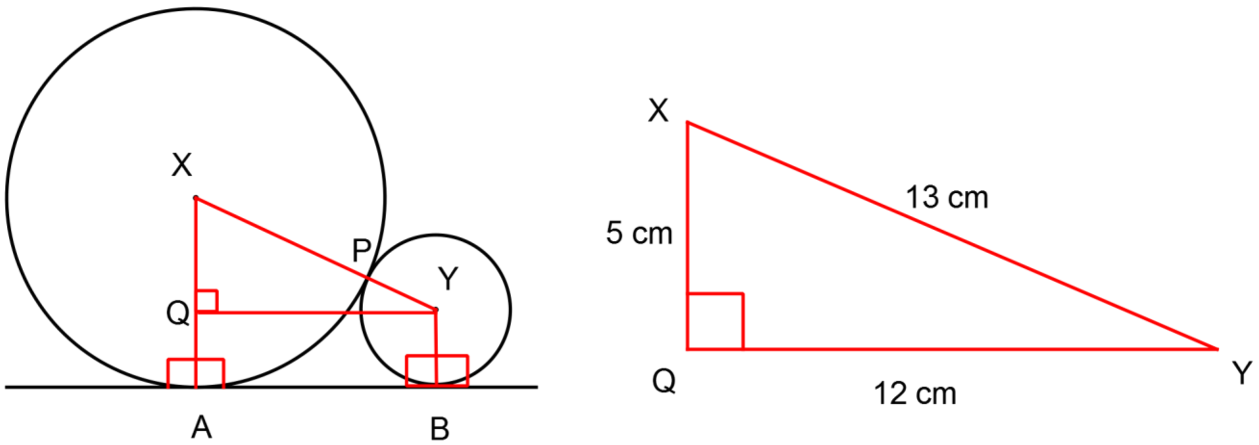
ii) We label a point Q on the radius AX such that $ABYQ$ is a rectangle.
Now XP = 9 cm and PY = 4 cm, so XY = 13 cm.
Also AX = 9 cm and BY = 4 cm, so QX = 9 – 4 = 5 cm.

The resulting triangle XQY has a hypotenuse of 13 cm and side QX = 5 cm.
We use Pythagoras to find the distance QY.

$$QY = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12 \text{ cm.}$$

Because $ABYQ$ is a rectangle, AB = QY.

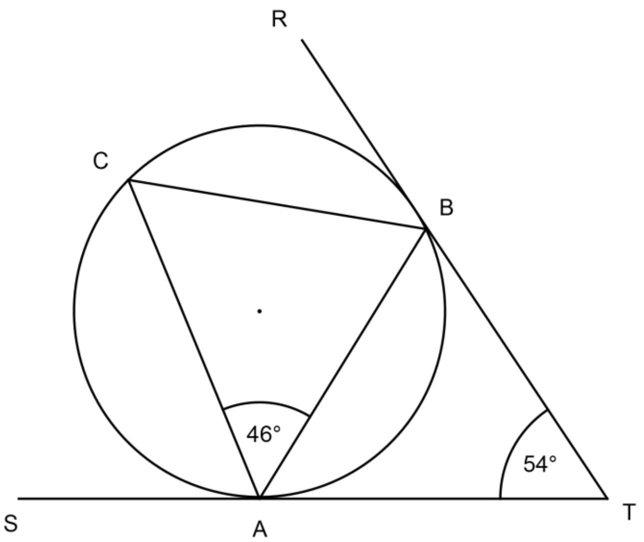
Hence the distance AB = 12 cm.



Example (8): The points *A*, *B* and *C* lie on the circumference of a circle.
The lines *SAT* and *RBT* are tangents to the circle at points *A* and *B* respectively.
These tangents meet at *T*.

Angle *CAB* = 46° and angle *BTA* = 54°.

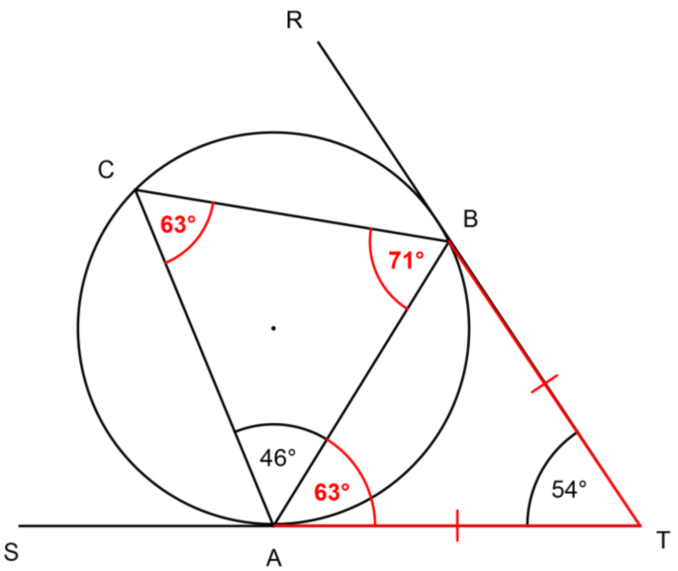
Find angles *BAT* and *ABC*, justifying your answers.



Tangents originating from the same point are equal in length so *TA* = *TB*.
Triangle *ATB* is therefore isosceles, where angles *BAT* and *ABT* = ½(180-54) = 63°.

Additionally, ∠ *ACB* = ∠ *BAT* = 63° by the alternate segment theorem.

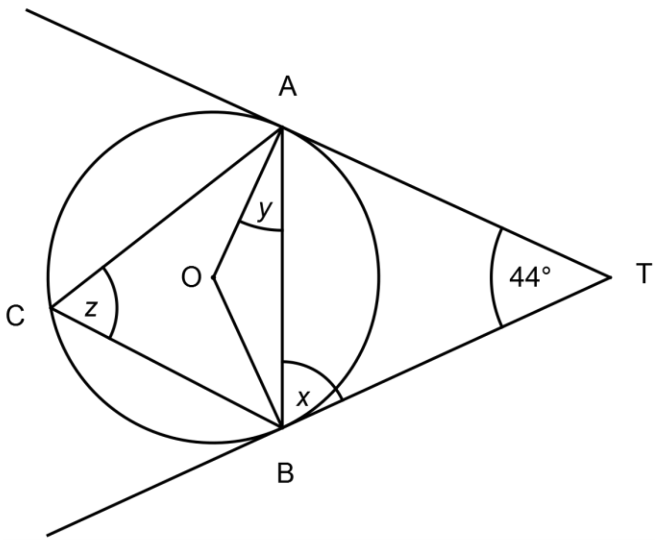
So ∠ *ABC* = (180 – (63 + 46))° = 71°.



Example (9): *A*, *B* and *C* are points on the circumference of a circle whose centre is *O*.

TA and *TB* are tangents to the circle
Angle *ATB* = 44°.

Find angles *x*, *y* and *z*, giving reasons for your answers.



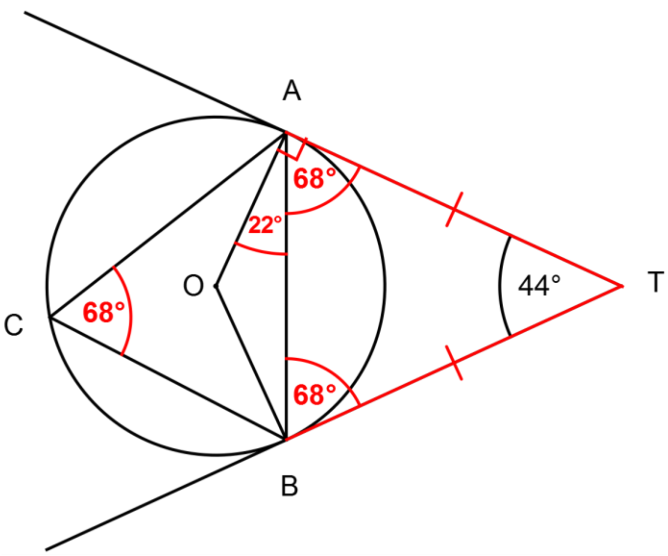
Tangents originating from the same point are equal in length so *TA* = *TB.*
Triangle *ATB* is therefore isosceles, where angles *BAT* and *ABT* (labelled *x*) = ½(180-44) = 68°.

Also ∠ *OBA* = ∠ *OAB* (labelled *y*) because Δ *AOB* is isosceles where *OB* and *OA* are radii. both radii), Additionally, ∠ *OBT* = 90° because the tangent and radius at *B* meet at right angles.

We then find *y* by subtraction, i.e. (90-68)° = 22°.

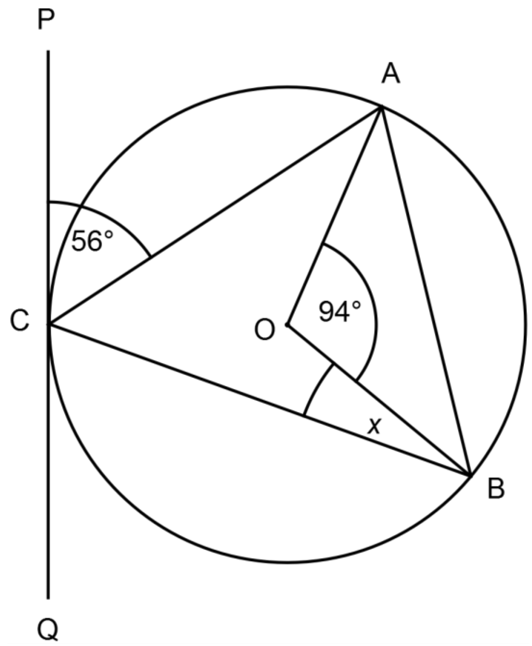
The angle *ACB* (labelled *z*) is equal to angle *x* by the alternate segment theorem, i.e. it is also 68°.

We could have also calculated ∠ *BOA* = (180 – 44)° = 136°. The size of angle *ACB* (labelled *z*) is half of that, or 68°, using the properties of angles at the centre.



Example (10): Points *A*, *B* and *C* lie on the circumference of a circle centred on *O*.
PQ is a tangent to this circle at *P*.
$\angle PCA = 56°$ and $\angle AOB = 94°$.

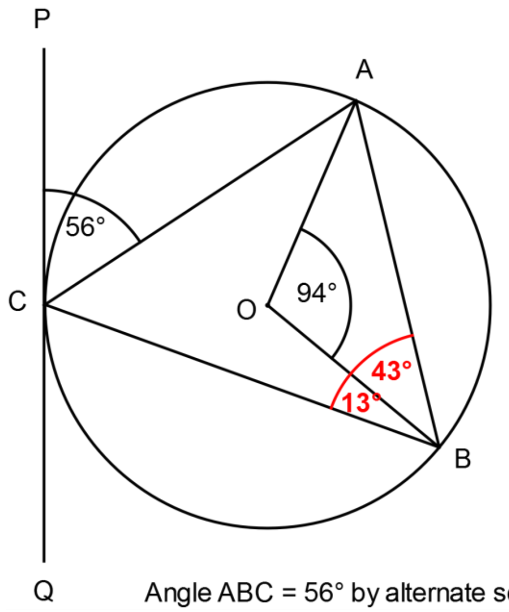
Calculate angle *OBC* (labelled *x*), showing all working.



We use the alternate segment theorem to deduce that $\angle ABC = 56°$. This angle is further divided up into angles *ABO* and *OBC*.

Since Δ *AOB* is isosceles,
$\angle ABO = ½(180\text{-}94) = 43°$.

Hence $\angle OBC$ (labelled *x*) = $(56 – 43)° = 13°$.

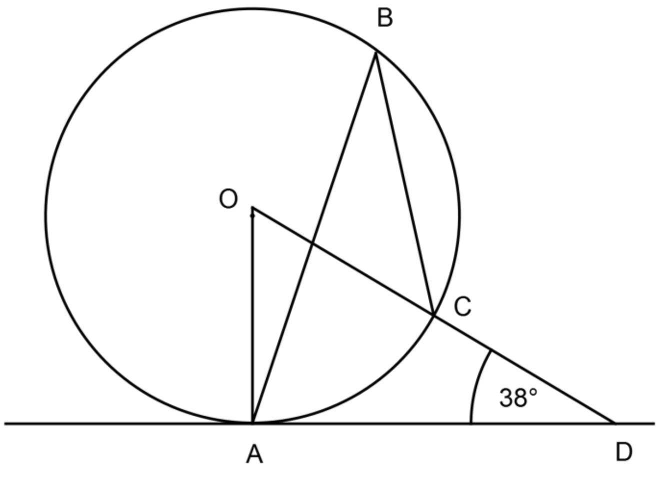


Angle ABC = 56° by alternate segment theorem

Example (11): The circle below is centred on *O*, and points *A*, *B* and *C* lie on its circumference.

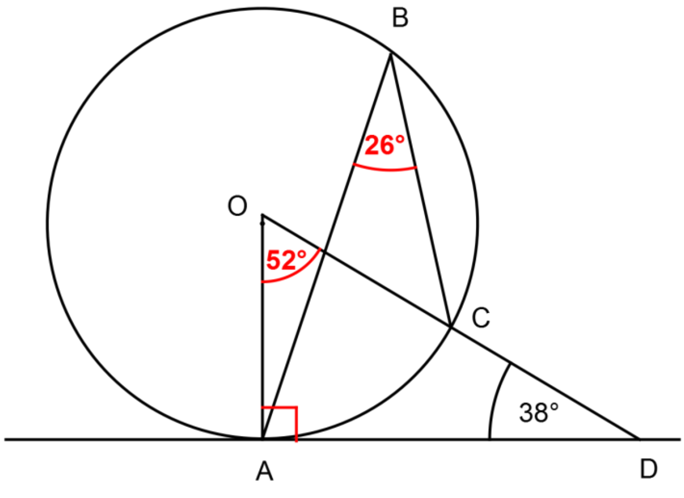
DCO is a straight line, *DA* is a tangent to the circle and $\angle ADO = 38°$.

i) Calculate angle *AOD.*
ii) Calculate angle *ABC*, giving full reasons for your answer.

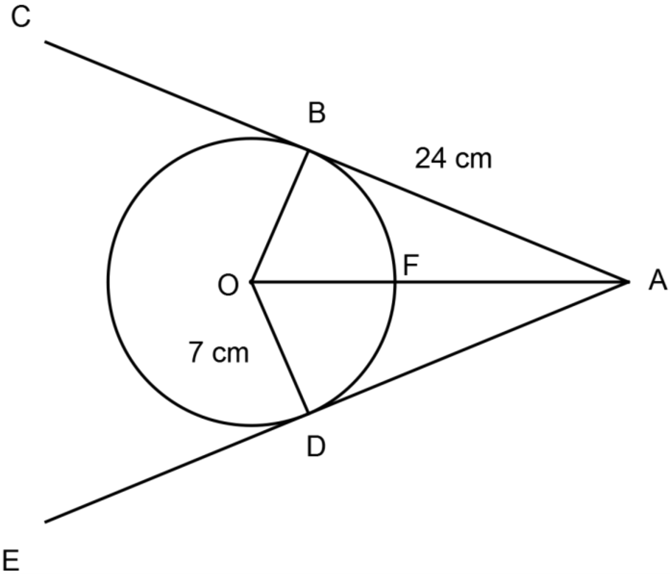


The tangent *AD* and the radius *OA* meet at 90° at point *A*, so $\angle OAD = 90°$.
Hence $\angle AOD = (90 - 38)° = 52°$

We then use properties of angles at the centre to deduce that the angle *ABC* at the circumference is half the angle *AOD* at the circumference, so $\angle ABC$ is half of 52° or 26°.



Example (12): The lines *ABC* and *ADE* are tangents to the circle centred on *O*. *OD* = 7 cm and *AB* = 24 cm.
F is the point where *AO* meets the circumference.
Work out the distance *AF*.



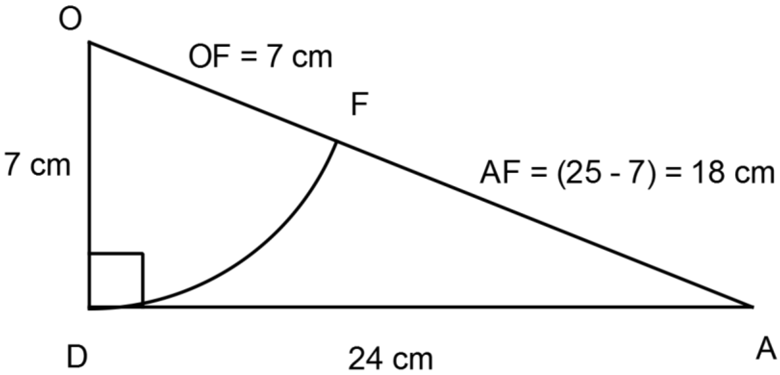
Tangents *AB* and *AD* are equal in length, as they originate from the same point *A*. Therefore *AD* = 24 cm.

Also, $\angle ODA$ = 90° because the tangent and radius at *D* meet at right angles.
Hence Δ *ODA* is right-angled.

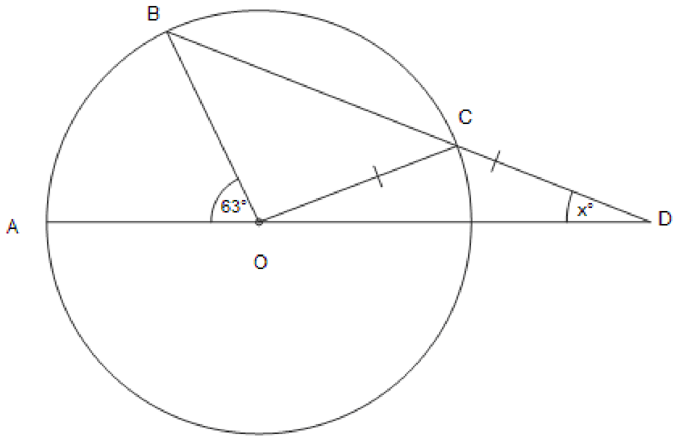
Using Pythagoras, we find that

$$OA = \sqrt{24^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25 \text{ cm.}$$

The distance *OF* is 7 cm, being a radius of the circle, so *AF* = *OA* – *AF* = (25 – 7) cm = 18 cm.



Example (13): In the circle below, $\angle AOB = 63°$.
The diameter through *A* and *O* is extended to point *D* such that *OC* = *CD*. Find angle *x*.



Because Δ *OCD* is isosceles, $\angle COD = x°$ and $\angle OCD = (180-2x)°$.

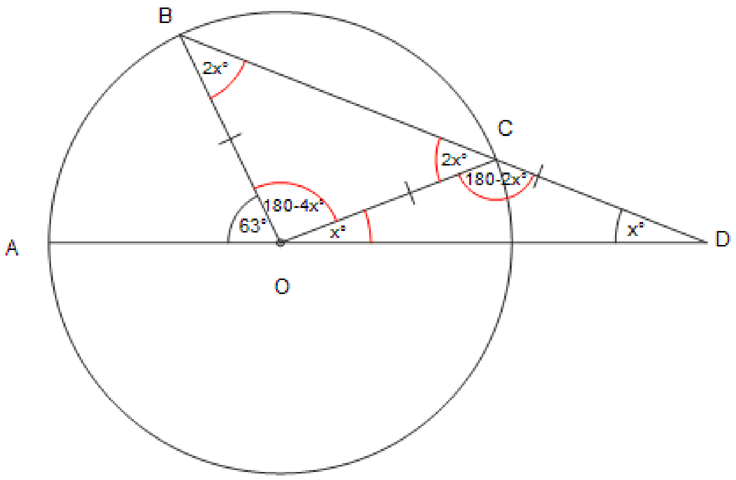
Hence $\angle BCO = 2x°$ (180° in a straight line), and, because Δ *OCB* is isosceles (*OB* , *OC* are radii) , it follows $\angle OBC = 2x°$ as well.

Hence $\angle BOC = 180-4x°$.

Since *AOD* is a straight line, the sum of angles *AOB, BOC* and *COD* must equal 180°. This leads to the linear equation

$63 + 180-4x + x = 180$, so
$63 - 3x = 0$, ∴ angle **$x = 21°$.**

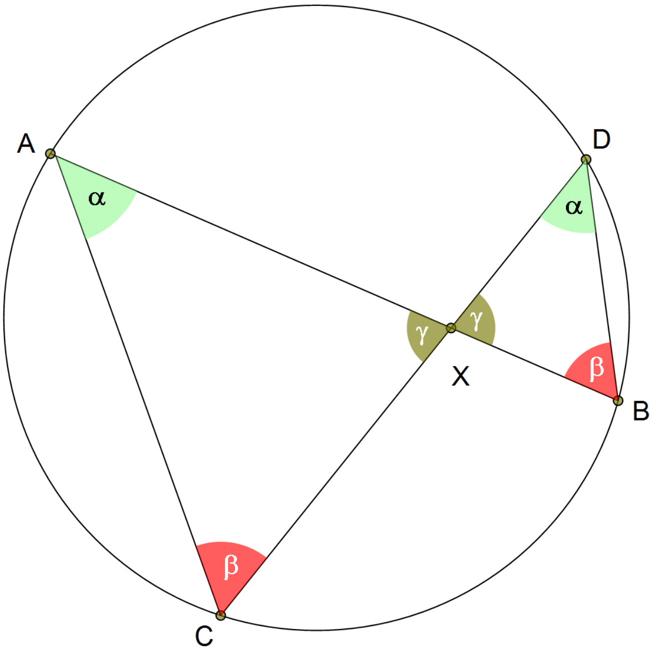
The angles of Δ *OCD* are thus 21°, 21° and 138°, while those of Δ *BOC* are 42°, 42° and 96°.



An important result (IGCSE).

AB and *CD* are two chords which intersect at point *X*, which is not necessarily the centre of the circle.

Prove that the products (*AX*)(*XB*) and (*CX*)(*XD*) are equal.



Proof.

Angles *CAX* and *XDB* (marked α) are equal because they are both subtended by the segment *CB*.

Angles *ACX* and *XBD* (marked β) are equal because they are both subtended by the segment *AD*.

Angles *AXC* and *DXB* (marked γ) are equal because they are vertically opposite.

From the three results above, the triangles *AXC* and *DXB* are similar.

Sides *CX* and *XB* correspond, as they are opposite the angles marked α.

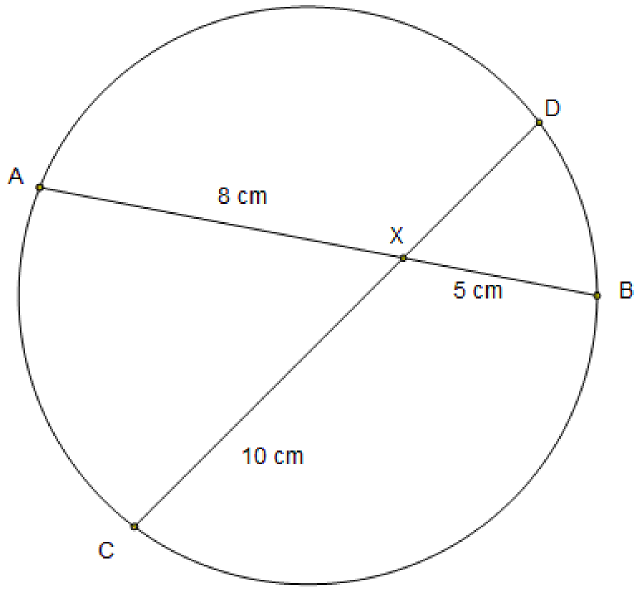
The ratio of their lengths, $\dfrac{CX}{XB}$, is a constant value *k*.

Sides *AX* and *XD* correspond, as they are opposite the angles marked β.

The ratio of their lengths, $\dfrac{AX}{XD}$, takes the same value *k*, as triangles *AXC* and *DXB* are similar.

Hence $\dfrac{CX}{XB} = \dfrac{AX}{XD}$, and cross-multiplying, (*AX*)(*XB*) = (*CX*)(*XD*).

Example (14): In the circle shown, *AX* = 8 cm, *XB* = 5 cm and *CX* = 10 cm. Find the length of the chord *CD*.



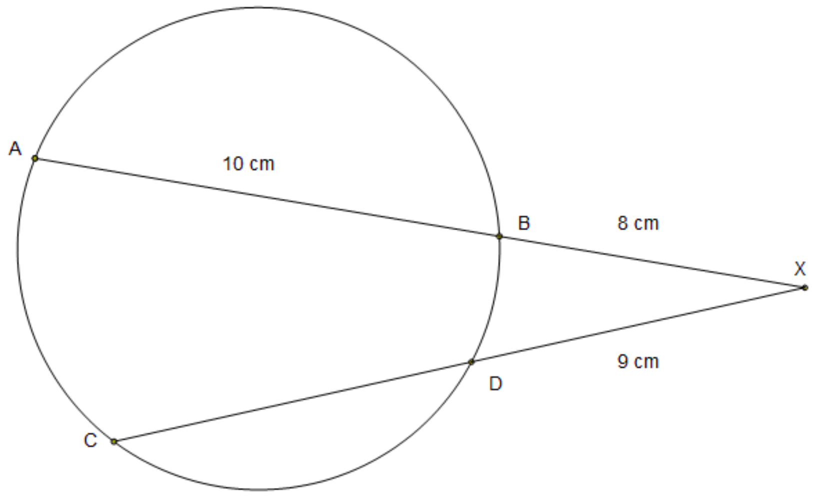
Now, (*AX*)(*XB*) = 8 × 5 = 40, so (*CX*)(*XD*) = 40.

Therefore *XD* = $\dfrac{40}{10}$ = 4 cm.

Thus the length of chord *CD* is 10 + 4 = 14 cm.

This rule also holds when the chords intersect outside the circle, as the following examples will show.

Example (15): In the circle shown, AB = 10 cm, XB = 5 cm and XD = 9 cm.
Find the length of the chord CD.



Now, $AX = 10 + 8 = 18$, so $(AX)(XB) = 18 \times 8 = 144$. Since $(CX)(XD) = 144$, $CX = \frac{144}{9} = 16$ cm.

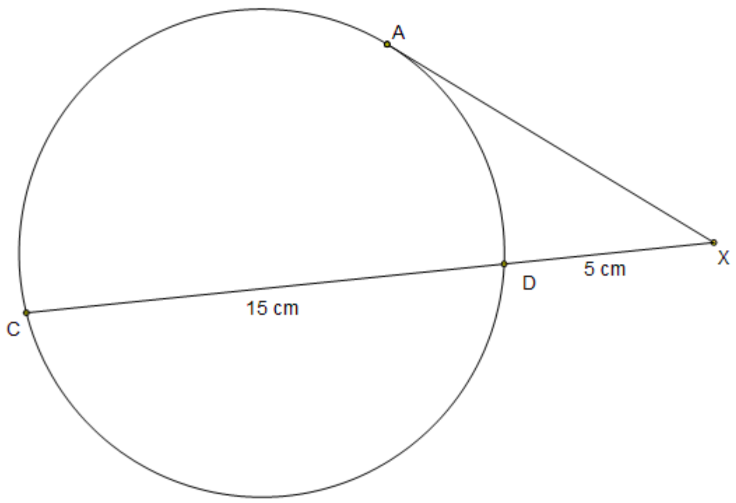
Hence the chord CD is 16 – 9, or 7 cm, long.

Additionally, if the chord AB degenerates into a tangent at point A, the product $(AX)(XB)$ becomes $(AX)(XA)$ or $(AX)^2$. In other words, we have $(AX)^2 = (CX)(XD)$.

Example (16): In the circle shown, CD = 15 cm and XD = 9 cm.
Find the length of the tangent AX.

Here, $CX = 15 + 5 = 20$, and $(CX)(XD) = 20 \times 5 = 100$.

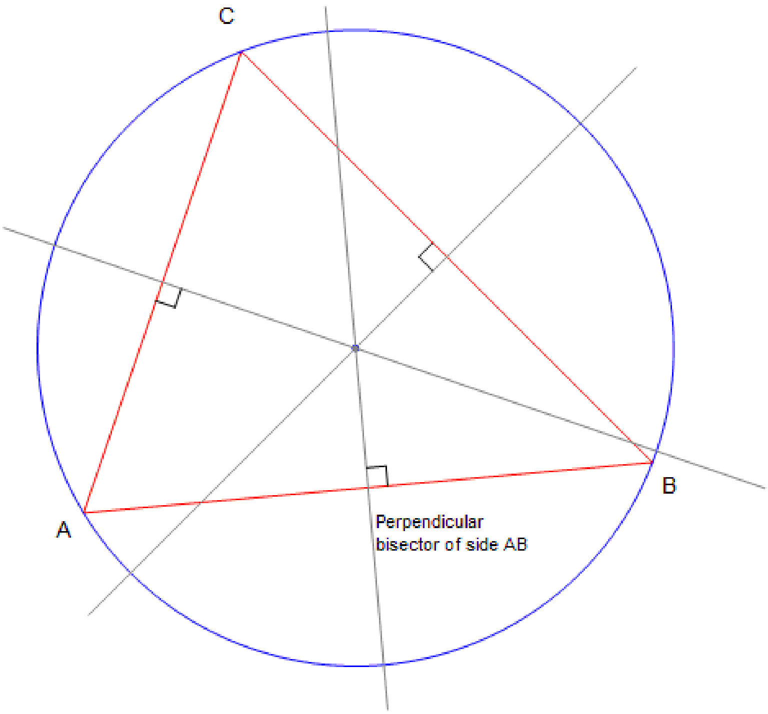
Since $(AX)^2 = 100$, $AX = \sqrt{100}$ or 10 cm, i.e. the tangent AX is 10 cm long.



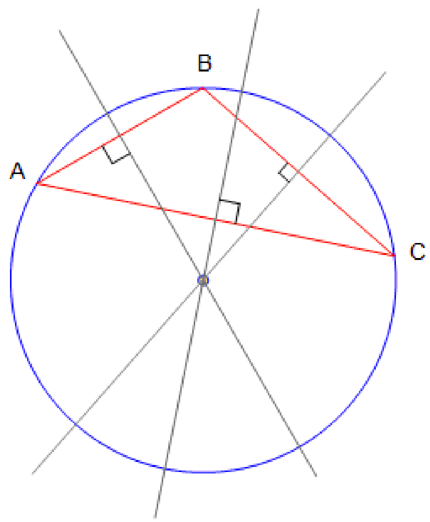
Circles and triangles.

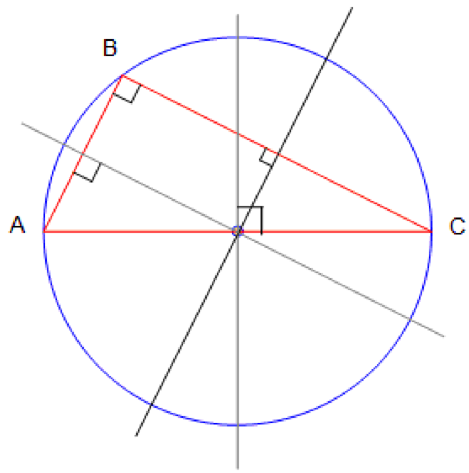
The **circumcircle** of a triangle is the circle passing through all of its vertices. Its centre is the intersection of the perpendicular bisectors of its sides.

If the triangle is acute-angled, the circumcentre will be inside the triangle.

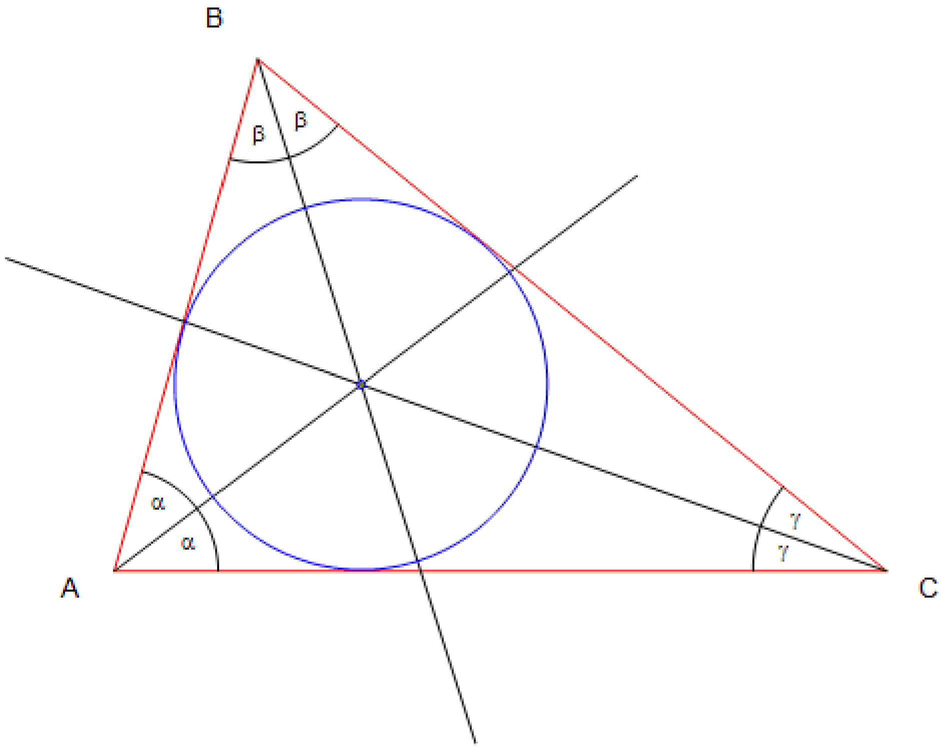


When we have an obtuse-angled triangle, the centre of the circumcircle lies outside the triangle, and with a right-angled triangle, the circumcentre lies on the midpoint of the hypotenuse.





The **incircle** of a triangle is the largest circle that can be inscribed within it. Its centre is the intersection of the bisectors of its angles, and the three sides of the triangle are tangents to it.



Note that the tangents to the incircle do not generally coincide with the angle bisectors.

To complete the discussion, a line joining the midpoint of a triangle's side to the opposite vertex is called a **median**. The three medians of the triangle meet at a point called the **centroid** of the triangle, which, incidentally, is also the triangle's centre of gravity.

The midpoints of sides AB, BC and CA are at points P, Q and R.

The medians are AQ, BR and CP and they intersect at the centroid X.