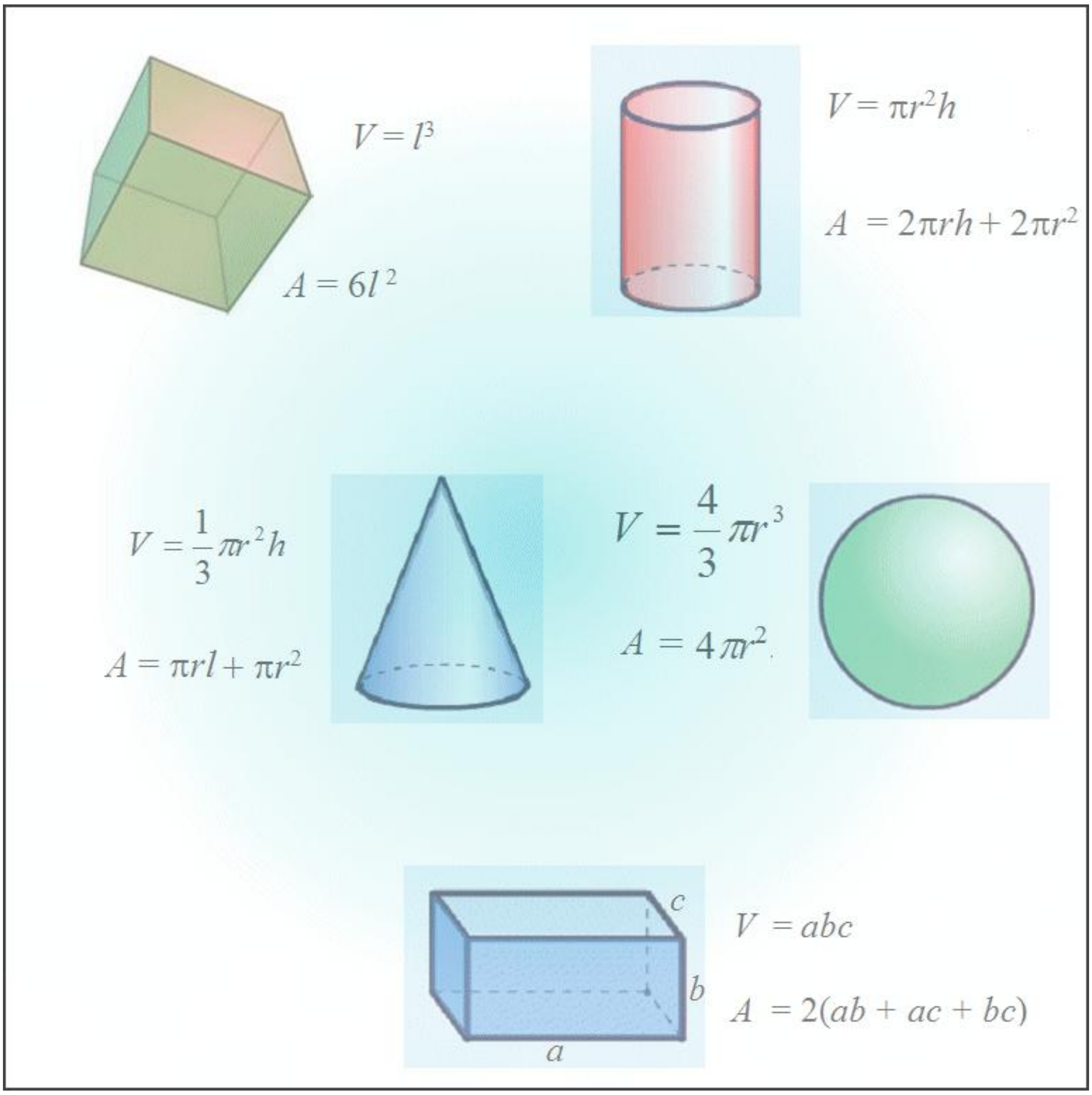


M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

SOLID SHAPES



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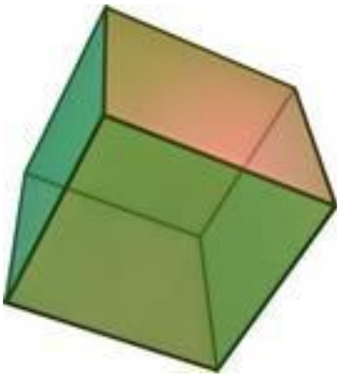
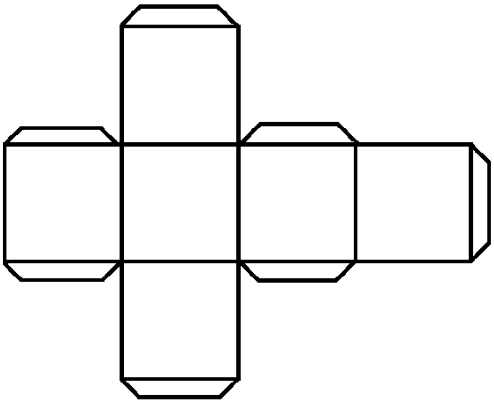
SOLID SHAPES

The cube.



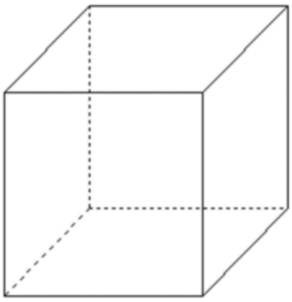


The cube is the most familiar of the 'regular' solids. It has 6 square **faces**, 8 corners or **vertices** (singular **vertex**) and 12 **edges**.

To the right of the figure of the cube is its **net**. The net of a solid can be formed by cutting along certain edges and unfolding the solid to give a flat 'map'.

To make a solid cube from the net, cut out the net, score the edges and cement the tabbed edges to the plain ones so that three faces meet at a vertex.

Because all the edges of a cube are equal in length and all the faces are squares, the formulae for its area and volume are especially simple:



Volume of cube = l^3 where l is the length of an edge.
Surface area of cube = $6l^2$, again where l is the length of an edge.

Example (1): Find the surface area and volume of a cube whose sides are 5 cm long.

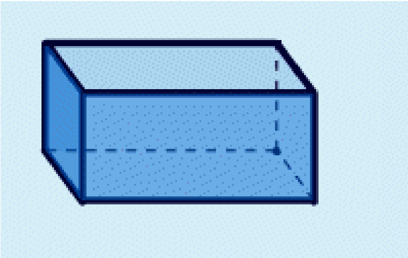
The volume of the cube is 5^3 cm^3 = 125 cm^3.
The surface area is 6×5^2 cm^2 = 150 cm^2.

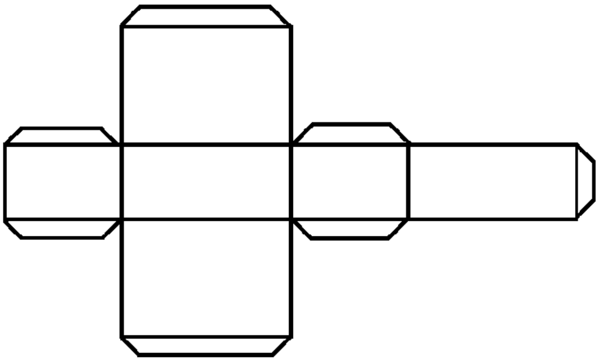
Example (2): An open cubic box (without lid) has an inside surface area of 3.2 m^2. Find its volume.

Because we are told that the box has no lid, there are **five** faces making up the surface area.
One face therefore has a surface area of 0.64m^2, and therefore a side of $\sqrt{0.64}$ metres or 0.8m.

The volume is therefore 0.8^3 m^3, or 0.512m^3, or 512 litres.

The cuboid.

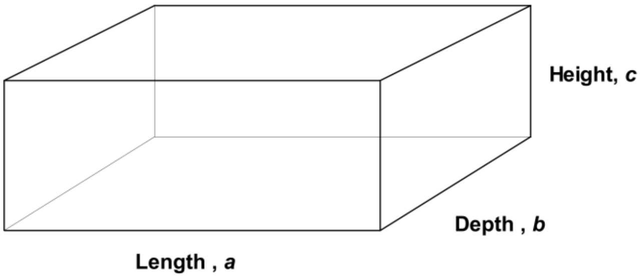




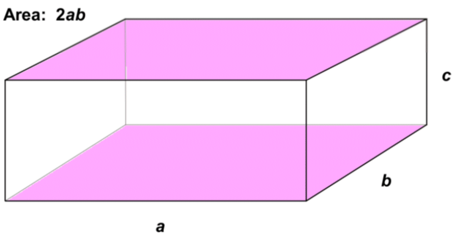
A **cuboid** is related to the cube, but is slightly less regular, with at least one pair of rectangles for faces rather than having six square faces like the cube.

Cuboids are very familiar in everyday life – most boxes are of that shape.

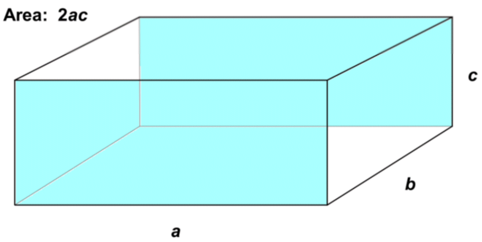
A cuboid has the same number of faces, vertices and edges as a cube, but now only opposite pairs of faces are equal.

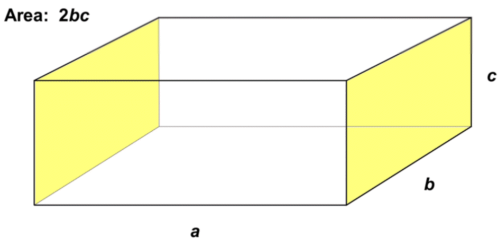


One such pair has a total area of $2ab$, (length × depth).



There are two other pairs, one with combined area $2ac$, (length × height) and a third pair with combined area $2bc$, (depth × height).





The formulae for the cube can be adapted as follows:

Volume of cuboid = abc where a, b and c are the length, depth and height.

Surface area of cuboid = $2(ab + ac + bc)$; again a, b and c are the length, depth and height.

Example (3): Find the surface area and volume of a cuboid whose dimensions are 8 cm × 5 cm × 4 cm.

The volume of the cuboid is 8 × 5 × 4 cm^3 = 160 cm^3.
The surface area is 2((8 × 5) + (8 × 4) + (5 × 4)) cm^2 = 2 × (40 + 32 + 20) cm^2 = 184 cm^2.

Example (4): A cuboidal room is 5 metres long, 4 metres wide and 2.5 metres high.
The ceiling and walls require two coats of paint, with a covering power of 12.5 m^2 per litre.
Work out the amount of paint needed after deducting 7 m^2 for doorways and windows.

Area of ceiling = length × width = 5 × 4 m^2 = 20 m^2.

Areas of walls = (perimeter × height) =
2 × ((length × height) + (width × height)) = 2 × ((5 × 2.5) + (4 × 2.5)) = 45 m^2.

This gives a total area of 65 m^2, but we must deduct 7m^2 for doorways and windows, giving the total area to be painted as 58 m^2.

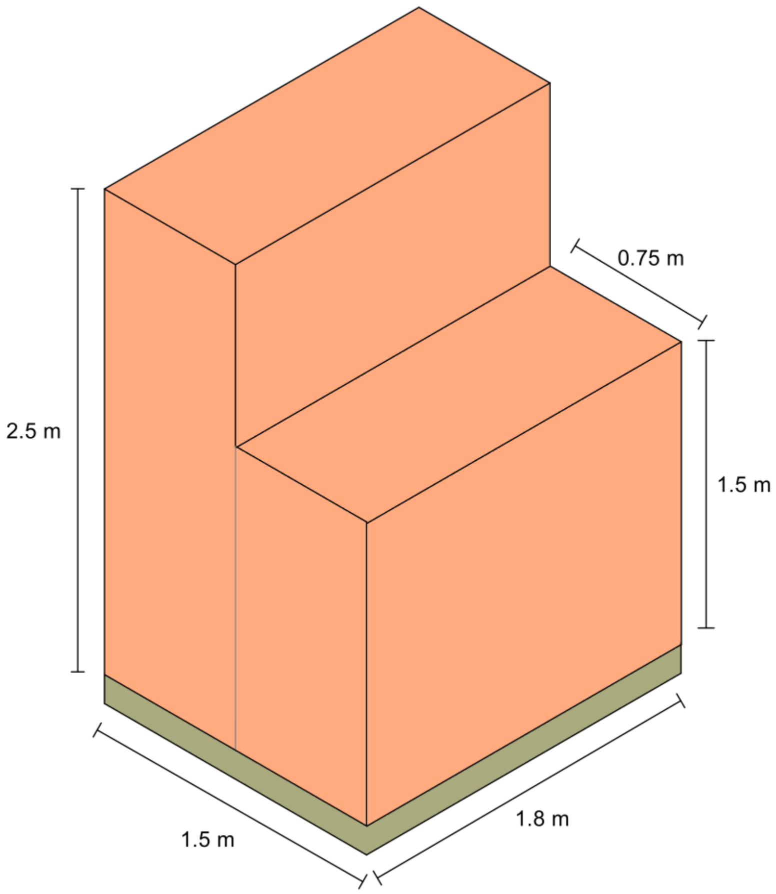
This area requires painting twice, so we need paint to cover 116 m^2. Since 1 litre of paint covers 12.5 m^2, we will need $\frac{116}{12.5}$ litres, or 9.28 litres.

Two 5-litre cans will therefore be enough !

Example (5): Mike works in a wholesale warehouse and has to build a promotional display stack of cases of drinks cans. The cases measure 45 cm x 30 cm x 12.5 cm.

He has a base plinth 1.8 metres long and 1.5 metres deep on which to stack the cases, so that the final display looks like this.

The cases are stacked to a height of 1.5 metres at the front half and 2.5 metres at the back half of the display.

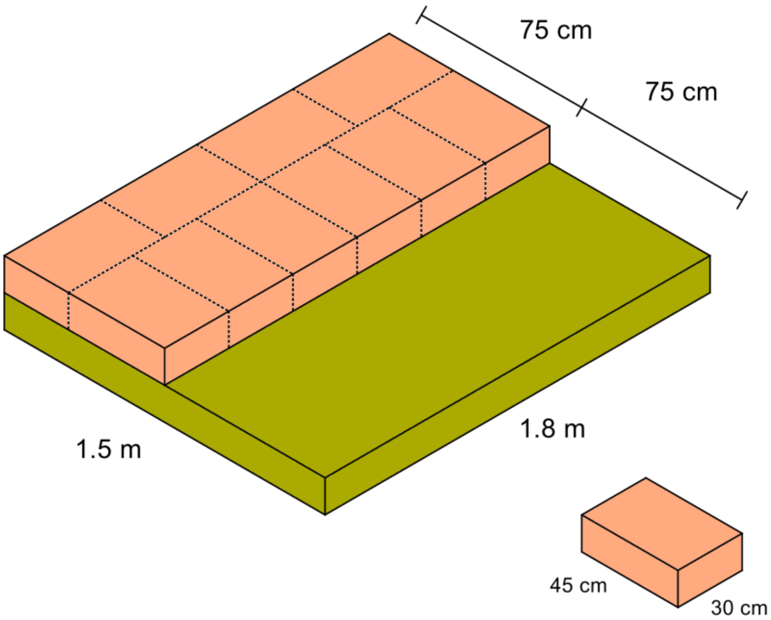


i) Show, with the aid of a diagram, that Mike can stack 20 cases per layer to cover the plinth, split horizontally in two groups of 10 cases, for the back and front of the stack.

ii) Hence, or otherwise, calculate the total number of cases in the stack.

i) The diagram shows how Mike can fit ten cases on one half of the plinth to form the bottom row of the rear half of the stack to form a layer 180 cm × 75cm . (We will use centimetres for consistency).

Note that 180 is a common multiple (though not the L.C.M.) of 45 and 30, as $4 \times 45 = 6 \times 30 = 180$. He can therefore arrange a row of four cases with their long sides forward and six cases with their short sides forward. Although 75 is neither a multiple of 45 or 30, it can be obtained by adding those two numbers, so there are no gaps in the layout.



ii) Treating the front and rear halves of the stack as separate cuboids, we know that there are 10 cases in each layer. To find the number of layers needed, we simply divide the height of the stack by the height of a single case.

For the rear half of the stack, we have $\frac{250}{12.5}$ or 20 layers of cases, making 200 cases.
(We are working in centimetres here).

For the front half of the stack, we have $\frac{150}{12.5}$ or 12 layers of cases, making 120 cases.

Mike needs 200 + 120, or 320 cases, to build the stack.

We simplified the working in part ii) by using the fact that 10 cases made up a layer.

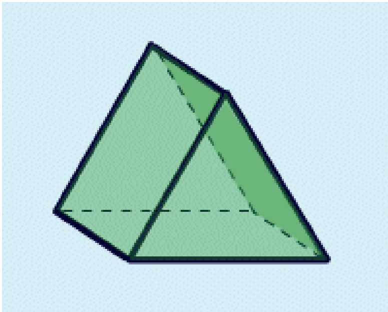
Alternatively, we could have worked out the volume of the whole stack and divided by the volume of a single case.

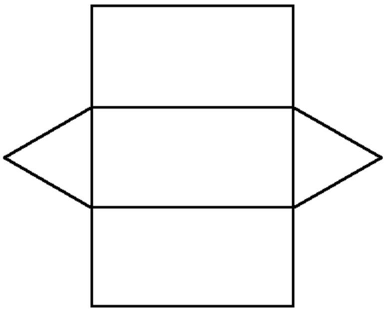
The rear half of the stack has a volume of $(2.5 \times 1.8 \times 0.75)$ m^3 = 3.375 m^3 and the front half has a volume of $(1.5 \times 1.8 \times 0.75)$ m^3 = 2.025 m^3. The total volume of the stack is (3.375 + 2.025) = 5.4 m^3.

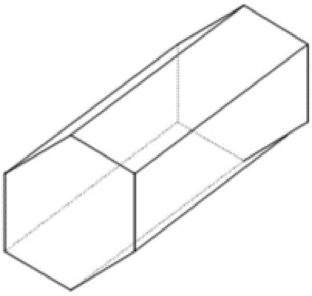
The volume of one case is $(0.45 \times 0.3 \times 0.125)$ m^3 ,

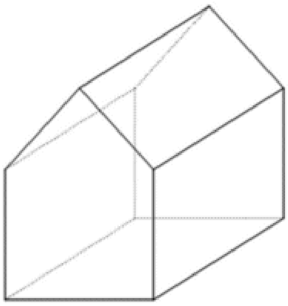
so the stack contains $\frac{5.4}{0.45 \times 0.3 \times 0.125} = 320$ cases.

The prism.









A **prism** is any solid of constant cross-section.

The side faces are all rectangles (they can be squares) and the end faces are congruent.

(The net above is of a triangular prism, seen in the physics lab and a certain brand of Swiss chocolate !)
The hexagonal prism is found in the honeycomb, and in an unused pencil.

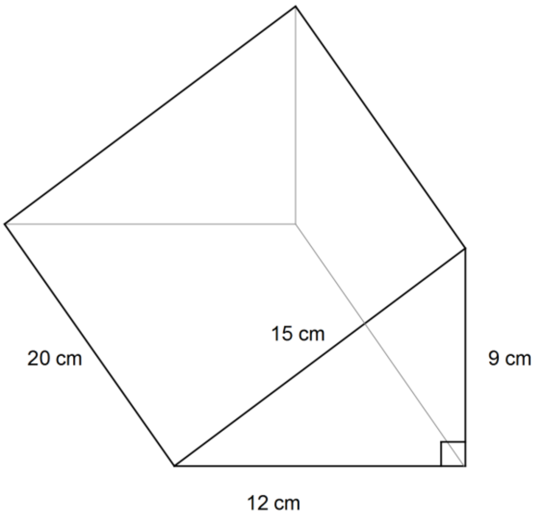
To find the volume of a prism, we multiply the cross-sectional surface area by the length.

Volume of prism = surface area of end face × length.

Example (6a): A triangular prism has two right-angled triangles for its end faces, and is 20 cm long. The sides of the end faces are 9 cm, 12 cm and 15 cm. Find the volume of the prism.

Because the triangular end faces are right-angled, with a base of 12 cm and a height of 9 cm, the surface area of a single face = $\frac{1}{2} \times 12 \times 9$ cm^2 = 54 cm^2.

Since the length of the prism is 20 cm, its volume is therefore 54×20 cm^3 = **1080 cm^3.**



Example (6b): Find the total surface area of the triangular prism in example (6a).

We have already calculated the area of a single triangular end face as 54 cm^2. The two of them thus have a total area of 108 cm^2.

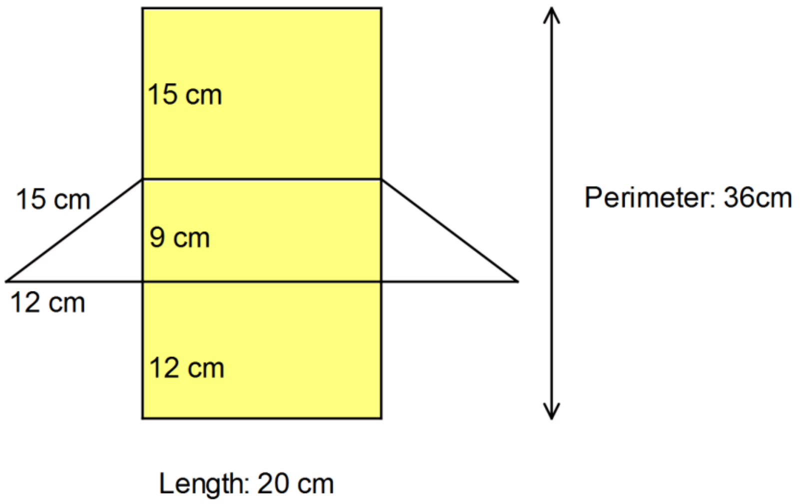
We now calculate the areas of the rectangular side faces.
The base has an area of 12×20 $\text{cm}^2 = 240$ cm^2.
The vertical face has an area of 9×20 $\text{cm}^2 = 180$ cm^2.
The sloping face has an area of 15×20 $\text{cm}^2 = 300$ cm^2.

The three side faces have a combined area of $(240 + 180 + 300)$ $\text{cm}^2 = 720$ cm^2.
Adding the area of the end faces gives the total surface area of the prism as 720 + 108, or **828 cm^2.**

Some of the calculations could have been simplified by adding the side lengths of the triangular end faces to give the perimeter, and then multiplying the perimeter by the length of the prism.

The three side faces have a combined area of $(12 + 9 + 15) \times 20$ $\text{cm}^2 = 720$ cm^2.

The net of the prism shows how we could treat the three rectangular side faces as if they were a single rectangle, of dimensions (length × perimeter of ends).

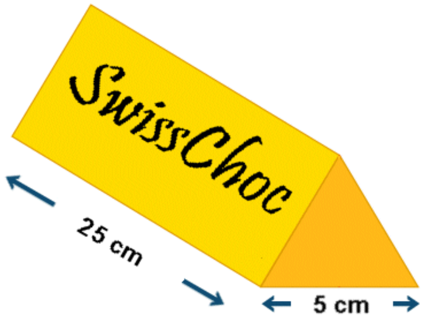


Hence: Surface area of prism = (2 × surface area of ends) + (length × perimeter of ends).

(The perimeter of the end face on the net is equal to the combined widths of the rectangular side faces).

In the case of the “circular prism”, otherwise known as the cylinder, the perimeter is the circumference.

Example (7) : SwissChoc bars are sold in cardboard boxes as shown on the right. The box is a prism whose end faces are equilateral triangles.



Given that the surface area of each triangular end face is 10.8 cm^2,

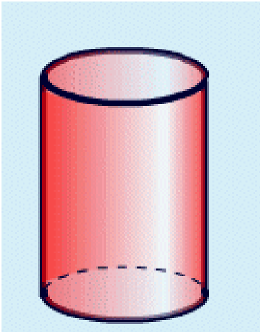
i) calculate the volume of the box to the nearest cm^3;
ii) calculate its total surface area to the nearest cm^2.

i) Since the volume of a prism = surface area of end face × length, the volume of the container is (10.8×25) cm^3 = **270 cm^3**.

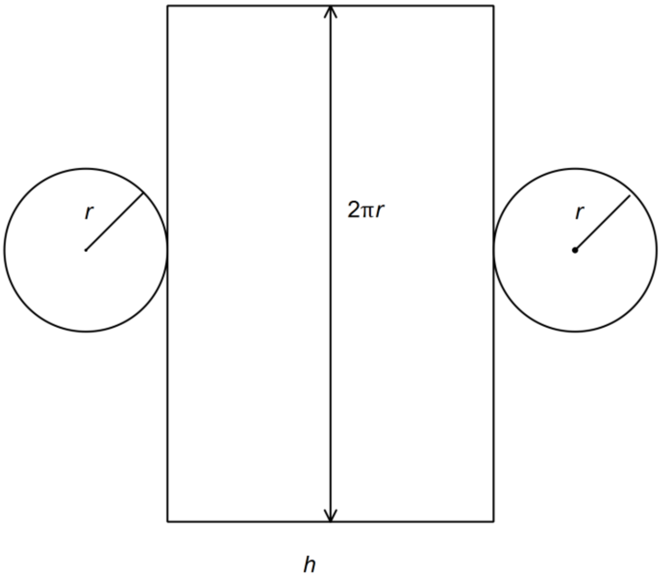
ii) The two end faces have a total area of 2×10.8cm^2 = 21.6 cm^2.
The three side faces have a combined area of $(3 \times 5) \times 25$ cm^2 = 375 cm^2.

Therefore the total surface area of the prism is $(21.6 + 375)$ cm^2 = **397 cm^2**.

The cylinder.



A **cylinder** is a special case of a prism, where the end faces are circles.



A cylinder has two end faces, and a single continuous curved "face", like the label around a can. If we were to cut such a label, its length would be equal to the perimeter, or circumference, of the circle and its height would be the same as that of the cylinder.

See the net of a cylinder of radius r and height h shown right. Remember that the circumference is given by the formula $C = 2\pi r$.

Thus, the total surface area of cylinder = $2\pi rh + 2\pi r^2$ or $2\pi r(r + h)$.

The "$2\pi rh$" refers to the curved 'side' and the "$2\pi r^2$" refers to the circular ends.

Example (8): A cylinder has a height of 12 cm, and each end face has a radius of 5 cm. Calculate the volume and total surface area, leaving the results in terms of π.

The volume is given by the formula $V = \pi r^2 h$, where $r = 5$ and $h = 12$.
Hence the cylinder has a volume of $(\pi \times 5^2 \times 12)$ cm^3 or **300π cm^3**.

The two circular end faces have a total area of $2\pi r^2$ or $(2\pi \times 5^2)$ cm^2 or 50π cm^2.
The curved "side" has an area of $2\pi rh$ or $(2\pi \times 5 \times 12)$ cm^2 or 120π cm^2.

The total surface area of the cylinder is 50π + 120π cm^3 = **170π cm^2**.

Example (9): A cylindrical drum is 1.2m tall and has a diameter of 0.8m. Find its capacity in litres. (1000 litres = 1 cubic metre.)

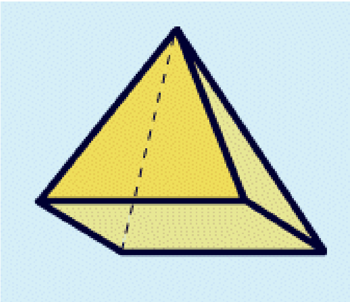
The radius is half the diameter, i.e. 0.4m, and so the volume of the cylinder is given by
$\pi \times 0.4^2 \times 1.2$ m^3 = 0.603m^3 or 603 litres.

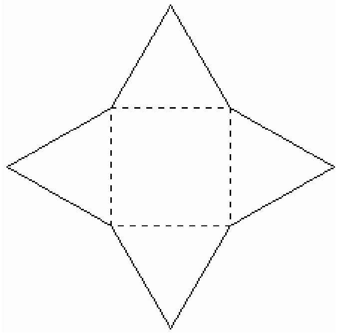
Example (10): A gas storage cylinder is 12m tall with a diameter 9.5m.
If 1 litre of paint covers 12m^2, how much paint will be required to paint its top and side ?

The area of the curved side is $2\pi rh$ m^2, or $2\pi \times 12 \times 4.75$ m^2, or 358m^2.
The area of the top is $\pi \times 4.75^2$ m^2 or 71m^2.

∴ The total area to be painted is 429m^2, requiring about 36 litres of paint.

The pyramid.

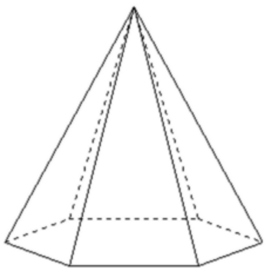




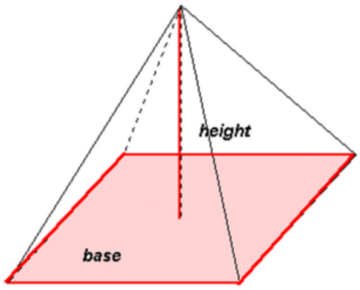
A **pyramid** is a solid with a polygonal base and sloping triangular faces, which all meet at an apex. Although the example above (with its net) is regular, with a square base and all its triangular faces equilateral, this does not need generally need to be the case – see the hexagonal example below.

The formula for the volume of a pyramid is $V = \frac{1}{3}Ah$ where *A* is the area of the base and *h* is the **perpendicular** height (see diagram right).

This holds true whether the pyramid is upright or slanted.

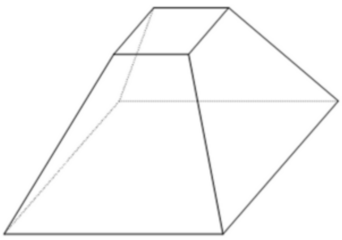


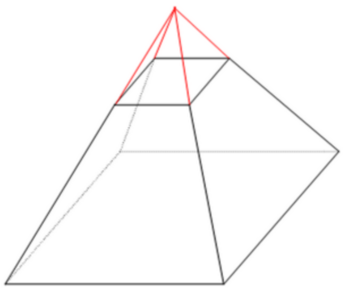
Hexagonal pyramid



If we were to make a cut parallel to the base of the pyramid, we will have two resulting solids, one of which will be a smaller, similar pyramid.

The other section, minus the smaller pyramid, is called a **frustum** (see left).



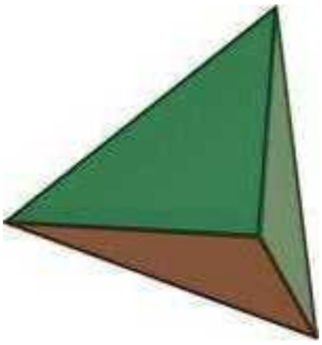


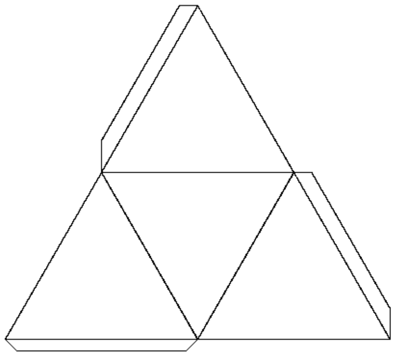
As the section removed is itself a pyramid similar to the main one , the formula for the volume of a frustum can be given as

$V = \frac{1}{3}(AH - ah)$ where *A* and *H* are the base area and (perpendicular) height of the large pyramid, and *a* and *h* are the base area and height of the smaller missing section.

If the height of neither pyramid is given, then we must apply ratios to the height of the frustum.

The tetrahedron.





A tetrahedron is a special case of a pyramid with a triangular base.

A regular tetrahedron (shown above with its net) has 4 equilateral triangles for its faces, with 4 vertices and 6 edges.

Because a tetrahedron is a pyramid, the formula for its volume is $V = \frac{1}{3}Ah$ where *A* is the area of the base and *h* is the perpendicular height.

Example (11): Candles are made in the shape of a square-based pyramid.
Their bases are 10cm square and their heights are 18cm.

How many such candles can be made out of a cuboidal block of wax measuring 1m × 0.6m × 0.4m, assuming no wastage of wax ?

The volume of the cuboidal block is 100 × 60 × 40 cm^3 = 240,000 cm^3 (Use consistent units)

The base area of one candle is 10 × 10 = 100 cm^2, and its height is 18 cm, and so the volume of wax used per candle is $V = \frac{1}{3} \times 100 \times 18 = 600$ cm^3.

$\therefore$ The number of candles that can be made from the block is $\frac{240000}{600} = 400$.

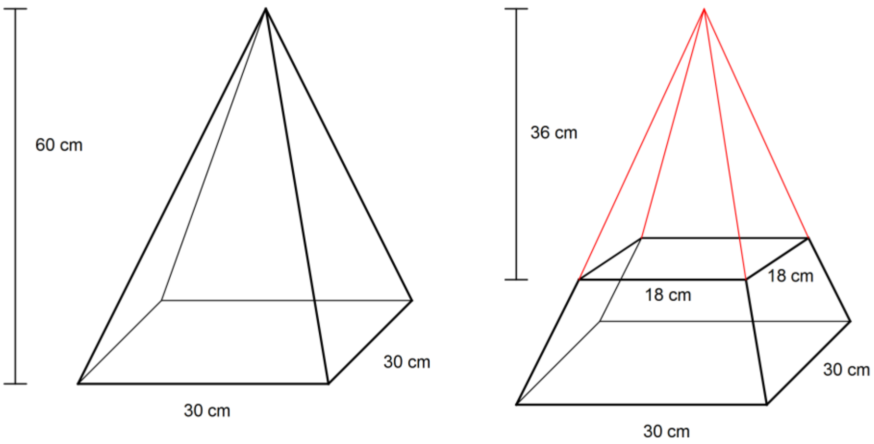
Example (12): A square pyramid has a height of 60 cm and a base side length of 30 cm. A cut is made parallel to the base of the pyramid such that a smaller pyramid of base side length of 18 cm is removed.

Find the volume of the remaining frustum in litres to 3 significant figures.

The base area A of the original pyramid is 30 × 30 = 900 cm^2, and its height H is 60 cm.

The smaller pyramid has a base of side 18 cm, which is $\frac{18}{30} = \frac{3}{5}$ of the base of the original pyramid.

Its height, h, is thus $\frac{3}{5} \times 60 = 36$ cm, its base area, a, is 18 × 18 = 324 cm^2.



Original pyramid *Resulting frustum*

The volume of the frustum is thus $V = \frac{1}{3}(AH - ah)$ where A = 900, H = 60, a = 324 and h = 36.

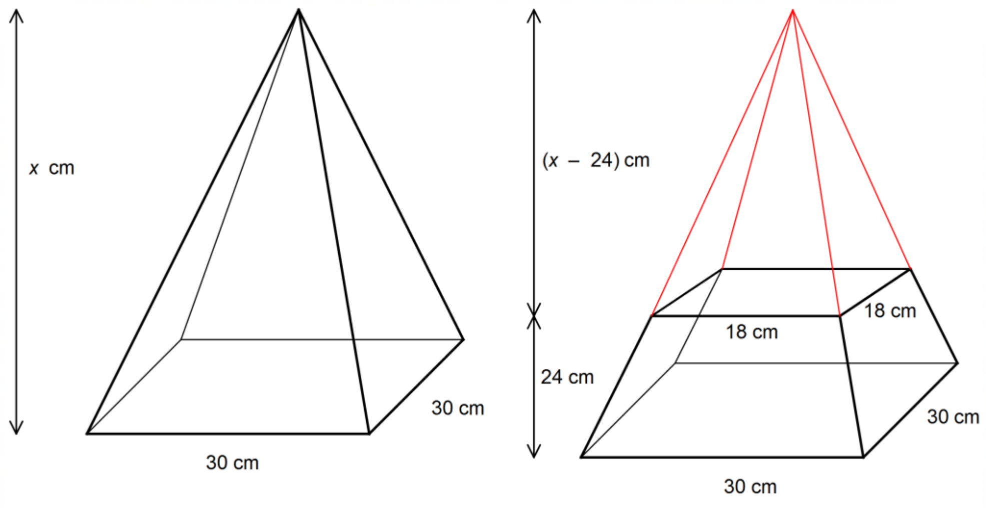
$$V = \frac{1}{3}(AH - ah) = \frac{1}{3}\left((900 \times 60) - (324 \times 36)\right) = \frac{1}{3}(54000 - 11664) = 14112 \text{ cm}^3$$

or **14.1 litres** to 3 significant figures.

Example (12a – variation) A square pyramid has a base side length of 30 cm. A cut is made parallel to the base of the pyramid such that a smaller pyramid of base side length of 18 cm is removed.

Given that the height of the resulting frustum is 24 cm, find its volume in litres to 3 significant figures.

The working is slightly different here, because we are not given either the height of the original pyramid, nor that of the smaller removed pyramid.



Original pyramid *Resulting frustum*

We need some additional algebra, so let x be the height of the original pyramid.
Since the frustum has a height of 24 cm, the removed pyramid has a height of (x – 24) cm.

The pyramids are similar, so their heights and bases will be in the same ratio, or in other words, the ratios $x : 30$ and $(x - 24) : 18$ will be equivalent.

Therefore $\frac{x}{30} = \frac{x-24}{18}$, or $18x = 30(x - 24)$, rearranging to $18x = 30x - 720$, then $12x - 720 = 0$

and finally to $x = 60$.

The height of the large pyramid is thus 60 cm, and that of the removed pyramid is (60-24) cm, or 36 cm. These heights will hence be denoted respectively by H and h to use in the formula.

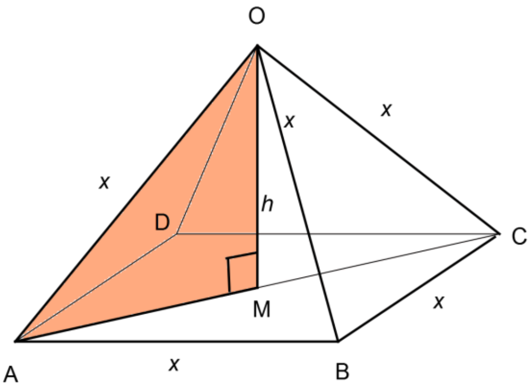
The base area A of the original pyramid is 30×30, or 900 cm^2, and the base area a of the smaller pyramid is 18×18, or 324 cm^2.

The volume of the frustum is thus $V = \frac{1}{3}(AH - ah)$ where $A = 900$, $H = 60$, $a = 324$ and $h = 36$.

$$V = \frac{1}{3}(AH - ah) = \frac{1}{3}((900 \times 60) - (324 \times 36)) = \frac{1}{3}(54000 - 11664) = 14112 \text{ cm}^3$$

or **14.1 litres** to 3 significant figures.

Example (13) (Harder):



This example includes work with surds, so revise that section if necessary.

A pyramid has a square base *ABCD*, and the four sloping triangular faces are all equilateral, meeting at *O*. The distance *OM* is the vertical height.
The.point *M* is at the centre of the base,.and the edges of the pyramid are of length *x*.

i) Find the exact length of the diagonal *AC* and state the length *AM*.
ii) Hence find an expression for the vertical height *h* in terms of *x*, in an exact form.
iii) From part ii), find an expression in terms of *x*, again in exact form, for the volume of the pyramid.

iv) Calculate the volume of a square-based pyramid whose edges are all 12 cm long, giving the result to the nearest cubic centimetre.

i) The diagonal *AC* can be found by Pythagoras : $AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}\,x$

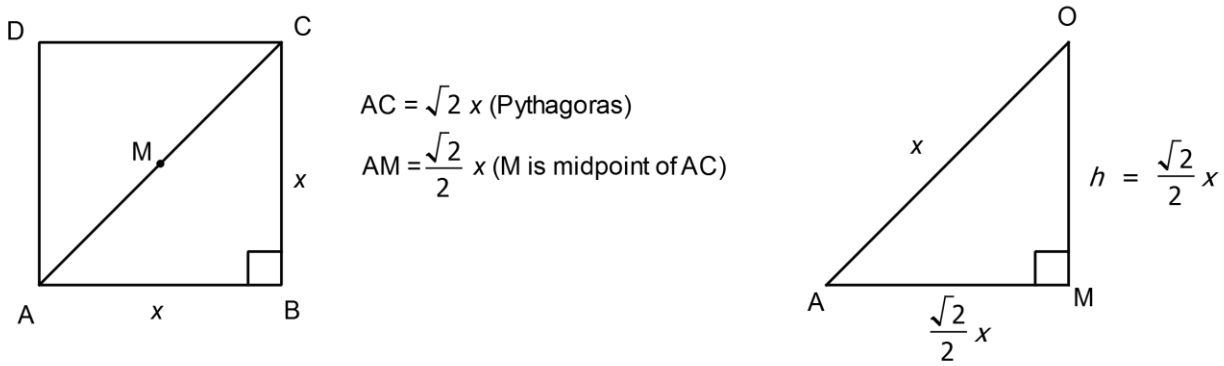
Also, *AM* is half *AC*, so its length is $\frac{\sqrt{2}}{2}x$.

ii) Applying Pythagoras again, the vertical height *OM* (also labelled *h*) is:

$$OM = \sqrt{(OA)^2 - (AM)^2} = \sqrt{x^2 - \frac{1}{2}x^2} = \sqrt{\frac{1}{2}x^2} = \frac{1}{\sqrt{2}}x = \frac{\sqrt{2}}{2}x.$$

(Rationalise the denominator in the last step !)

Note how *h* = *OM* = *AM*, with triangle *OMA* isosceles.



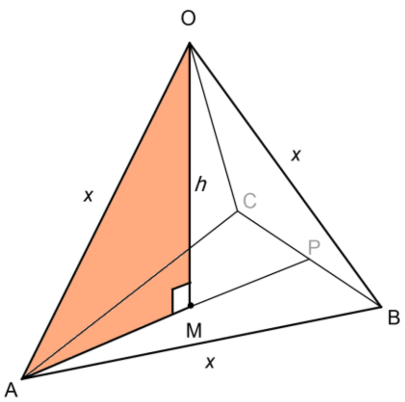
iii) The base area *A* of the pyramid is simply x^2, and its vertical height is $h = \frac{\sqrt{2}}{2}x$.

Its volume is $V = \frac{1}{3}Ah$ or $V = \frac{1}{3}x^2\left(\frac{\sqrt{2}}{2}x\right) = \frac{\sqrt{2}}{6}x^3$.

iv) Using the formula $V = \frac{\sqrt{2}}{6}x^3$ and setting *x* = 12 cm, the volume of the pyramid is

$V = \frac{\sqrt{2}}{6} \times 12^3 = 288\sqrt{2}$ = **407 cm³** to the nearest cubic centimetre.

Example (14) (Harder, Grade 9) : The figure on the right is a regular tetrahedron whose edges are all of length *x*.
Its base is the face *ABC*, its apex is at *O*, and its vertical height is *OM*.
The line *AP* connects point *A* to the midpoint of *CB*, and we are given that *AM* is two-thirds of *AP*.



Again, there is a lot on surds here !

i) Find the exact distances *AP* and *AM*.

ii) Find the exact area of the base *ABC*.

iii) Show that the height *h* of the tetrahedron is $\frac{\sqrt{2}}{\sqrt{3}}x$.

iv) Hence show that the volume of a regular tetrahedron can be given by $V = \frac{\sqrt{2}}{12}x^3$.

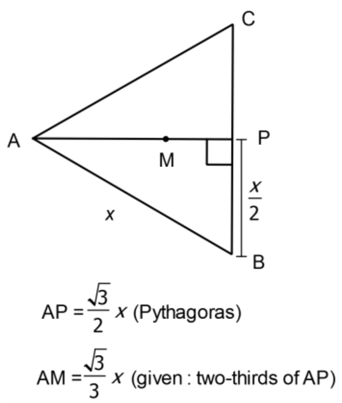
i) By Pythagoras, $AP = \sqrt{(AB)^2 - (BP)^2} = \sqrt{x^2 - \left(\frac{x}{2}\right)^2} = \sqrt{\frac{3}{4}x^2} = \frac{\sqrt{3}}{2}x$.

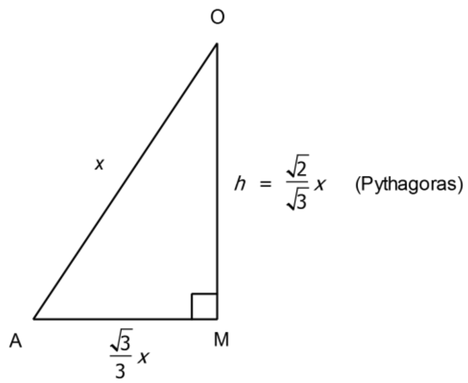
AM is two-thirds of *AP*, so $AM = \frac{2}{3} \times \frac{\sqrt{3}}{2}x = \frac{\sqrt{3}}{3}x$.

ii) The area of the base *ABC* is half the base *BC*, times the height *AM*, or $\frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x$.

iii) We apply Pythagoras again to find the height *h*, shown as *OM*.

$$OM = \sqrt{(OA)^2 - (AM)^2} = \sqrt{x^2 - \left(\frac{\sqrt{3}}{3}x\right)^2} = \sqrt{x^2 - \frac{1}{3}x^2} = \sqrt{\frac{2}{3}x^2} = \frac{\sqrt{2}}{\sqrt{3}}x.$$





iv) The volume of the tetrahedron, like that of any pyramid, is one-third the base area multiplied by the height, so, using the results from parts ii) and iii) :

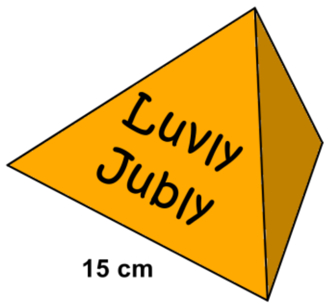
$$V = \frac{1}{3} \times \left(\frac{\sqrt{3}}{4}x\right) \times \left(\frac{\sqrt{2}}{\sqrt{3}}x\right) = \frac{\sqrt{2}}{12}x^3.$$

Example (14a) (Simplified)

A soft drink is sold in cartons in the shape of a regular tetrahedron.
Each edge of the carton is 15 cm long.

i) Calculate the surface area of a single face. (Hint : divide into two parts and use Pythagoras)

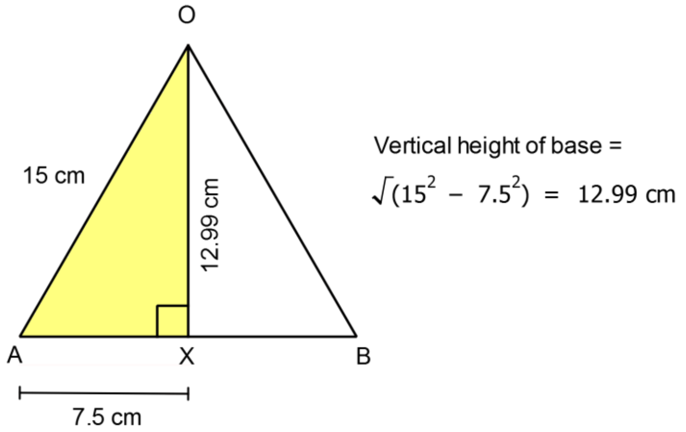
ii) Given that the vertical height of the carton is 12.25 cm, calculate its volume to the nearest cubic centimetre.



i) Firstly, we work out the area of a single face. Because the tetrahedron is regular, all of the faces are equilateral triangles, i.e. $OA = OB = AB$ = 15 cm.
The line OX is a perpendicular bisector of AB, so $AX = XB$ = 7.5 cm.
We then use Pythagoras to find the height OX.

$OX = \sqrt{(OA)^2 - (AX)^2} = \sqrt{15^2 - 7.5^2} = 12.99$ cm.

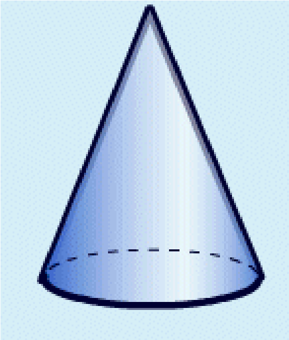
The area of a face is therefore $7.5 \times 12.99 = 97.4$ cm^2.
(The base AX of triangle OAX is already half the base of the equilateral triangle.)



ii) The volume of the tetrahedron, like that of any pyramid, is one-third the base area multiplied by the vertical height, so, using the results from part i) we get

$V = \frac{1}{3} \times 97.4 \times 12.25 = 398$ cm^3 to the nearest cubic centimetre.

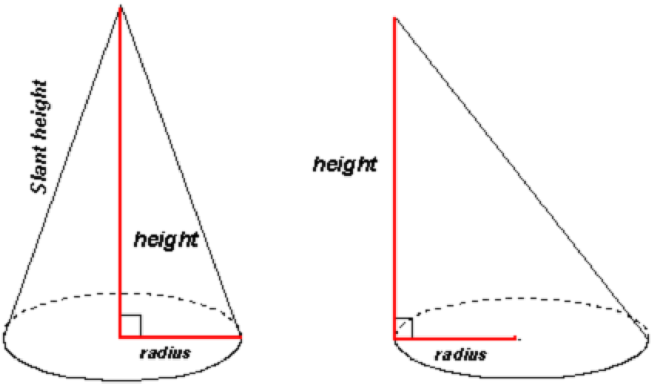
The cone.



The cone is another special case of a pyramid, this time with a circular base.

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where *r* is the radius of the base and *h* is the **perpendicular** height. Again, this holds whether the cone is upright or slanted.

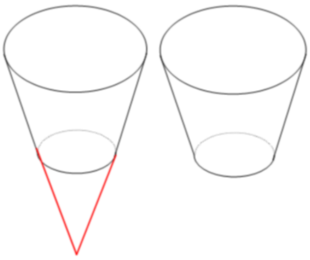
The surface area of an **upright** cone is given by $A = \pi rl$ where *r* is the radius and *l* is the slant height.
(This formula does not hold for slanted cones.)



Again, we can have a **frustum** of a cone analogous to that of a pyramid.

The section removed is itself a smaller similar cone, so the volume formula for a frustum is

$V = \frac{1}{3}\pi(R^2 H - r^2 h)$ where *R* and *H* are the base radius and height of the large cone, and *r* and *h* are the base radius and height of the smaller missing section.



Example (15): The candlemakers from Example (11) also manufacture a range of upright conical candles, with bases 14cm in diameter and height of 24cm.

How many such candles can be made out of a cylinder of wax of diameter 28 cm and height 40 cm assuming no wastage of wax ?

The volume of the wax cylinder is $V = \pi r^2 h$, where r = 14 (halve the diameter !) and h = 40.
Hence the cylinder has a volume of $(\pi \times 14^2 \times 40)$ cm^3 or **7840 π cm³**.

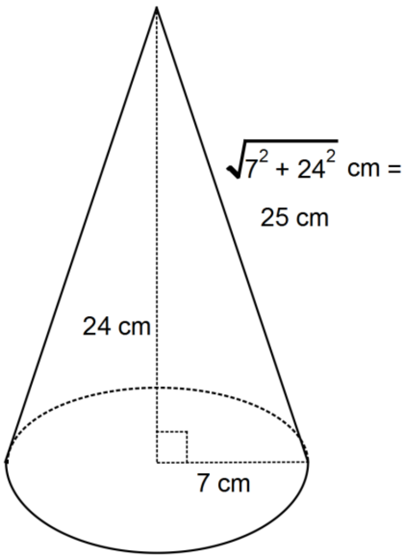
The base radius of one candle is 7 cm, hence the base area is 49π cm^2 and its height is 24 cm.

The volume of wax used per candle is thus $V = \frac{1}{3}\pi r^2 h$, or $V = \frac{1}{3} \times 49\pi \times 24 = 392\pi$ cm^3.

∴ The number of candles that can be made from the block is $\frac{7840\pi}{392\pi} = 20$.

Notice also how all the working was left in terms of π rather than using numerical approximations, as in the end we were able to cancel π out anyway !

Example (15a): The finished candles from Example (15) also require their curved surfaces coating with gold paint, though not their bases.



Calculate the total area that needs painting to 3 significant figures.

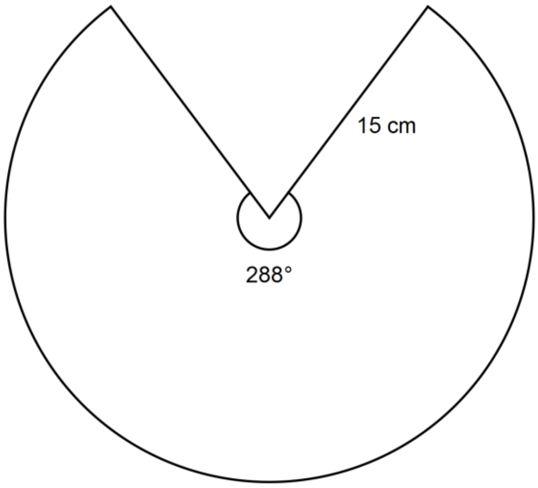
One candle has a surface area of $A = \pi rl$ where r is the radius and l is the slant height. We use Pythagoras to find l :

$$l = \sqrt{7^2 + 24^2} = 25$$

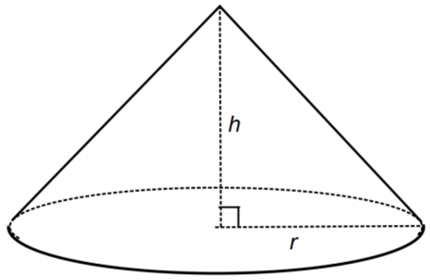
The curved surface area of one cone is therefore
$\pi \times 7 \times 25$ cm^2 = 175π cm^2, so the total curved surface area of all 20 of them is 20 × 175π cm^2 = 3500π cm^2 = 10996 cm^2.

∴ The total area to be painted = **11000 cm²** to 3 significant figures

Example (16): The circular sector (left) forms the net of curved surface of a cone. Its radius is 15 cm and its angle is 288°.



Net of curved surface



Folded cone

i) Find the area of the sector.

ii) Show that the radius of the base of the cone is 12 cm after the radii are joined together

The cone is then completed by having a circular base added to it.

iii) Find the total surface area and the volume of the completed cone.

i) Since $\frac{288}{360} = \frac{4}{5}$, the sector forms $\frac{4}{5}$ of a circle, and so its area is $\frac{4}{5}\pi r^2$.

The radius of the sector is 15 cm, so the area is is $\frac{4}{5} \times \pi \times 15^2 = 180\pi$ cm^2 or **565 cm^2** to the nearest cm^2. (This radius of 15 cm becomes the slant height of the cone.)

ii) The arc length is $\frac{4}{5} \times 2\pi r = \frac{8}{5} \times \pi \times 15 = 24\pi$ cm, which, in turn, becomes the circumference of the base of the cone.

The radius of the base of the cone therefore satisfies $2\pi r = 24\pi$, and hence r = **12 cm**.

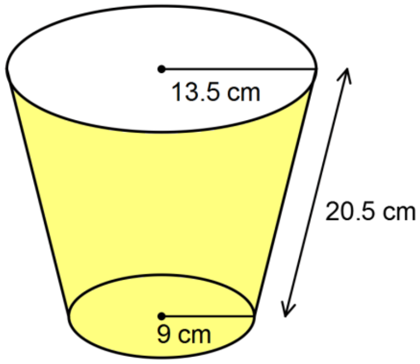
iii) The area of the base of the cone is $\pi \times 12^2$ $cm^2 = 144\pi$ cm^2**,** so the total surface area is $(180\pi + 144\pi) = 324\pi$ cm^2 = **1018 cm^2**.

In order to find the volume of the cone, we need to find the vertical height h first, but we know that the slant height l is 15 cm, and the radius r is 12 cm.

Using Pythagoras, $h = \sqrt{15^2 - 12^2} = \sqrt{81}$ cm = 9 cm.

The volume of the cone is therefore $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 12^2 \times 9 = 432\pi$ cm^3 = **1357 cm^3.**

Example (17) (Harder): A pail is in the shape of a frustum of a cone, and has a rim of radius 13.5 cm, a base of radius 9 cm and a slant height of 20.5 cm.



i) Use ratios and similar figures to show that the removed cone has a slant height of 41 cm.

ii) Show that the vertical height of the original larger cone is 60 cm.

iii) Hence find the capacity of the pail to 3 significant figures.

i) We are not given the slant height of the original cone, so we have to extend the sloping sides of the pail until they meet at the vertex of the original cone at point *O*, as in the diagram below left.

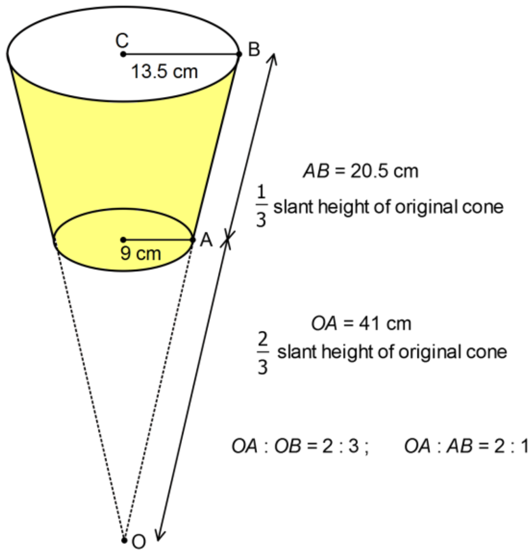
We can see that the original cone has a base radius of 13.5 cm and the removed cone has a base of radius 9 cm.

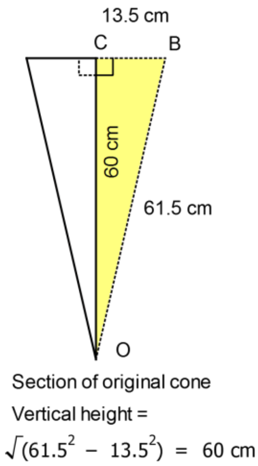
The similar removed cone therefore has a base two-thirds that of the original cone.

The cones' slant heights *OA* and *OB* are related in the same 2 : 3 ratio as their bases.

In other words, the lengths *OA* and *AB* are in the ratio 2 : 1.

Hence *OA* is twice *AB*, i.e. 2×20.5 cm, or 41 cm.





ii) From part i), the distance *OB* = 41 + 20.5 cm, or 61.5 cm. This is the hypotenuse of triangle *OCB*, which is half the vertical cross-section of the original cone, and where *C* is the centre of the base.
Using Pythagoras, the vertical height *OC* of the original cone is $\sqrt{61.5^2 - 13.5^2} = 60$ cm.

iii) The only other measurement required to find the volume of the frustum is the vertical height of the removed cone, which is two-thirds of 60 cm, or 40 cm.

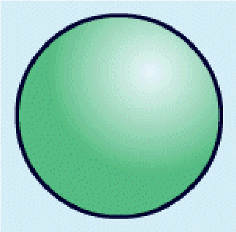
The volume of the frustum is $V = \frac{1}{3}\pi(R^2H - r^2h)$ where *R* and *H* are the base radius and height of the large cone, and *r* and *h* are the base radius and height of the removed section.

Substituting *R* = 13.5, *H* = 60, *r* = 9, *h* = 40, the volume of the frustum is

$$V = \frac{1}{3}\pi\left((182.25 \times 60) - (81 \times 40)\right) = \frac{1}{3}\pi(10935 - 3240) = 2565\pi \text{ cm}^3 \text{ or } 8058 \text{ cm}^3.$$

$\therefore$ The capacity of the pail is **8.06 litres** to 3 significant figures.

The sphere.



All points on the outside of a sphere are at a fixed radius from the centre.

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where r is its radius.

The surface area is given by $A = 4\pi r^2$.

Example (18): A world globe has a diameter of 40cm. Find its surface area and volume.

We substitute r = 20 (remember to use the radius !) into the formulae above to obtain

$V = \frac{4}{3} \times \pi \times 20^3$ cm^3 = 33,500 cm^3. $\therefore$ the volume of the sphere is 33.5 litres to 3 s.f.

The area of the sphere is $A = 4 \times \pi \times 20^2$ cm^2 = 5,030 cm^2.

Example (19):

i) Show that the total surface area of a hemisphere can be given by the formula $A = 3\pi r^2$.
ii) Hence calculate the total surface area of a hemispherical paperweight of diameter 8 cm.

i) The curved surface of a hemisphere is half the total surface of a sphere, i.e. $2\pi r^2$.
We then add the area of the circular base, namely πr^2, to obtain the total of $2\pi r^2 + \pi r^2 = 3\pi r^2$.

ii) Substituting r = 4 (halve the diameter !) into the formula $A = 3\pi r^2$, the total surface area of the paperweight is 48π cm^2 = **151 cm^2.**

Example (20): A kitchen bowl is hemispherical in shape and is designed to hold 3 litres of water. Find its diameter in centimetres.

Since the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, the volume of a hemisphere will be given as half that, i.e. $v = \frac{2}{3}\pi r^3$ where r is its radius.

Rearranging to make r the subject, we have $r = \sqrt[3]{\frac{3v}{2\pi}}$.

Given v = 3000 (1 litre = 1000 cm^3), $r = \sqrt[3]{\frac{9000}{2\pi}}$ = 11.27 cm.
That value is the radius; doubling it gives the diameter, namely 22.5 cm to 3 s.f.

(The corresponding formula to find the radius of a full sphere, given the volume, is $r = \sqrt[3]{\frac{3V}{4\pi}}$).

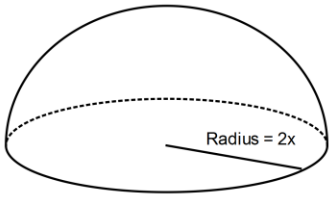
Example (21): The hemisphere and cone shown below have the same volume, but the radius of the hemisphere is twice that of the cone.

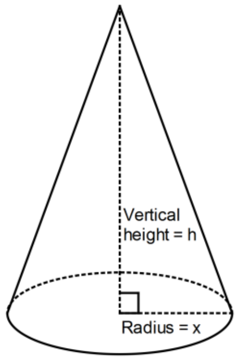
Express the vertical height of the cone in terms of its radius.

The volume of the hemisphere will be half that of a sphere, i.e. $V = \frac{2}{3}\pi r^3$ where *r* is its radius.

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$ where *r* is its radius and *h* is its vertical height.

Because the radius of the hemisphere is twice that of the cone, we must replace *r* with 2*x* in the hemisphere volume formula; for the cone, we replace *r* with *x*.





The volumes of the two solids are equal, so $\frac{1}{3}\pi x^2 h = \frac{2}{3}\pi (2x)^3$

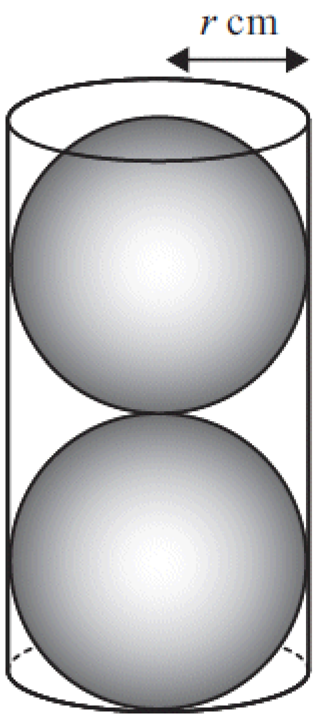
$\rightarrow \pi x^2 h = 2\pi (2x)^3$ (multiplying by 3)

$\rightarrow \pi x^2 h = 16\pi x^3$

$\rightarrow x^2 h = 16x^3$ (dividing by π)

$\rightarrow h = 16x$ (dividing by x^2) $\therefore$ The height of the cone is sixteen times its radius.

Example (22):



A hollow cylinder of height $4r$ cm exactly encloses two solid spheres of radius r cm.

The volume of space *not* occupied by the spheres is

$\frac{500}{3}\pi$ cm^3.

Find the radius of each sphere.

The volume of the cylinder is $V = \pi r^2 h$, but here $h = 4r$, so we can rewrite it as $V = 4\pi r^3$.

The volume of one sphere is $V = \frac{4}{3}\pi r^3$, so the combined volume of both is $\frac{8}{3}\pi r^3$.

We are also given that the difference between the volumes is $\frac{500}{3}\pi$.

Therefore $4\pi r^3 - \frac{8}{3}\pi r^3 = \frac{500}{3}\pi$

$\rightarrow 12\pi r^3 - 8\pi r^3 = 500\pi$

$\rightarrow 4\pi r^3 = 500\pi$

$\rightarrow 4r^3 = 500$

$\rightarrow r^3 = 125$

$\rightarrow r = 5$ cm.

$\therefore$ The radius of the spheres is 5 cm.

As in some other earlier examples, all the working was left in terms of π.

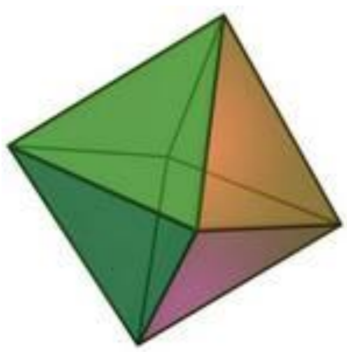
Other regular solids.

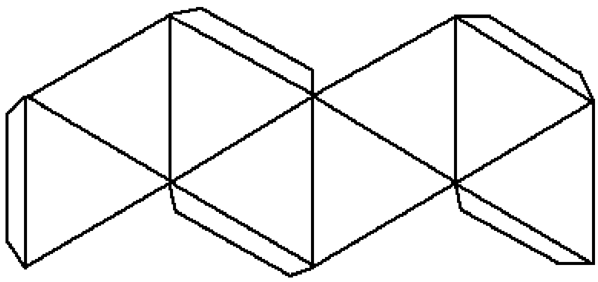
The cube and the regular tetrahedron are two solids whose faces are all regular and identical and also arranged in the same way around each vertex.

(Incidentally, the technical name for a cube is a **regular hexahedron** as it has six faces.)

There are in fact three other such regular solids; two of them have equilateral triangles as faces, and the final one has regular pentagons.

The regular octahedron.





The regular octahedron has 8 equilateral triangles as its faces, 6 vertices and 12 edges.
(Compare these to the cube's 6 faces, 8 vertices and 12 edges).
A regular square-based pyramid forms half of an octahedron.

Example (23): In Example (13), we obtained the formula $V = \frac{\sqrt{2}}{6}x^3$ for the volume of a regular square-based pyramid.

i) State the formula for the volume of a regular octahedron whose edges are *x* units long.

ii) Hence find the mass of a crystal glass ornament in the shape of a regular octahedron of edge length 6 cm, given that the density of the glass is 3.2 g / cm^3. Give your answer to the nearest gram.

i) A regular square pyramid is half of an octahedron, so we just multiply the pyramid formula by 2:

Volume of regular octahedron = $V = \frac{\sqrt{2}}{3}x^3$.

ii) Using the formula $V = \frac{\sqrt{2}}{3}x^3$ and substituting x = 6 cm, the volume of the octahedral ornament is

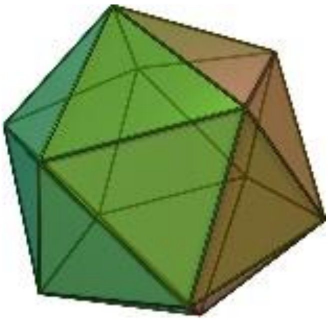
$V = \frac{\sqrt{2}}{3} \times 6^3 = 72\sqrt{2}$ = 101.8 cm^3.

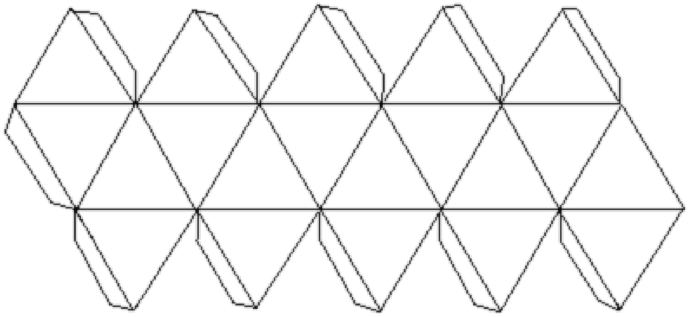
We need to calculate the mass of the ornament, so we multiply the volume by the density :

Mass of ornament = (101.8 × 3.2) g = **326g** to the nearest gram.

The next two solids are included for completeness only, and will not be featured in exam questions.

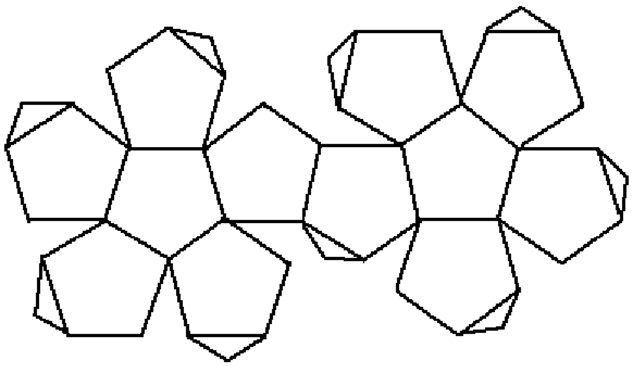
The regular icosahedron.





Another regular solid with equilateral triangles for faces is the icosahedron, with 20 of them.
It has 12 vertices and 30 edges.

The regular dodecahedron.



This is the last regular solid with 12 regular pentagons for faces, 20 vertices and 30 edges.