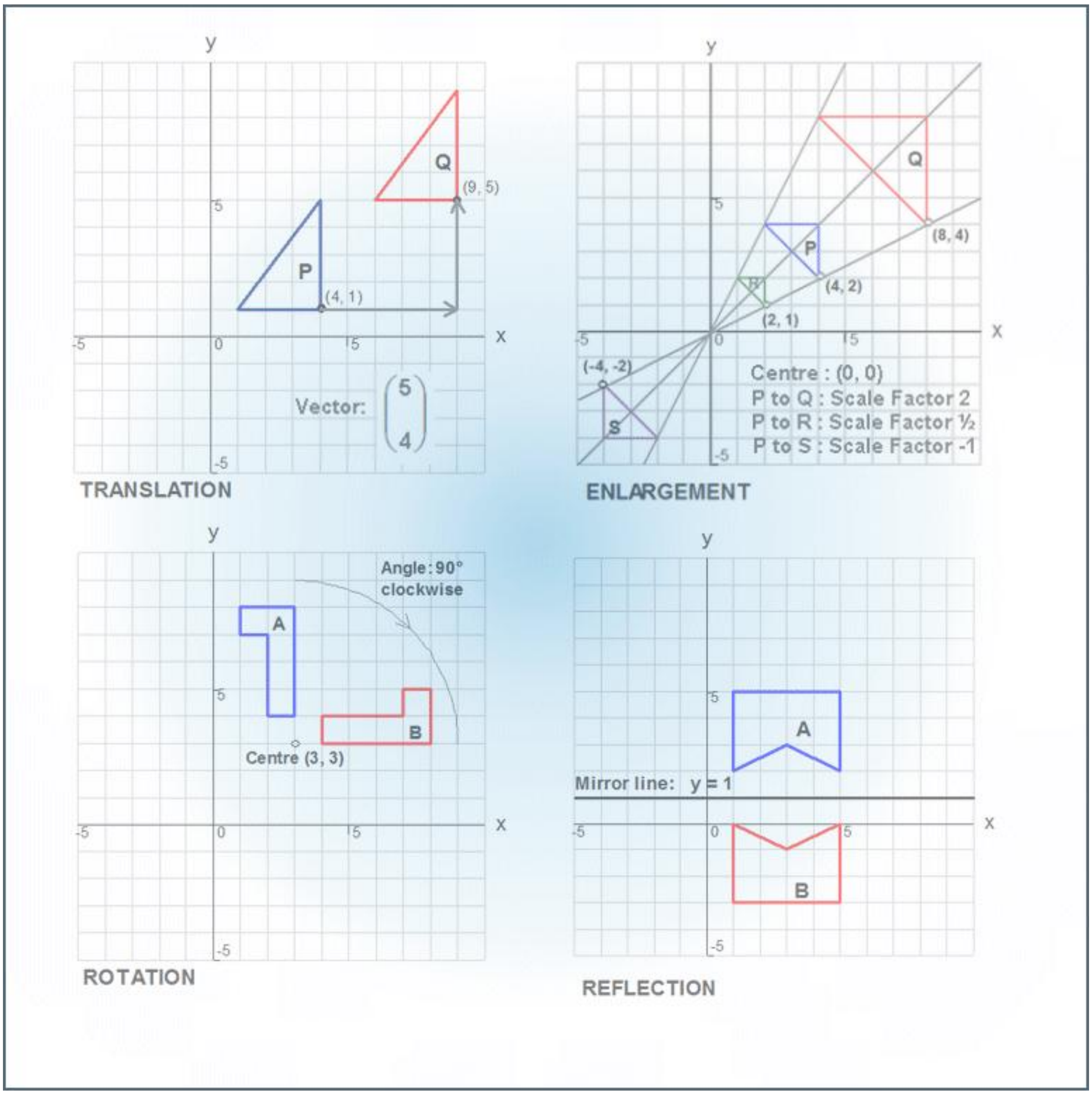
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

TRANSFORMATIONS



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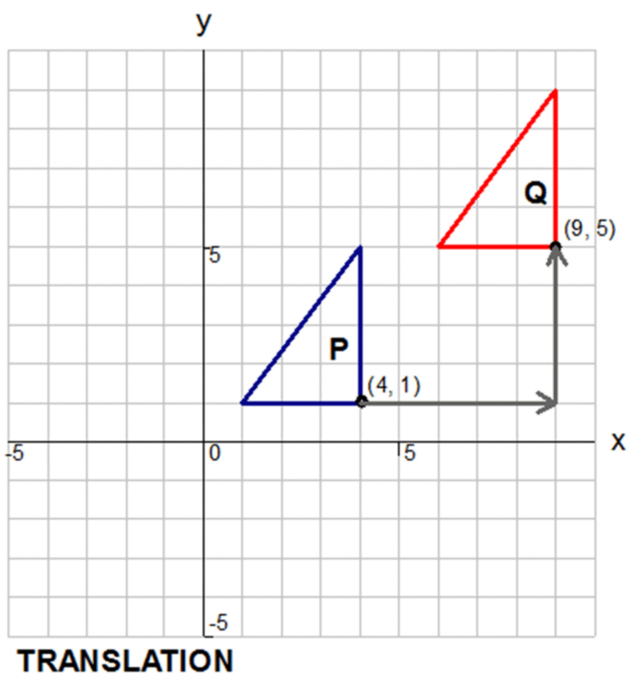
TRANSFORMATIONS.

There are four main ways in which a point (or set of points) can be transformed within the plane. They are:

Translation
Enlargement
Rotation
Reflection

TRANSLATION.

In a translation, all points in the plane are moved by a constant quantity in both the *x*- and *y*- directions. Triangle **Q** is a translation of triangle **P**.
Note that the shape, size and orientation of each triangle remains unchanged.



Example (1):
If we were to take the point (4, 1) on triangle **P**, we find that it has moved to the point (9, 5) on triangle **Q**.

All other points on the same triangle have been moved 5 units right and 4 units up. For example, the upper vertex of triangle **P** is at (4, 5) and the corresponding vertex of triangle **Q** is at (9, 9).

In other words, each point on triangle **P** has had its *x*-coordinate increased by 5 and its *y*-coordinate increased by 4 to map it to its corresponding point on triangle **Q**.

$\therefore$ Any point on **P** would map to (x+5, y+4) on **Q**.

This brings us to the idea of using **vectors** - quantities with magnitude and direction. There are several ways of expressing those quantities, but the most convenient one is the column vector.

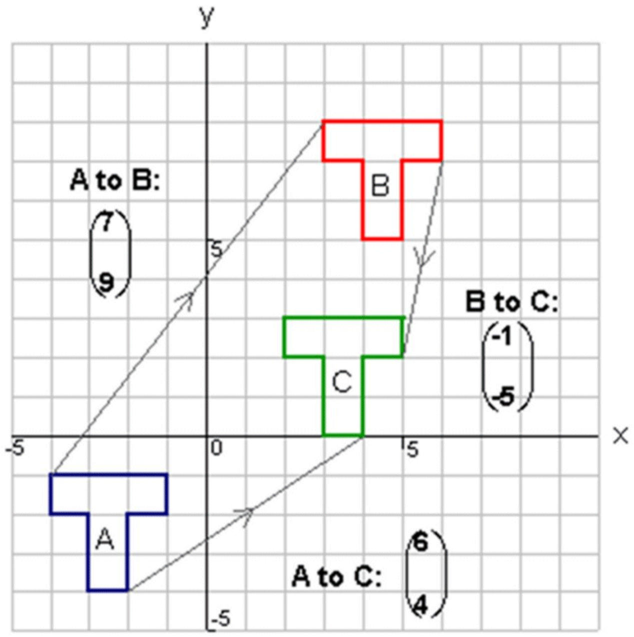
From the above example, triangle **Q** is a mapping of triangle **P** by the translation whose column vector representation is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$. The upper figure is the change in x and the lower one the change in y.

A positive change in $x \rightarrow$ move figure to the right
A negative change in $x \rightarrow$ move figure to the left
A positive change in $y \rightarrow$ move figure up
A negative change in $y \rightarrow$ move figure down

Example (2): Translate the figure **A** by the vector $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$. Draw and label the transformed figure **B**.

Next, translate figure **B** by the vector $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ to give figure **C**. Draw and label this figure.

What vector translates figure **A** directly to figure **C** ?



We move figure **A** 7 units right and 9 units up to get figure **B.**

Thus, the point (-4, -1) on the left top bar of the T is translated to the point (3, 8).

Then we move figure **B** one unit left and 5 units down to get figure **C**. The point (6, 7) on figure **B** is translated to point (5, 2) as per the arrow.

Looking at the arrow showing the direct translation from **A** to **C**, we can see that the point (-2, -4) is moved to (4, 0) – an increase of 6 in x and of 4 in y.

In vector form, the translation from **A** to **C** is $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

This mapping could have been worked out by vector addition:

$$\begin{pmatrix} 7 \\ 9 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

ENLARGEMENT.

The enlargement requires two properties to define it; the **centre of enlargement** and the **scale factor**.

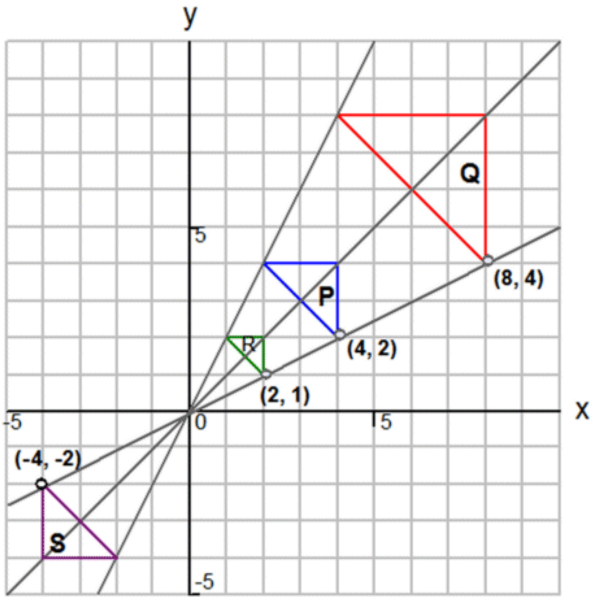
The scale factor is the number by which **lengths** of the original figure are multiplied.

If we are given both the original figure and the enlarged one, we can draw lines through corresponding pairs of points. Those lines meet at the **centre of enlargement**.

Example (3):

Look closely at triangles **P** and **Q**, taking particular notice of the points (4, 2) and (8, 4).

If we draw three lines joining corresponding vertices of triangles **P** and **Q**, we will find that they pass through the centre of enlargement, here the origin.



ENLARGEMENT

We can also see that the vertices of **Q** are twice as far from the origin as those of **P**. This can be seen in the doubling of the *x*- and *y*-coordinates – compare point (8, 4) on **Q** with point (4, 2) on **P**.

The sides of **Q** are thus twice as long as those of **P**, and thus the scale factor of the enlargement mapping **P** to **Q** is 2.

(Remember that the scale factor needs to be **squared** when comparing areas **–** thus the area of triangle **Q** is four times the area of **P !**) .

If the enlargement scale factor is positive and less than 1, then the transformed figure will be smaller than the original. Triangle **R** is an enlargement of scale factor ½, again centred about the origin. This time the coordinates of the vertices are half those of the original – e.g. the point (4, 2) on **P** maps to (2, 1) on **R.**
Note again how the vertices are all on the same line passing through the origin.

In general, for any enlargement of scale factor *k* passing through the origin, any point (*x*, *y*) on the original figure transforms to the point (*kx*, *ky*).

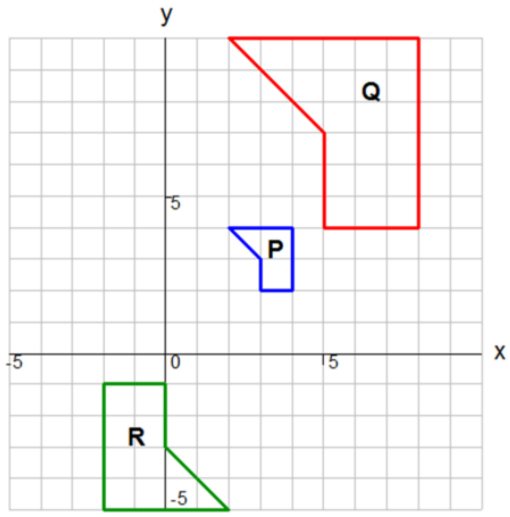
The above case also holds true for negative enlargements. In a negative enlargement, the transformed figure appears on the opposite side of the centre of enlargement to the original.

Triangle **S** is an enlargement of triangle **P** by a scale factor of –1. The triangle is the same size as **P** but is on the opposite side of the origin. In fact, it is equivalent to a 180° rotation of **P** about the origin.

Enlargements not centred on the origin.

When an enlargement is not centred on the origin, constructions are slightly trickier.

Example (4): Figure **P** is transformed to figures **Q and R** by two enlargements about the same centre. Find the common centre and the scale factor of each enlargement.

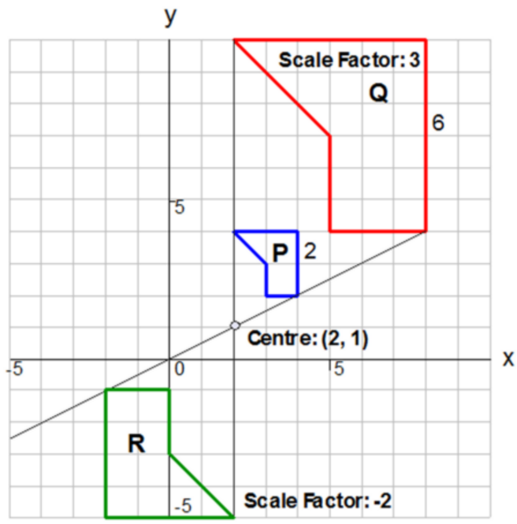


The height of figure **P** is 2 units and that of figure **Q** is 6 units, so we can deduce that the scale factor for **Q** is 3.

Figure **R** is upside-down relative to **P** and its height is 4 units, so its scale factor is –2.

To find the centre of enlargement, we take one point from **P** and draw a line which passes through this point and the corresponding point on **Q** (and for that matter **R**). One such line passes through (2, 4), (2, 10) and (2, -5).
(It is actually the line $x = 2$.)

Next, draw another line passing through different corresponding points on **P**, **Q** and **R**. This line passes through the points (4, 2), (8, 4) and (-2, -1).



The centre of enlargement is the point of intersection of those two lines – here its coordinates are (2, 1).

The scale factor can also be visualised as follows; we can take points on one connecting line and note their distances from the centre.

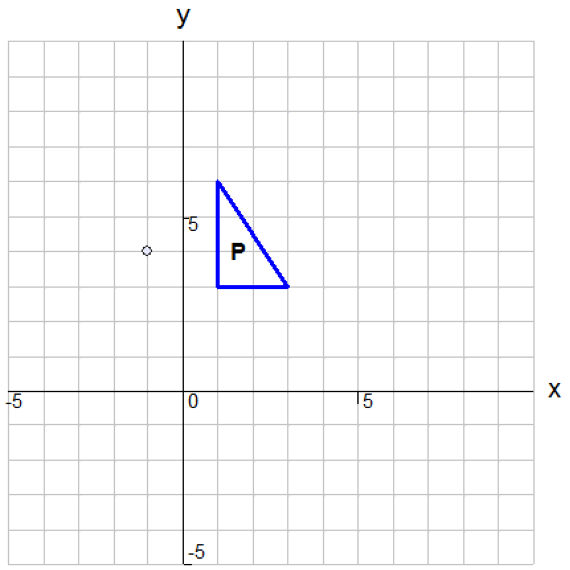
The point (2, 4) on **P** is 3 *y*-units from the centre at (2 ,1).

The corresponding point (2, 10) on **Q** is 9 *y*-units away – a factor of 3 times as far as **P**.

Similarly, point (2, -5) on **R** is –6 *y*-units away – a factor of –2 times as far as **P**.

It is also worth noting that any enlargement with a scale factor of -1 about a point (x,y) is equivalent to a rotation of 180° about that same point.

Example (5): Enlarge the triangle **P** by a scale factor of 2, where the centre of enlargement is given as the point (-1, 4).
Label the enlarged triangle **Q**.

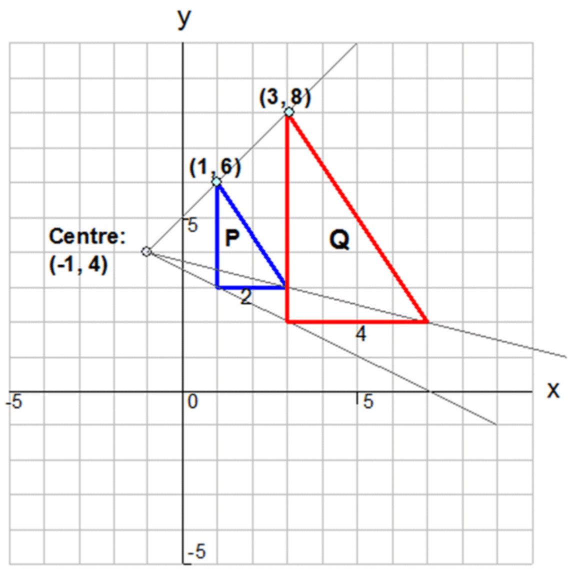


The first stage is to draw lines radiating from the point (-1, 4) and passing through each of the vertices of the triangle – construction shown on right.

Next, we work out where to plot the transformed points on those construction lines.

To get from the centre at point (-1,4) to the point (1,6) on triangle **P** requires a move of 2 units in the *x*-direction and 2 in the *y*-direction. Since **Q** is enlarged with scale factor 2, the corresponding point on **Q** must be twice as far from the centre of enlargement – a move of 4 units in *x* and 4 units in *y*.

This gives the required point at (3, 8).



The other two points on the original triangle **P** are treated in the same way, where the corresponding points on triangle **Q** are twice as far from the centre.

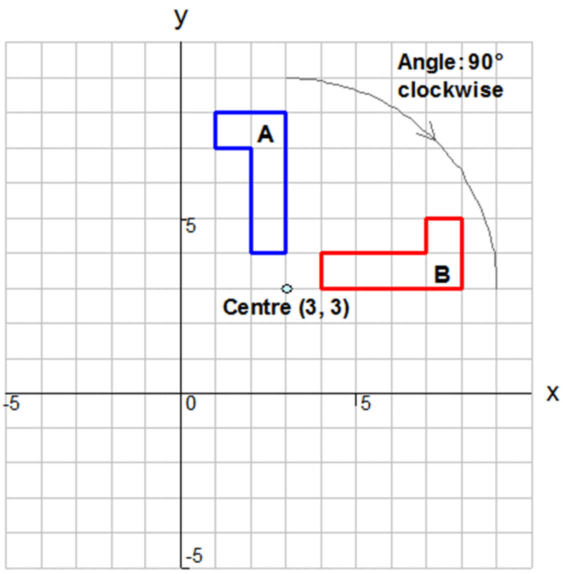
Also, the orientation of the triangle **Q** is identical to that of P, only the side lengths being twice as large.

We could therefore draw the vertical side of **Q** (6 units long) from the point (3, 8).

ROTATION.

In a rotation, all points in the plane are rotated by a specified angle about a specified centre. These two quantities define the rotation.

Example (6): Figure **A** is transformed to figure **B** by a rotation of 90° clockwise about the centre of rotation; here it is the point (3, 3)

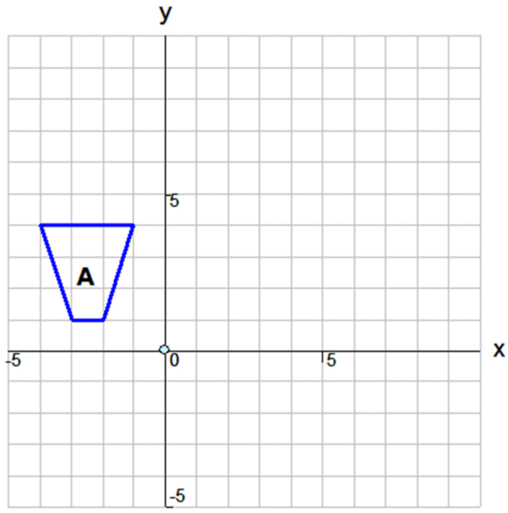


ROTATION

Rotations about the origin.

Example (7): Rotate the figure **A** through 90° clockwise about the origin. Label this rotated figure **B.**

Then, follow up by rotating **B** through 90° to give figure **C**, and finally rotate again through 90° to obtain figure **D.**

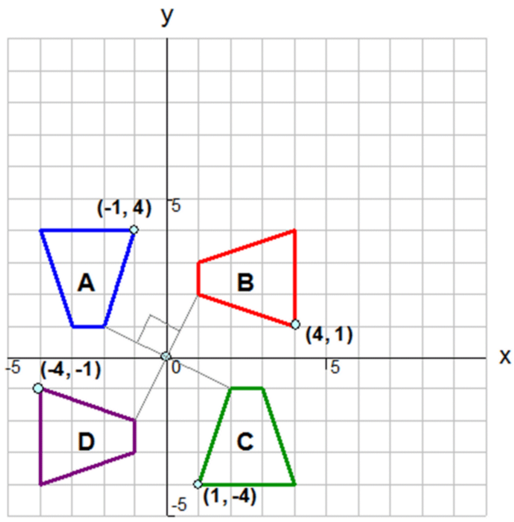


Select point (-1, 4) on figure **A.** What points does it transform to on the other three figures ?
Find a general formula for each of those rotations about the origin.

Tracing paper is helpful here – we trace figure A, the axes and the origin onto the paper, and then pivot the traced figure 90° clockwise to obtain **B**.

Another clockwise 90° turn is equivalent to a half turn (180°) of figure **A** to give figure **C**.

The final quarter-turn is equal to a 270° clockwise rotation of **A** (or 90° anticlockwise !) to give figure **D.**



The point (-1, 4) on **A** transforms to (1, -4) on **C** – in other words, a rotation of 180° about the origin reverses the signs of the *x*- and *y*- coordinates without affecting their numerical values.

∴ The general point (*x*, *y*) maps to (-*x*, -*y*) after a rotation of 180° about the origin.

When it comes to the 90° clockwise turn, the *x*-coordinate has its sign changed and then the numerical values of *x* and *y* are exchanged.
Thus (-1, 4) transforms to (4, 1)

∴ The general point (*x*, *y*) maps to (*y*, -*x*) after a clockwise rotation of 90° about the origin.

For 90° anticlockwise turns, the *y*-coordinate has its sign changed before exchanging the numerical values of *x* and *y*. Thus (-1, 4) transforms to (-4, -1).

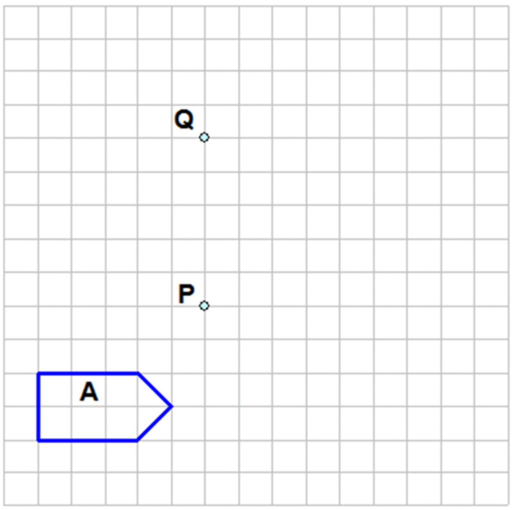
∴ The general point (*x*, *y*) maps to (-*y*, *x*) after an anticlockwise rotation of 90° about the origin.

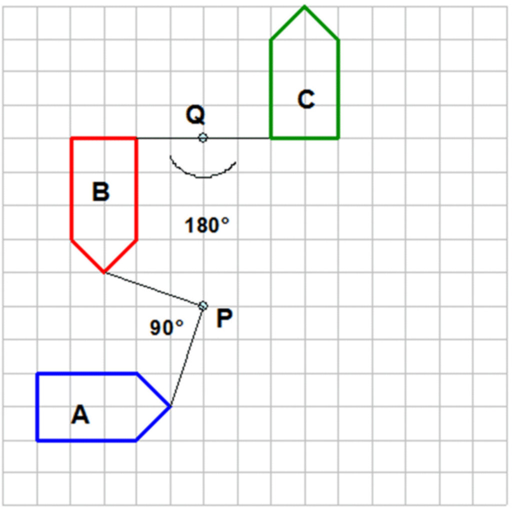
Rotation about points other than the origin.

The neat results of Example (8) apply only to rotations about the origin.

When rotating figures about points other than the origin, or on an unmarked grid, it is enough just to draw the rotated figure with the help of tracing paper.

Example (8): Rotate the shape **A** by 90° clockwise about the point **P**. Label this shape **B** and then rotate **B** by 180° about the point **Q** to give shape **C.**



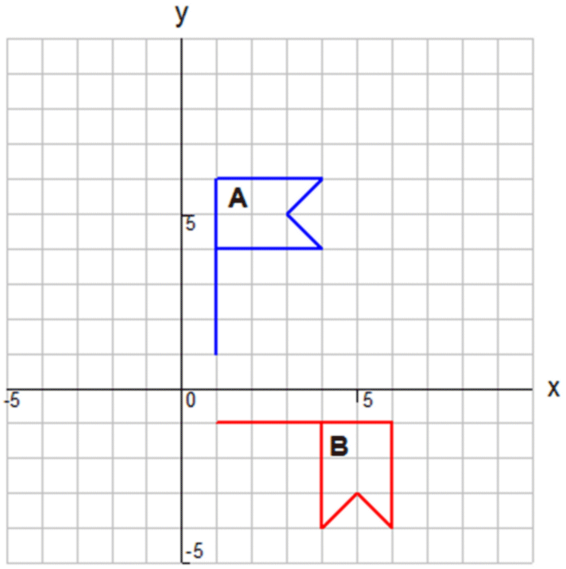


Firstly, we trace shape **A** and then pivot the tracing paper 90° clockwise about point **P**. The traced shape will be at position **B**, so we then copy shape **B** into place.

Next we trace shape **B** and then give the tracing paper a half turn about point **Q**. The traced shape will be at position **C**, so we then copy shape **C** into place.

Finding details of a rotation from a diagram – 'trial' method.

Example (9): Flag **B** is a rotation of flag **A**. Give its centre and angle of rotation.



We first look at the 'directions' of the flags. We can see that flag **B** looks like a clockwise rotation of flag **A** by 90°.

Next we try to deduce the centre of rotation. We will first try the easiest case, namely the origin.

Looking at the 'foot' of each flagpole, we can see that each one is the same distance away from the origin (1 diagonal square) ; also they make a right angle with the origin.

$\therefore$ Flag **B** is a clockwise rotation of flag **A** by 90° about the origin.

Example (10): Describe in full the rotation mapping triangle **P** to triangle **Q.**

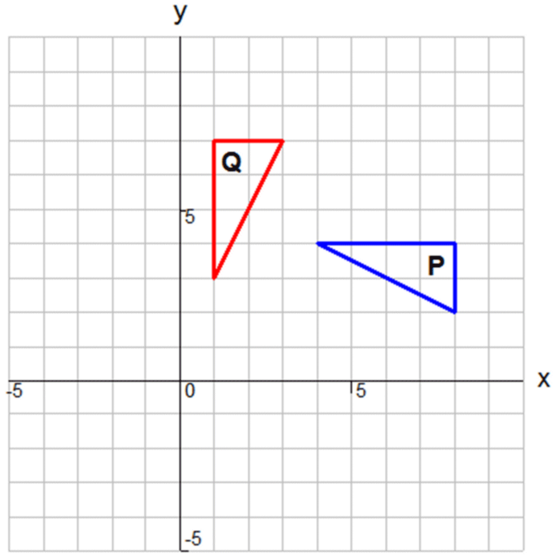
By looking at the way the two triangles are oriented, it can be seen that **Q** looks like a 90° anticlockwise rotation of **P**.

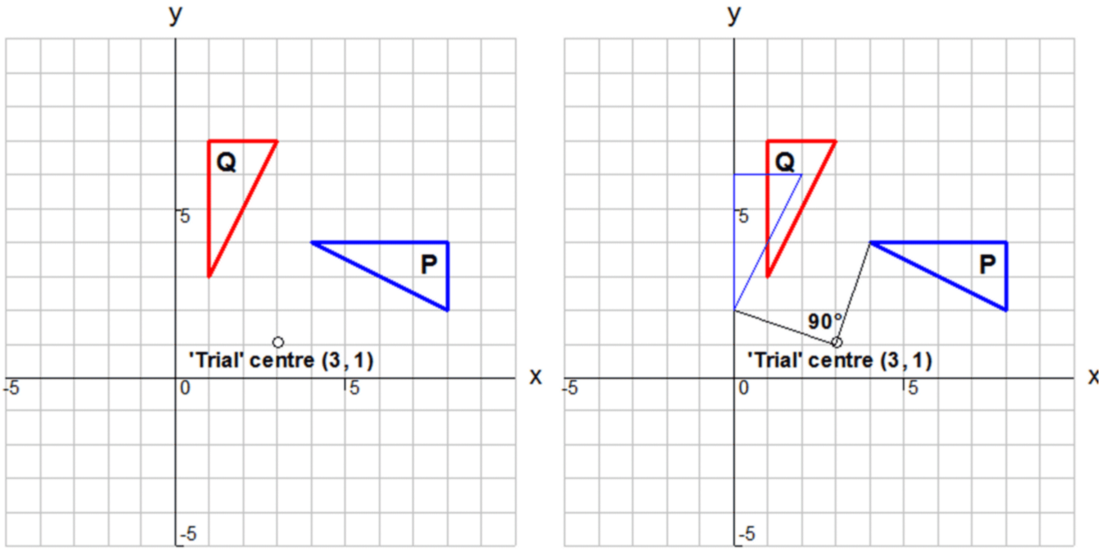
The centre is less obvious to find, because the point (4, 4) on **P** is much further from the origin than the point (1, 3) on **Q**. Also, the lines joining the points (1, 3) and (4, 4) to the origin do not look perpendicular.

The centre cannot be the origin, so we must make some 'trial' attempts at the answer.

We simply trace the triangle, and then pivot the tracing paper 90° anticlockwise about the 'trial' centre of rotation.

If the traced triangle does not coincide with **Q**, then we must try again !





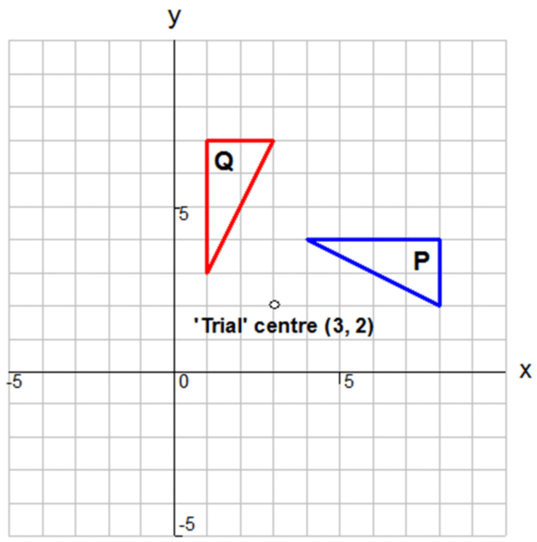
One 'try' is shown in the diagrams above. We guess that the centre of rotation might be (3, 1). We therefore trace triangle **P** onto paper and give it a 90° turn anticlockwise about the point (3, 1).

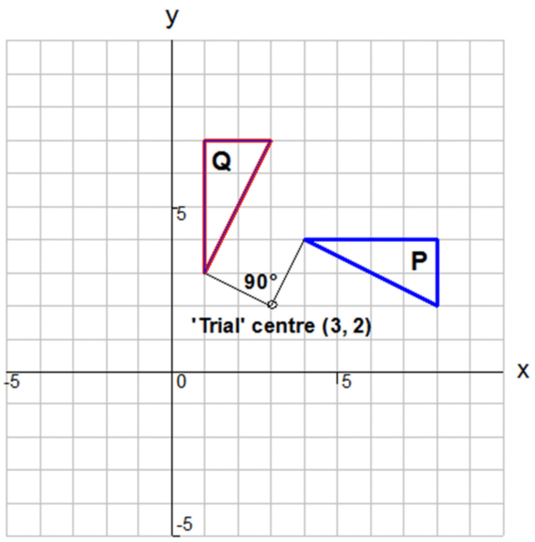
Unfortunately, the traced triangle does not coincide with **Q**, so we must try again.

We therefore try the nearby point of (3, 2). This time, pivoting the traced triangle by 90° anticlockwise about (3, 2) does give the required result.

The points on the traced triangle now coincide with those on **Q**.

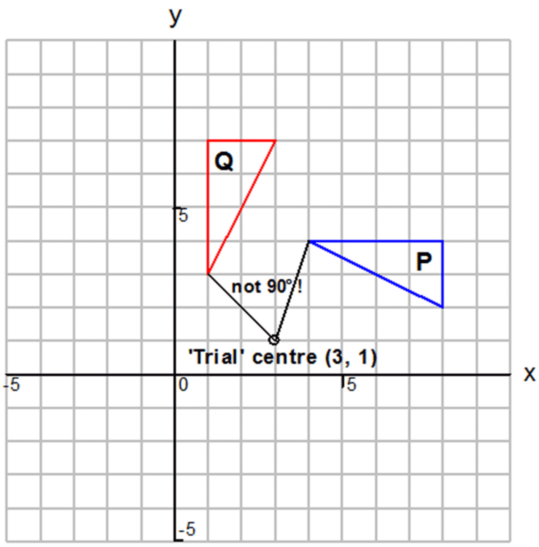
∴ Triangle **P** is transformed to triangle **Q** by a rotation of 90° anticlockwise about the point (3,2).





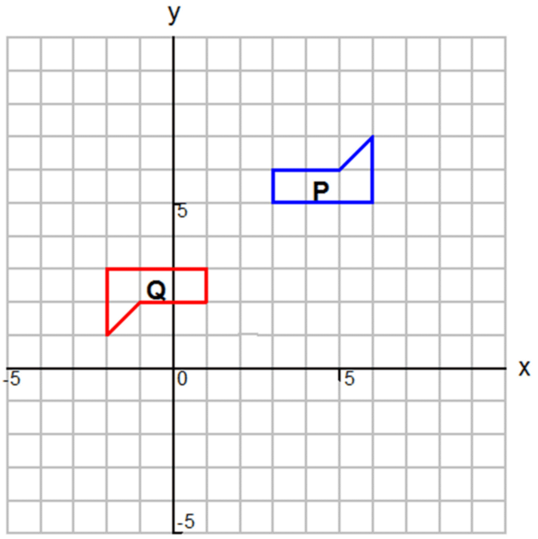
Notice how the corresponding vertices on triangles **P** and **Q** form an isosceles right-angled triangle with the centre of rotation. This is true for rotations of 90° either clockwise or anticlockwise.

By comparison, the 'trial' centre of rotation at (3,1) did not produce a right-angled isosceles triangle.



Sometimes it might be possible to find the full details of a rotation more easily.
Rotations of 180° are easy to recognise, as shown below.

Example (11): Describe in full the rotation mapping figure **P** to figure **Q**.



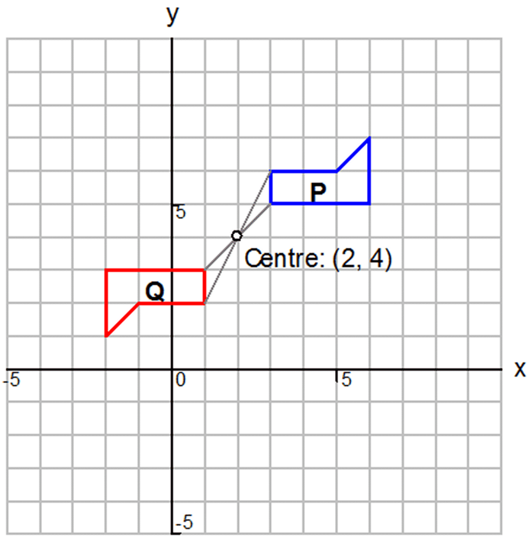
Here, we can see that figure **Q** is upside-down compared to figure **P**, so we can say immediately that **Q** is a rotation of **P** through 180° (either clockwise on anticlockwise – direction is immaterial) .

For a rotation through 180°, it is easy to find the centre.

We begin by drawing a line between two corresponding points on **P** and **Q**, say (3,5) and (1.3).

We then choose another pair of points, say (3,6) and (1,2).

The intersection of the two lines, here the point (2,4), is also the centre of the rotation, and incidentally, the midpoint of each line between the pairs of points.

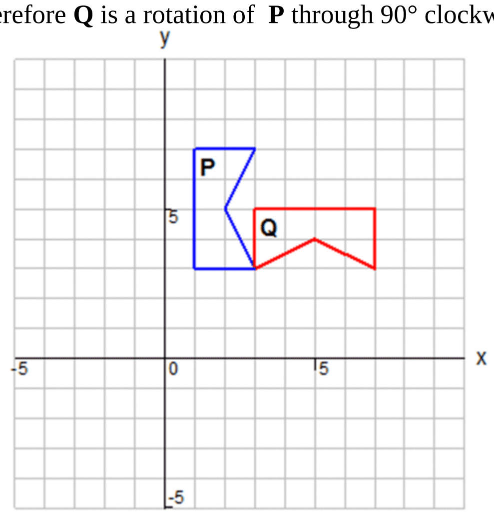


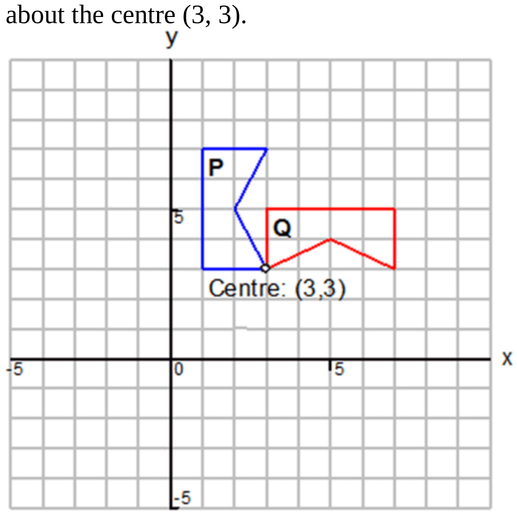
Hence figure **Q** is a rotation of figure **P** through 180° about the point (2, 4).

Example (12): Describe in full the rotation mapping figure **P** to figure **Q**.

All rotations are specified by an angle and the fixed centre, and here we have a point on the original figure **P** which has remained unchanged in position after the transformation. This point, at (3, 3), is therefore the centre of the rotation. We can also see that **P** has been rotated clockwise through 90°.

Therefore **Q** is a rotation of **P** through 90° clockwise about the centre (3, 3).





In general, if a rotation leaves any point on a given figure unchanged in position, then that point is at the centre of rotation.

Finding details of a rotation from a diagram – systematic method.

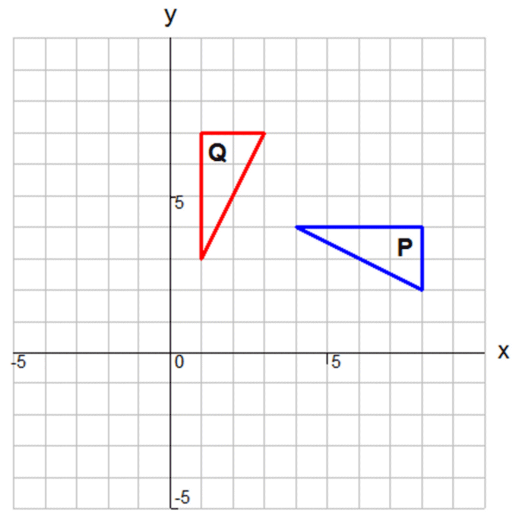
We need to use a ruler and compass to find the centre, and maybe a protractor to find the angle. The process looks difficult with all the constructions, but it is straightforward. Its other advantage is that it is not limited to 'obvious' angles of rotation such as 90°.

Example (13): (Repeat of Ex 10): Describe in full the rotation that transforms triangle **P** to triangle **Q.**

By looking at the way the two triangles are oriented, it can be seen that **Q** looks like a 90° anticlockwise rotation of **P**.

The centre is less obvious to find, because the point (4, 4) on **P** is much further from the origin than the point (1, 3) on **Q**. Also, the lines joining the points (1, 3) and (4, 4) to the origin do not look perpendicular.

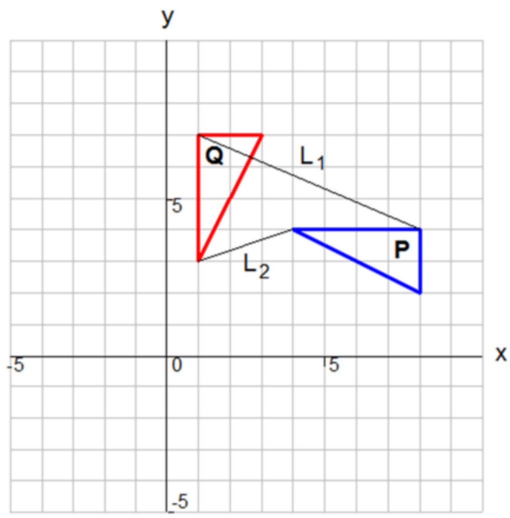
The centre cannot be the origin, so in Example (10) we made some 'trial' attempts at the answer.



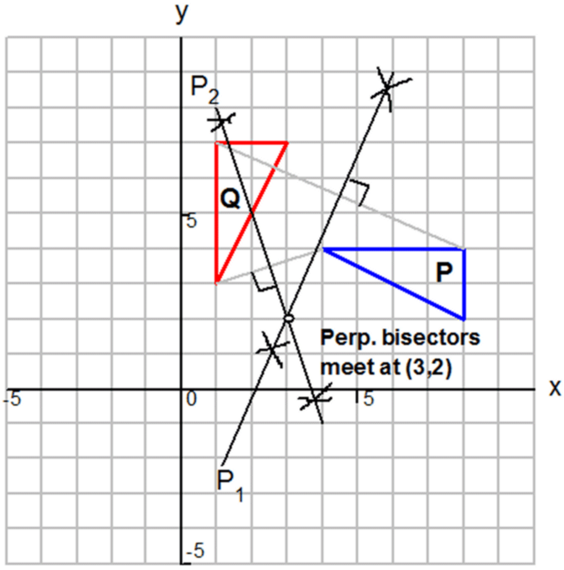
The first part is to find two pairs of corresponding points both before and after the rotation.

Here we have chosen (8, 4) on **P** and (1, 7) on **Q** and drawn line L_1 to join them.

The other pair of points, joined by line L_2, are (4, 4) on **P** and (1, 3) on **Q**.



The next stage is a slightly tricky one.

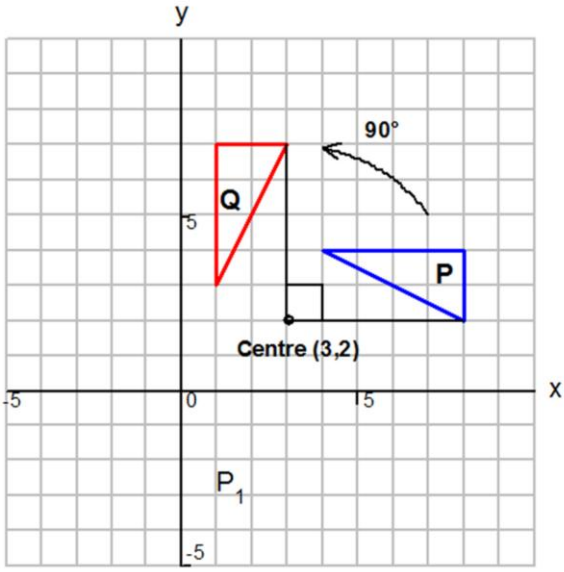


We must then bisect lines L_1 and L_2 and draw in their perpendicular bisectors, P_1 and P_2, using the ruler-and-compass methods in the "Measures and Constructions" document.

The point where the perpendicular bisectors meet is the actual centre of rotation, in this case it is the point (3, 2).

To find the angle of rotation, we draw two lines radiating from the centre and passing through corresponding points on **P** and **Q** respectively.

Here we have chosen (8, 2) on **P** and (3, 7) on **Q.** The lines joining each point to the centre at (3, 2) can easily be seen to be perpendicular, so the angle of rotation transforming **P** to **Q** is 90° anticlockwise.



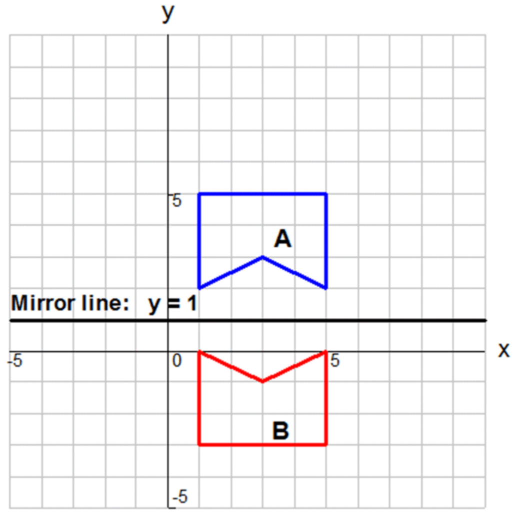
∴ Triangle P is transformed to triangle Q by a rotation of 90° anticlockwise about the point (3,2).

REFLECTION.

In a reflection, all points in the plane are mirrored in a straight line. The equation of that straight line defines the reflection.

For all reflections, the transformed point is on the opposite side of the reflection line and at the same perpendicular distance from it.

Example (14): Figure **A** is transformed to figure **B** by a reflection in the mirror line $y = 1$.



REFLECTION

Reflections in the axes.

The simplest reflections are those in the *x*-axis and the *y*-axis.

Example (15):

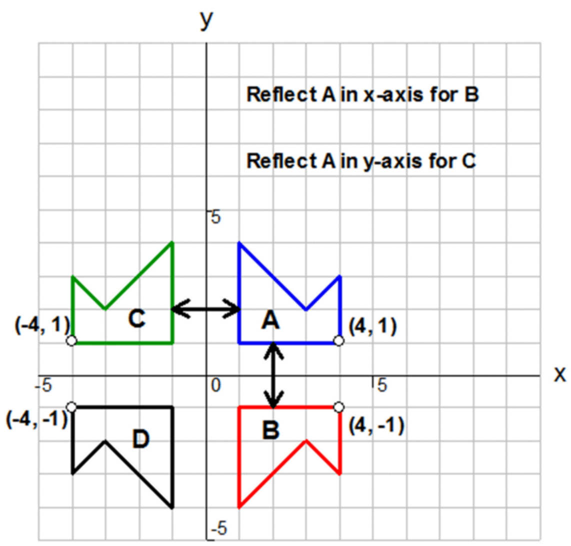
Reflect figure **A** in the *x*-axis to obtain figure **B.** What happens to the point (4, 1) after the reflection ?

Reflect figure **A** in the *y*-axis to obtain figure **C.** How is the point (4, 1) transformed after the reflection ?

What happens to figure **A** if the second reflection were carried out on the result **B** of the first reflection ?

As can be seen in the diagram, reflection in the *x*-axis reverses the sign of the *y* coordinate, whilst reflection in the *y*-axis reverses the sign of the *x*-coordinate.

The double reflection reverses the signs of *both* coordinates – identical to rotating by 180°.



Thus, if we were to reflect figure **C** in the *x*-axis, or figure **B** in the *y*-axis, we would have figure **D**.

Reflection in the *x*-axis maps any point (*x*, *y*) to (*x*,-*y*).
Reflection in the *y*-axis maps any point (*x*, *y*) to (-*x*, *y*).

Reflections in the main diagonals.

Almost as straightforward are reflections in the lines $y = x$ and $y = -x$.

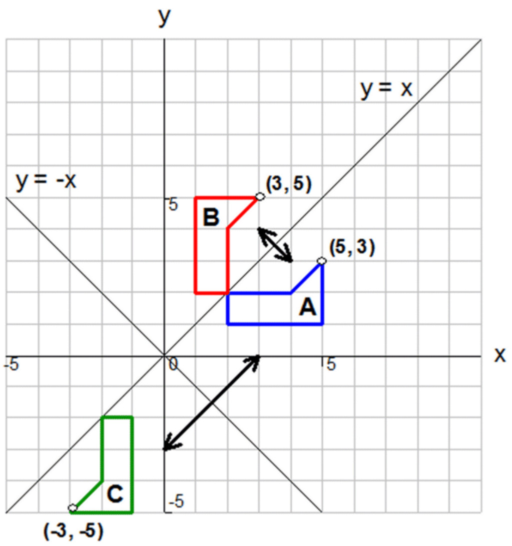
Example (16): Reflect figure **A** in the line $y = x$ to give figure **B.** How is point (3, 5) transformed ?

Then reflect figure **A** in the other main diagonal $y = -x$ to give figure **C.**

How is point (3,5) transformed now ?

This time, we might need to draw the shape, main diagonals and axes on tracing paper, and then turn the paper upside down while keeping the origin and required diagonal coincident.

Alternatively, we could take each point in turn and plot its reflection at the same distance but opposite direction from the mirror line.



After reflection in the main diagonal $y = x$, the point (5, 3) maps to (3, 5). In general, all reflected points will have their x and y coordinates exchanged. Notice that the point (2, 2) remains unchanged by the reflection.

After reflection in the main diagonal $y = -x$, the point (5, 3) maps to (-3, -5). In general, all reflected points will have their x and y coordinates exchanged and their signs reversed.

Reflection in the line $y = x$ maps any point (x, y) to (y, x).
Reflection in the line $y = -x$ maps any point (x, y) to $(-y, -x)$.

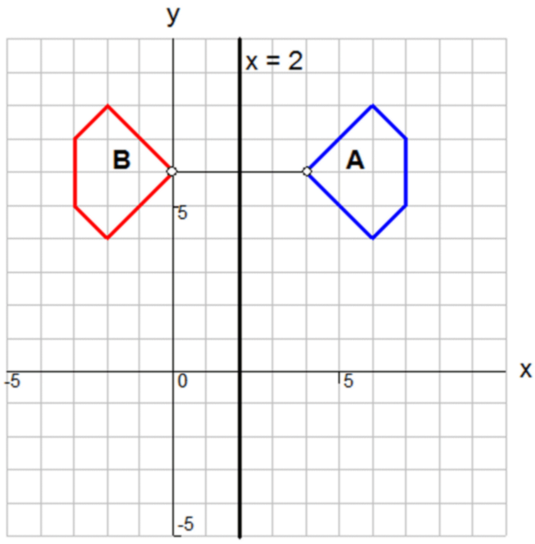
Example (17): Reflect figure **A** in the line $x = 2$.

We could draw the shape, origin, and the reflection line $x = 2$ on tracing paper, and then turn the paper upside down while keeping the x-axis and the reflection line coincident.

Alternatively, we could take each point in turn and plot its reflection at the same distance but opposite direction from the mirror line.

We have shown it for the point (4, 6) and its reflection (0, 6).

Note: if the reflection is in a horizontal line, e.g. $y = 2$, and tracing paper is used, then we must keep the y-axis and the reflection line coincident.



With most reflections, it is easy to identify if it is a vertical, horizontal, or along a main diagonal.

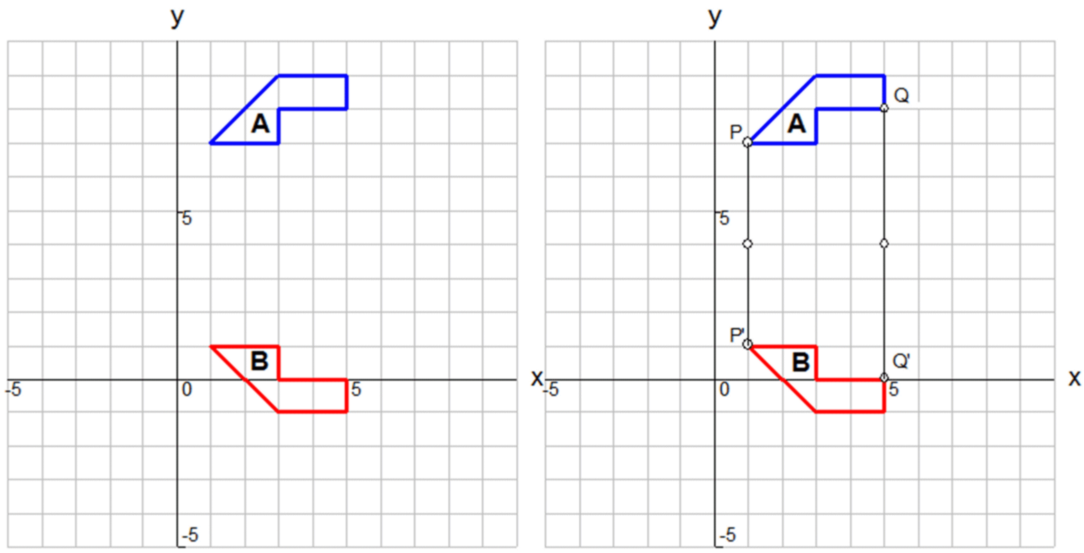
Looking at the last example, figures **A** and **B** form a 'left – right' pair with a vertical mirror line.

All 'vertical' reflections have the mirror line equation of $x = c$ where c is a constant number.
This is the general equation of a line parallel to the y-axis.
When $c = 0$, the mirror line is the y-axis itself.

'Horizontal' reflections have the mirror line equation of $y = c$, where the mirror line is parallel to the x-axis. When $c = 0$, the mirror line is the x-axis itself.

Diagonal reflections are generally limited to the two main diagonals, $y = x$ and $y = -x$.

Example (18): Figure **B** is a reflection of figure **A.** What is the equation of the reflection line ?



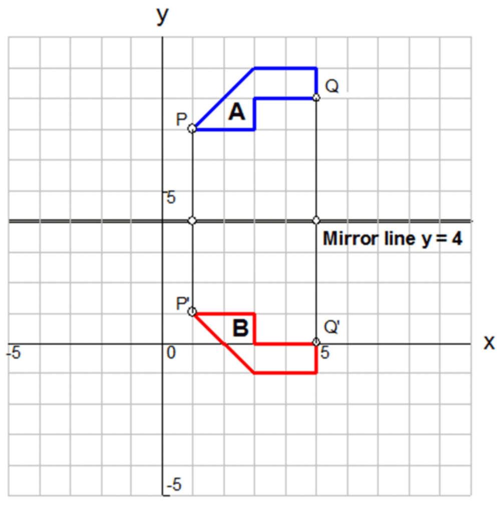
A glance at the diagram on the left suggests that figures **A** and **B** form a 'top-bottom' pair, and therefore the mirror line is horizontal, i.e. parallel to the *x*-axis. The mirror line therefore has an equation of $y = c$.

We therefore select two points on figure **A** and their reflections on **B**.
Call them P, Q, P', and Q'.

Next, we draw lines P P' and Q Q' and plot their midpoints.

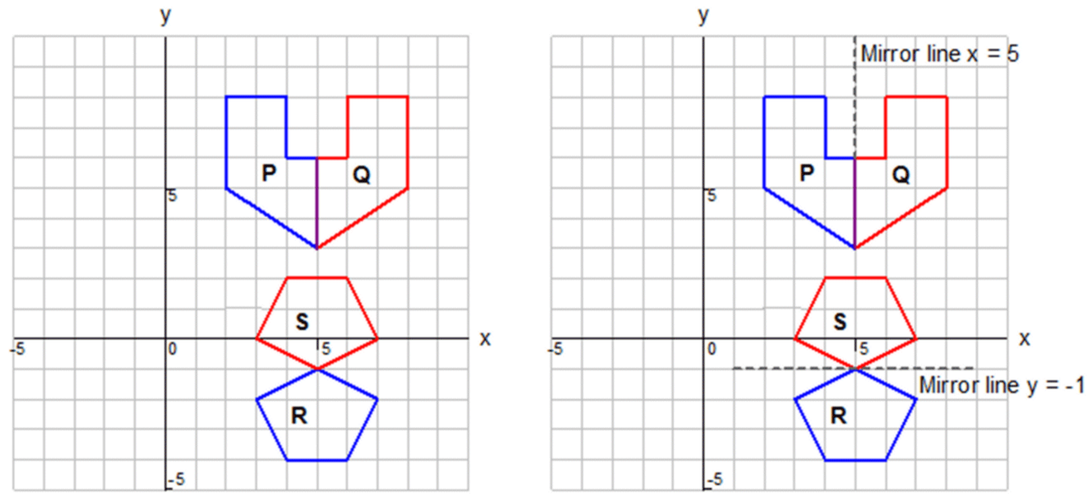
The line passing through both midpoints is the mirror line, $y = 4$.

∴ Figure A is transformed to figure B by a reflection in the line $y = 4$.



Because a reflection leaves all points on the mirror line unchanged, it is sometimes possible to find the equation of the mirror line very easily.

Example (19): Figures P and Q are reflections of each other, as are **R** and **S.** Find the equation of the mirror line in each case.



The first thing to notice about figures **P** and **Q** is that they have an edge in common. This common edge is therefore part of the mirror line $x = 5$.

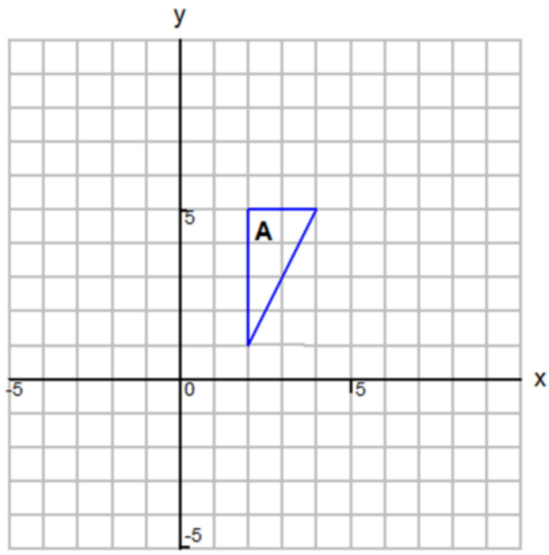
Figures **R** and **S** have a point in common rather than a line, but a similar reasoning applies. Because the reflection is a horizontal one, the mirror line must have an equation of $y = c$. The common point has a y-coordinate of -1, and so the equation of the mirror line is $y = -1$.

Composite Transformations.

We finish this section by looking at how transformations can be combined in various ways.

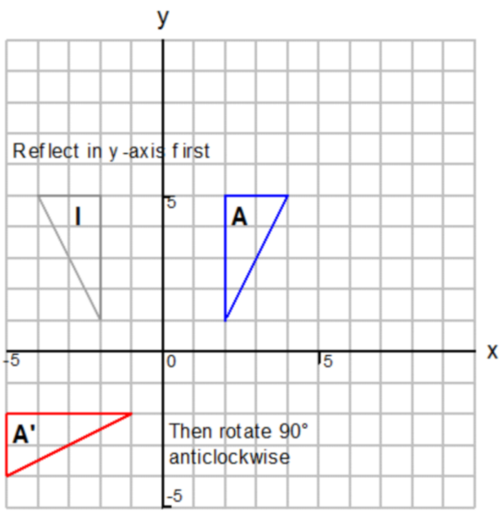
Example (20): Translate the given triangle **A** by first reflecting it in the *y*-axis, followed by a 90° anticlockwise rotation about the origin.
Label the intermediate triangle **I** and the final result **A'**.

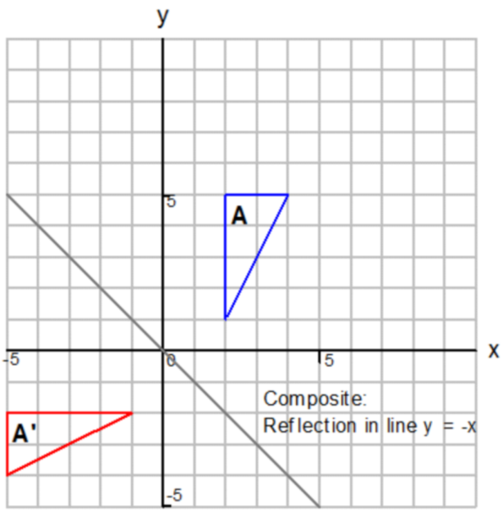
What is the single transformation mapping **A** to **A'** ?



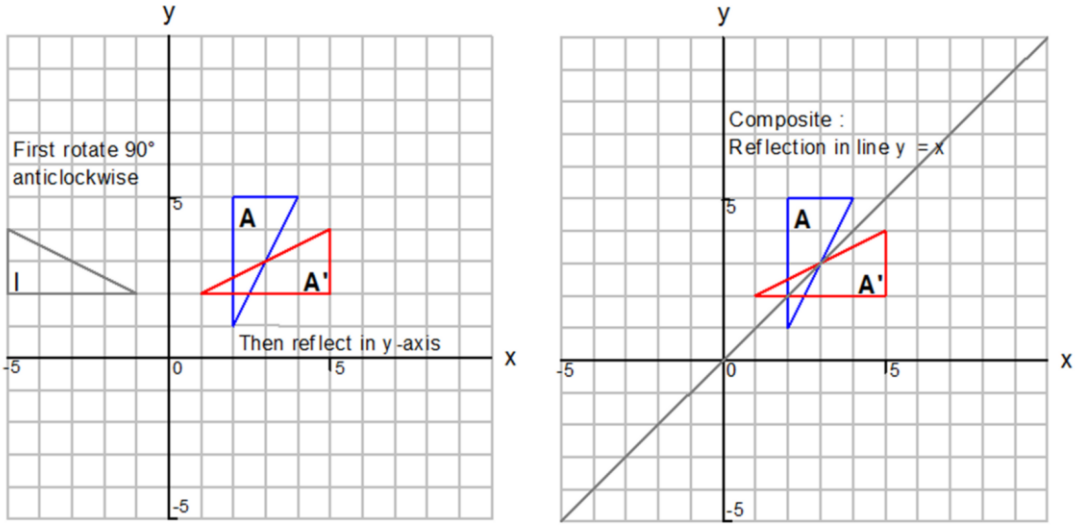
The reflection in the *y*-axis transforms triangle **A** to triangle **I**, and rotating the intermediate result **I** through 90° anticlockwise about the origin gives the final result **A'**.

This composite transformation is equivalent to the single transformation of reflection in the line $y = -x$.





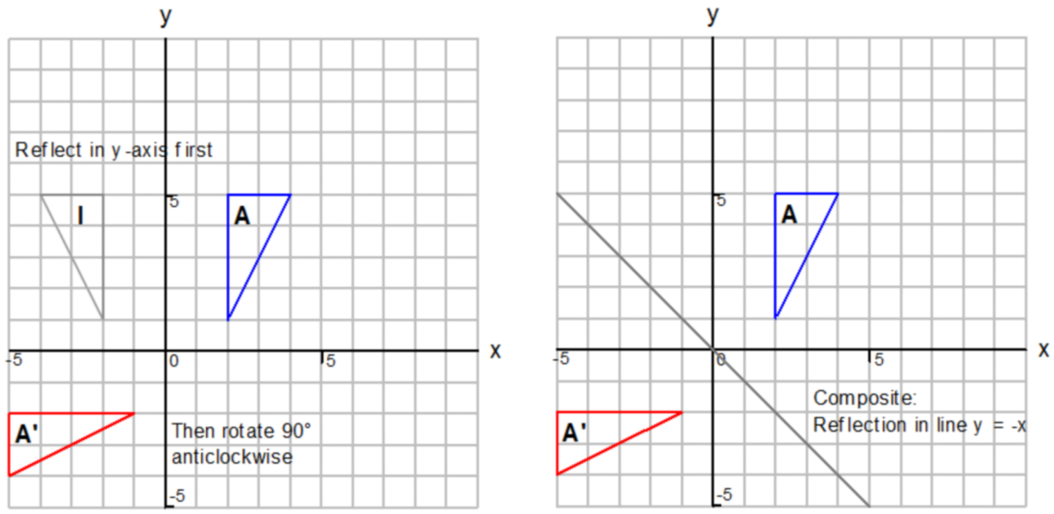
Example (21): Repeat Example (20), but this time translate triangle **A** by first rotating it 90° anticlockwise about the origin and then reflecting it in the *y*-axis.



What is the single transformation mapping **A** to **A’** ?

The 90° anticlockwise rotation about the origin transforms triangle **A** to the intermediate triangle **I**, and reflecting **I** in the *y*-axis gives the final result **A’**.

A glance at the diagrams above shows that the composite transformation is equivalent to the single transformation of reflection in the line $y = x$ and not the line $y = -x$ from the last example.



The last two examples show that the order in which transformations are carried out is important. The results are generally different in the two cases !