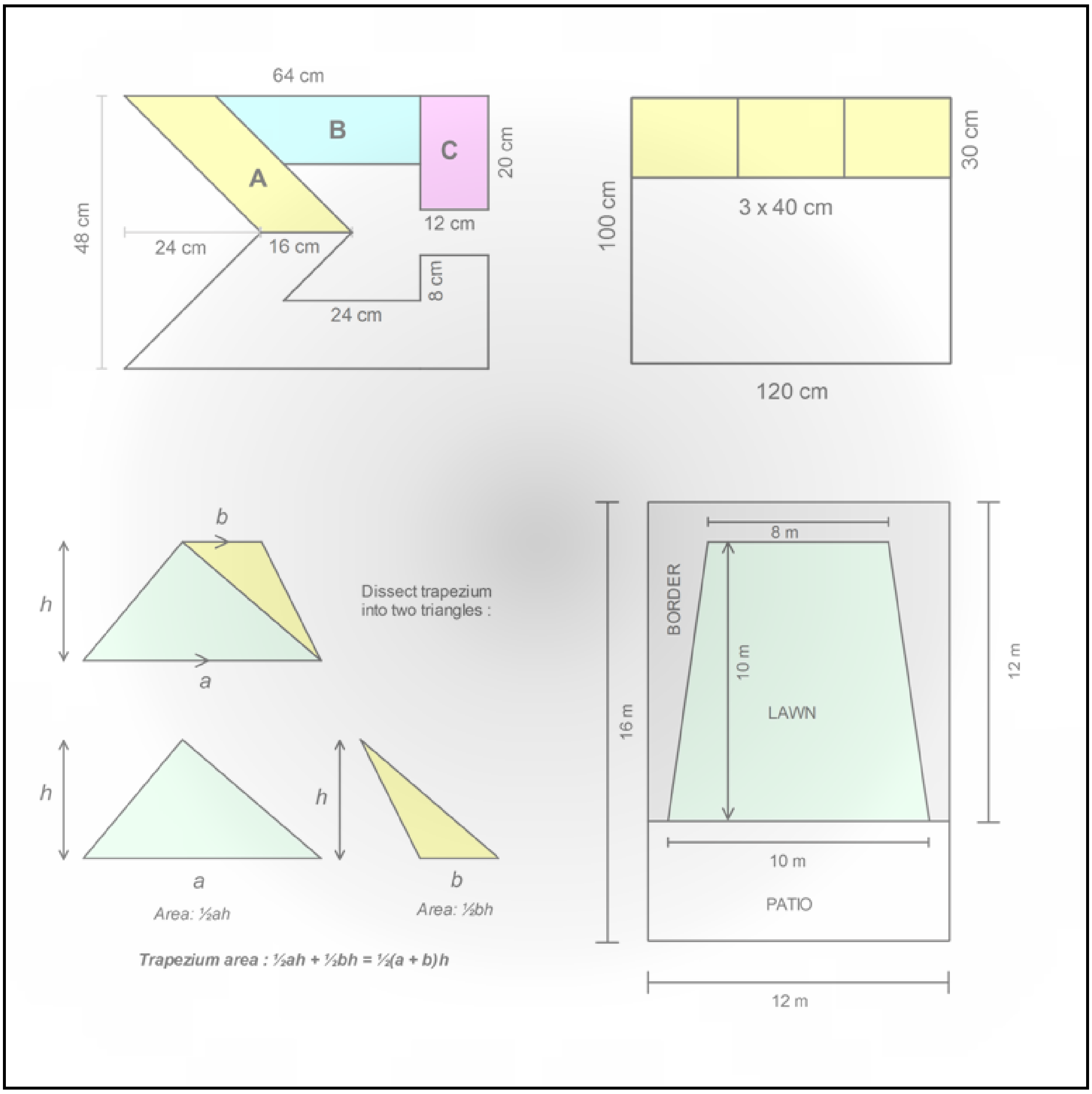
M.K. HOME TUITION

Mathematics Revision Guides

Level: GCSE Higher Tier

MEASURING SHAPES



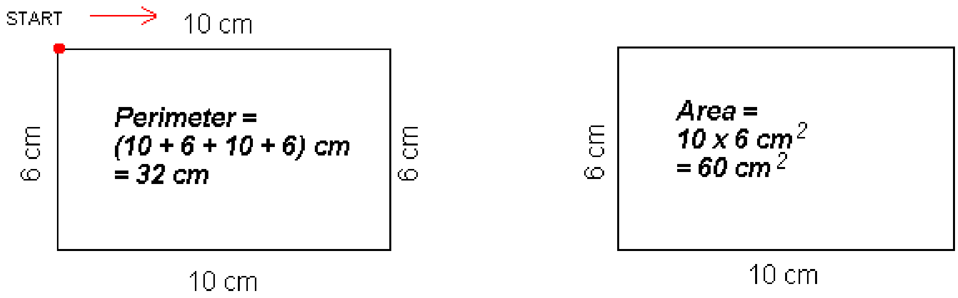
Version: 2.2 Date: 16-11-2015

MEASURING SHAPES

(Most of this section overlaps with Foundation Tier)

Perimeter and area.

Those two terms are sometimes confused – the examples below will explain the difference.

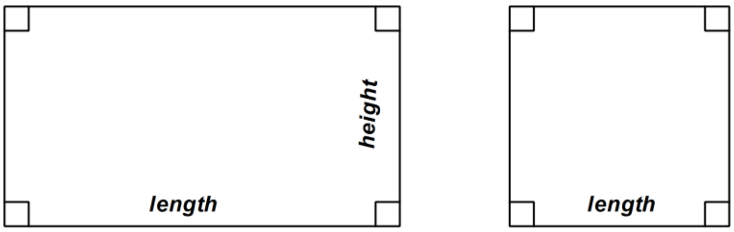


The **perimeter** of a shape is the distance around it, and is found by adding the side lengths together. Thus the perimeter of a rectangle measuring 10 cm by 6 cm is (10 + 6 + 10 + 6) cm or 32 cm. Since the opposite sides of a rectangle (or for that matter, any parallelogram) are equal in length, we can also say that the perimeter is $2 \times (10 + 6) = 32$ cm.

Notice that the perimeter is a **length** – it is measured in centimetres here.

The **area** of a shape is the amount of space it occupies in two dimensions – the 10 cm by 6 cm rectangle has an area of $10 \times 6 = 60$ cm^2. Notice that the area is measured in **square** centimetres.

The rectangle and square.



Area of rectangle = length x height **Area of square = length^2**

Example (1): Find the perimeter and area of i) a rectangle measuring 8 cm by 5 cm;
ii) a square of side 6 cm.

i) The rectangle has a perimeter of $2 \times (8 + 5) = 26$ cm, and an area of $8 \times 5 = 40$ cm^2.

ii) The square has four sides of 6 cm, so its perimeter is simply 6 cm $\times$ 4 , or 24 cm.
Its area is 6×6 cm^2, or 36 cm^2.

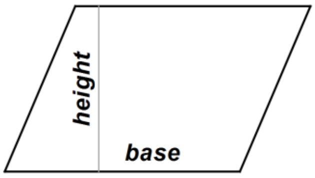
In general, the perimeter of a **regular** polygon is simply the side length multiplied by the number of sides. Thus a regular hexagon of side 5 cm has a perimeter of 30 cm.

(Areas of regular polygons other than squares are more complicated and only a few can be analysed at GCSE level.)

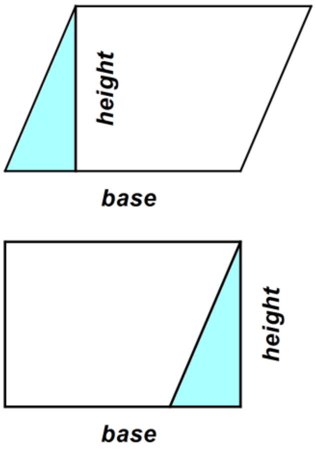
The parallelogram.

In the case of the parallelogram, the height is specifically the height **perpendicular** to the base, and not the length of the sloping side. The dissections in the right-hand diagrams demonstrate the formula.

(We remove a right-angled triangle from one side of the parallelogram and move it to the other).



Area of parallelogram = base x height

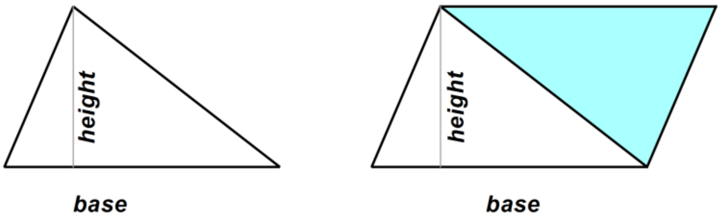


The triangle.

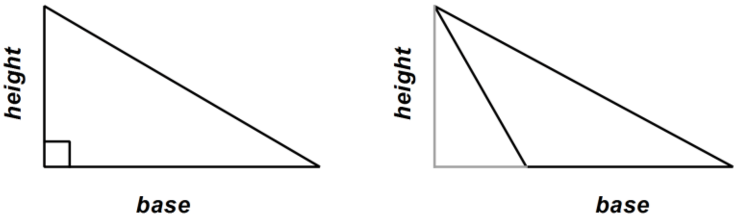
We again use the perpendicular height to measure triangles. The diagram upper right shows how we can derive the area formula from that of the parallelogram by "doubling up" the triangle.

The right-angled triangle is a special case, where the height and base correspond to the sides containing the right angle.

Note also how we need to extend the base of the obtuse-angled triangle to obtain the height.

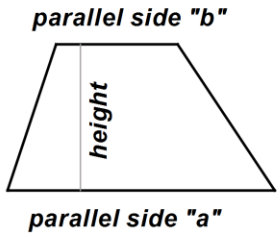


Area of triangle = ½ (base x height)



The trapezium.

With the trapezium, we must take the mean (average) of the two parallel sides and not the others !
Again the height is perpendicular to the two parallel sides.

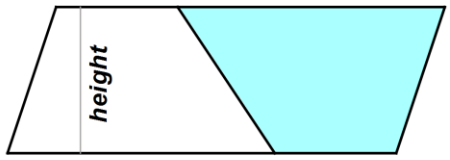


**Area of trapezium =
½ (side "a" + side "b") x height**

We can visualise this formula by joining together two identical trapezia to form a parallelogram. This resulting parallelogram has a base whose length is the sum of the two parallel sides and whose height is the same as that of the original trapezium.

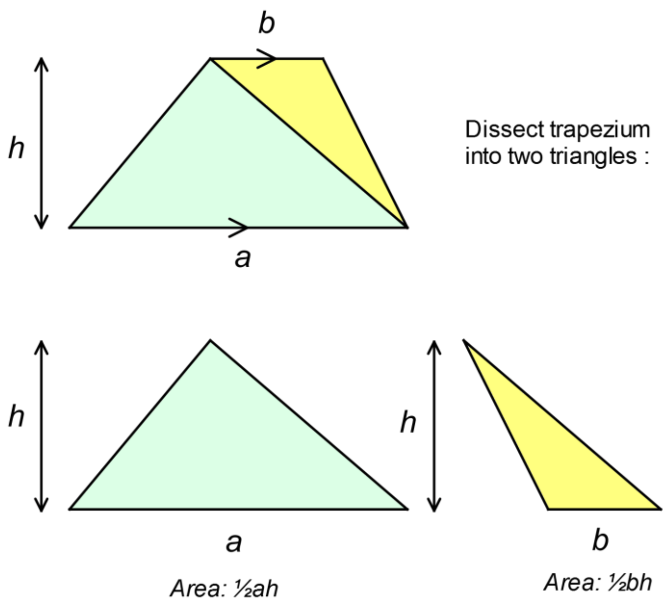
To find the area of the original trapezium, we have to halve the sum of the parallel sides before multiplying by the height.

(Double up to make a parallelogram)



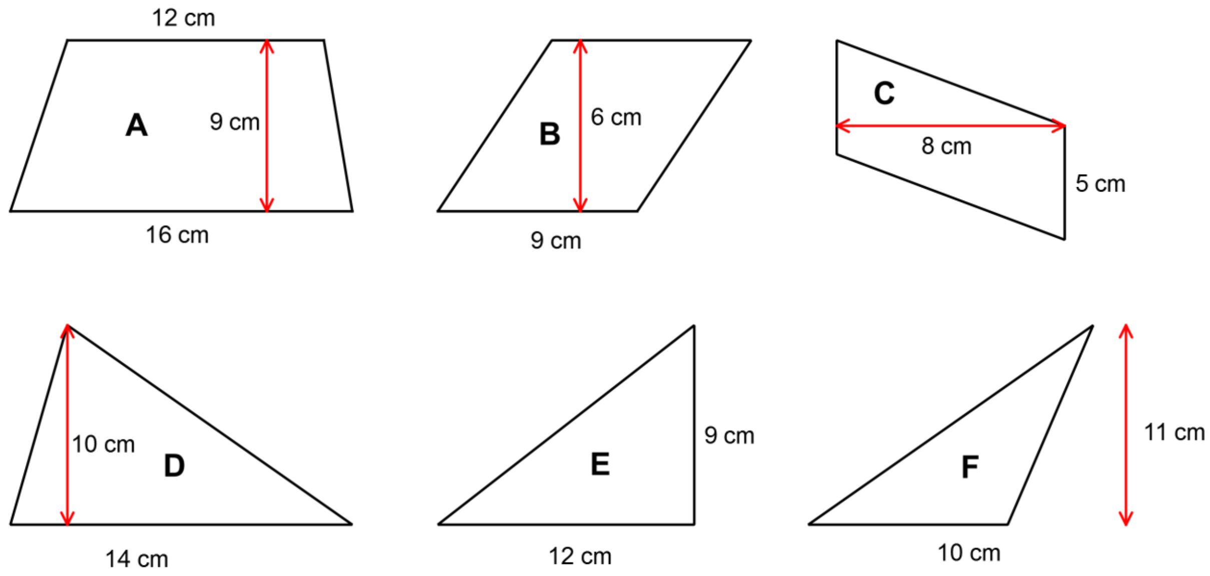
Total length : side "a" + side "b"

Another dissection is shown below, where we use the triangle area formula.



Trapezium area : ½ah + ½bh = ½(a + b)h

Example (2): Find the areas of the shapes below (not to scale).



Shape **A** is a trapezium whose parallel sides are 12cm and 16cm respectively, and whose height is 9 cm. Its area is therefore $\frac{1}{2}(12 + 16) \times 9$ cm^2 = **126 cm^2**.

Shape **B** is a parallelogram of base 9 cm and height 6 cm, so its area is (9×6) cm^2 = **54 cm^2**.

Shape **C** is also a parallelogram, but in this case it is better to treat the vertical as the base and the horizontal as the height. The base is 5 cm and the height 8 cm, so the area is **40 cm^2**.

Shape **D** is a triangle of base 14 cm and height 10 cm, and so its area is $\frac{1}{2}(14 \times 10)$ cm^2 = **70 cm^2**.

Shape **E** is a right-angled triangle of base 12 cm and height 9 cm, so its area is $\frac{1}{2}(9 \times 12)$ cm^2 = **54 cm^2**.

Shape **F** is a triangle of base 10 cm and height 11 cm, and so its area is $\frac{1}{2}(10 \times 11)$ cm^2 = **55 cm^2**.

Example (3): Find the areas of: i) a rectangle measuring 8 cm × 9 cm; ii) a square of side 7 cm; iii) a triangle of base 10 cm and perpendicular height 12 cm; iv) a trapezium of height 6 cm, and whose parallel sides are 7 cm and 15 cm long.

i) The area of the rectangle is (8 × 9) cm^2 = **72 cm^2**.
ii) The area of the square is 7^2 cm^2, or **49 cm^2**.
iii) The area of the triangle is ½(10 × 12) cm^2 = **60 cm^2**.
iv) The area of the trapezium is ½(7 + 15) × 6 cm^2 = **66 cm^2**.

Sometimes a question might quote the area or perimeter of a figure, and ask for one of the other measurements.

Example (4): Find the following:
i) the short side of a rectangle of area 42 cm^2 and long side of 7 cm
ii) the base of a parallelogram of area 36 cm^2 and perpendicular height 4 cm
iii) the perpendicular height of a triangle of area 40 cm^2 and base 8 cm
iv) the base of a triangle of area 48 cm^2 and perpendicular height 6 cm
v) the side of a square of area 100 cm^2

i) The short side of the rectangle is $\frac{42}{7}$ cm or **6 cm**. (Divide area by long side.)

ii) The base of the parallelogram is $\frac{36}{4}$ cm or **9 cm**. (Divide area by perpendicular height.)

iii) Half of the base of the triangle is 4 cm, so the perpendicular height is $\frac{40}{4}$ cm or **10 cm**. (Divide area by one-half of the base.)

iv) Half of the base of the triangle is $\frac{48}{6}$ cm or 8 cm, so the base is **16 cm**. (Divide area by perpendicular height and double.)

v) By inspection, the square root of 100 is 10, so the sides of the square are **10 cm** long.

Example (5): The perimeter of a rectangle is 26 cm and its area is 40 cm^2. Find the lengths of the sides, given that they are a whole number of centimetres.

We are looking for two numbers which give 40 when multiplied together, and half of 26, or 13, when added together. Such a pair of numbers is 8 and 5, so the rectangle measures 8 cm × 5 cm.

(Check : area = 8 × 5 = 40; perimeter = 2 × (8 + 5) = 26.)

Example (6): A trapezium has an area of 72 cm^2 and a height of 8 cm. The longer of the two parallel sides is 11 cm long. Find the length of the shorter side.

The area of a trapezium is the mean of the two parallel sides multiplied by the height, so the mean of the parallel sides here is (area ÷ height) , or $\frac{72}{8}$ cm, i.e. 9 cm.

If the mean of the two parallel sides is 9 cm, their sum must be twice that, or 18 cm.
The longer parallel side is given as 11 cm, and so the shorter one must be **7 cm**.

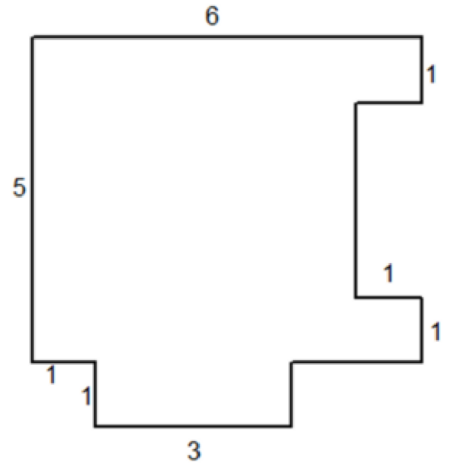
Finding perimeters and areas of compound shapes.

Many complex shapes can be broken up into simpler ones, such as rectangles and triangles, which makes area calculations easier.

Example (7): Find the perimeter and area of the room whose plan is shown on the right. All angles are right angles, and lengths are quoted in metres.

The first apparent problem here is that four of the lengths are missing.

We therefore label the corners on the diagram and use reasoning to find the missing sides.



To find the distance CD, we notice that it is parallel to EF and equal in length , ∴ **CD = 1 m.**

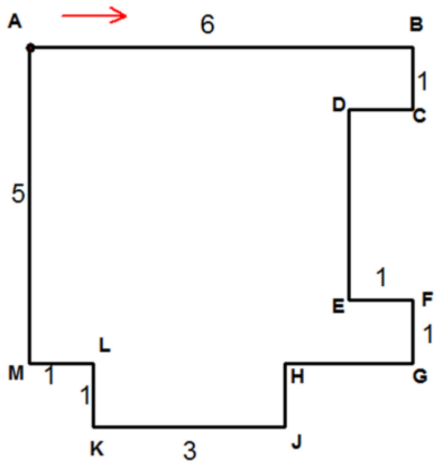
We see that AM = 5 m, so BC + DE + FG = 5 m
Since BC = 1 m and FG = 1m, **DE = 3 m.**

Also, we can see that AB = 6 m, and therefore
GH + JK + LM = 6 m. As JK + LM = 4 m , **GH = 2 m.**

Finally, HJ = KL = 1 m.

The perimeter of the room is therefore AB + BC + CD.... + MA, or

(6 + 1 + 1 + 3 + 1 + 1 + 2 + 1 + 3 + 1 + 1 + 5) metres, i.e. **26 metres**.



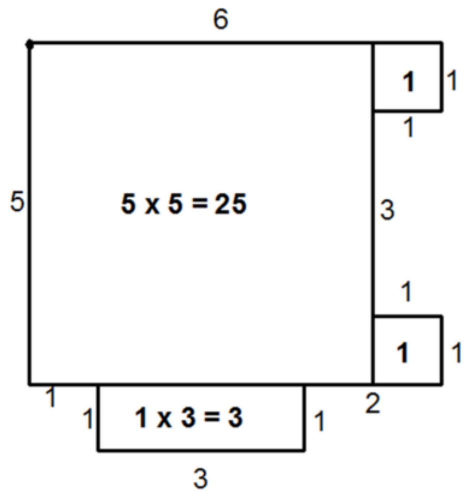
The area can be found by splitting the figure into rectangles.

The method shown right is one of many possible ones, and is probably the easiest at GCSE.

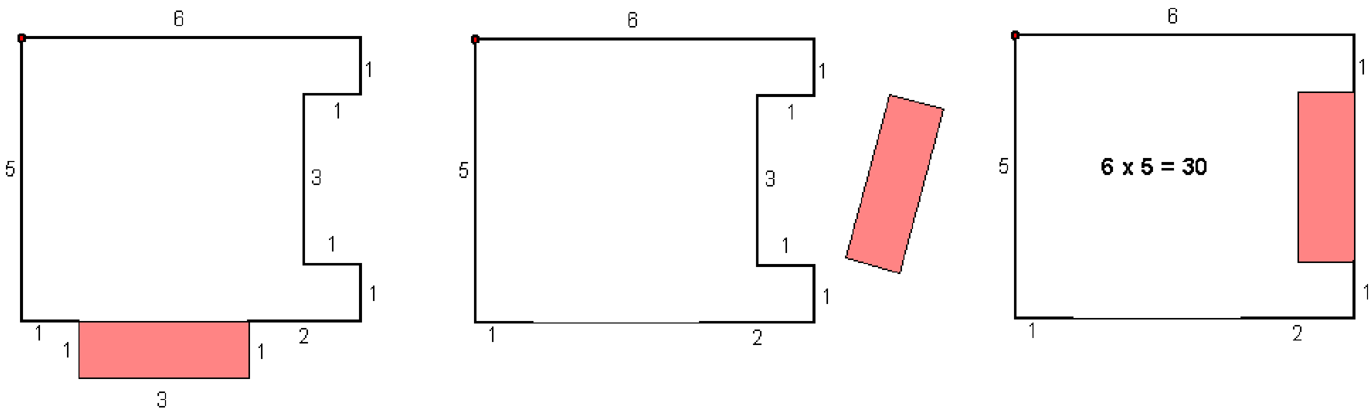
The largest section is a 5 × 5 m square, remembering that 1 m has been cut off the 6 m side.

The remaining sections are a 3 × 1 m rectangle and two 1 m squares.

The complete area of the room is 25 + 3 + 1 + 1 m^2 or **30 m^2.**

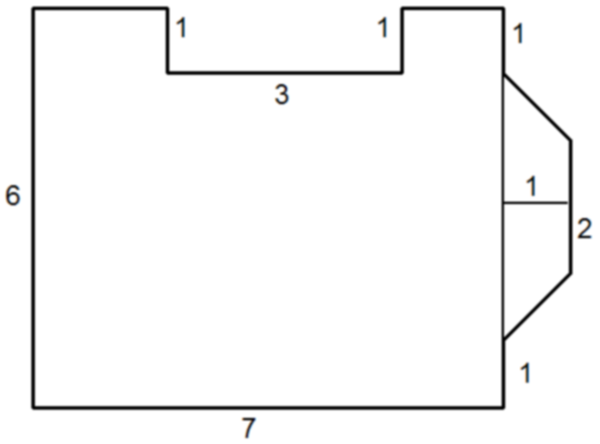


Another, more elegant, method is shown below:



Example (8): Find the area of the bay-windowed room below (lengths in metres):

This time we have two unspecified lengths along the upper edge, the diagonal part of the bay, and the width of the bay itself.

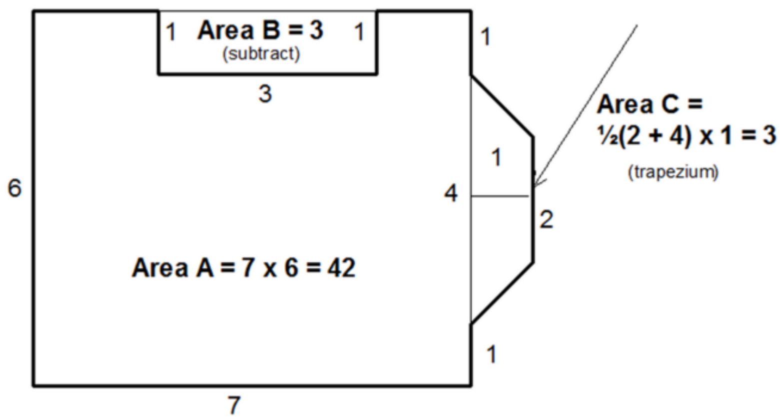


The best way of looking at this shape is to visualise it as a 7 × 6 rectangle with a smaller 3 × 1 rectangle removed, plus a bay section.

The bay plus the two short 1 m sections must add up to the 6 m of the opposite wall, so the bay is 4 m at its maximum.

The bay is in fact a trapezium 1 m high and with parallel sides of 2 m and 4 m.

To work out the area, we first treat the main room as area A, namely a 7 × 6 m rectangle with area 42 m^2.



Next, we subtract the small area B, namely 3 m^2, and add back the bay area C, also 3 m^2.

The total area of the room is
(42 – 3 + 3) m^2 = 42 m^2.

Interestingly, the fact that the lengths along the upper edge are not specified does not prevent us from finding the area here !

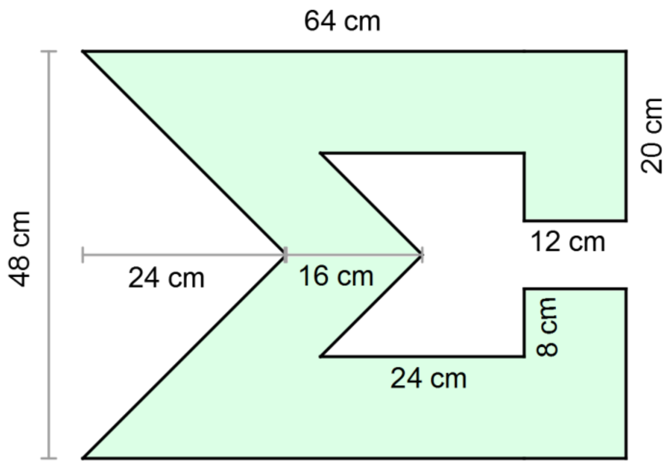
The method of dividing up into simpler shapes is necessary if the diagram has no grid, or is not drawn accurately.

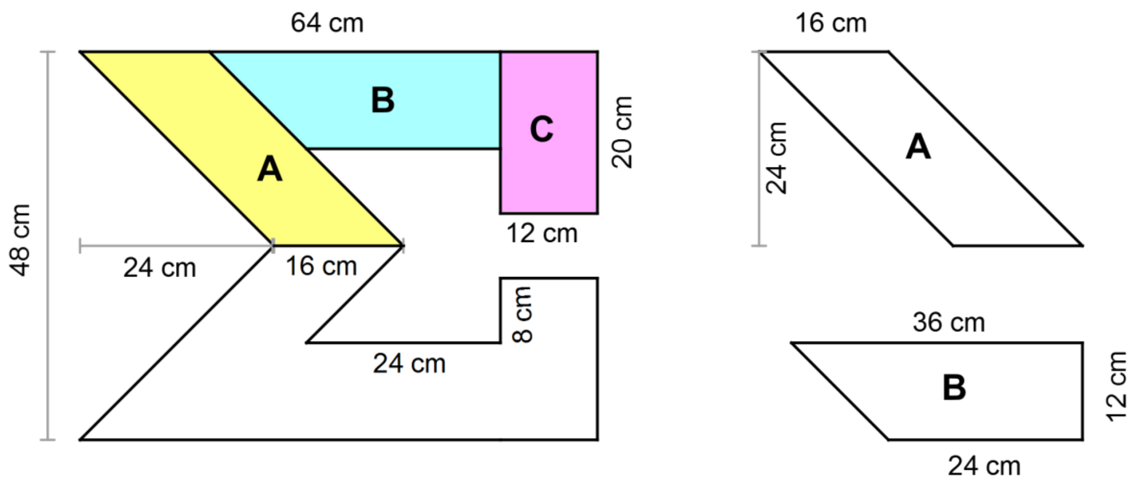
Example (9): Sigma Solutions want to display their logo (shown right) in their head office reception room.

Calculate the area of the logo, given that it has a horizontal line of symmetry.

The dissection shown below is only one of many possible ones !

The logo can be split into two parallelograms '**A**', two trapezia '**B**' and two rectangles '**C**' in mirror-image pairs due to its symmetry.





The rectangles '**C**' are the easiest to deal with : each measures 20 cm $\times$ 12 cm for an area of 240 cm^2. Combined, they make up **480 cm^2** of the total area.

One parallelogram '**A**' has a base of 16 cm and a height of 24 cm (half the height of the whole logo), giving an area of 384 cm^2, so two of them make up **768cm^2** of the total.

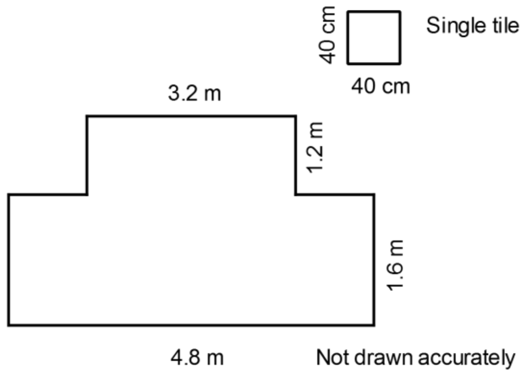
The trapezium '**B**' is the trickiest to work out, as only the shorter parallel side of 24 cm is given. We have to subtract the base of parallelogram '**A**' (16 cm) and the short side of rectangle '**C**' (12 cm) from the total logo length of 64 cm to find the longer parallel side, measuring 36 cm.

The height of 12 cm is obtained by subtracting the overhang of 8 cm from the long (20 cm) side of rectangle '**C**'.

Substituting into the trapezium area formula we have ½(24 + 36) $\times$ 12 cm^2 or 360 cm^2 for the area of one trapezium. Doubling up, the two trapezia have an area of **720 cm^2**.

The total area of the logo is therefore (480 + 768 + 720) = **1968 cm^2**.

Example (10): Julie wishes to tile her dining-room floor, as per the diagram on the right.



The tiles are square, measuring 40 cm × 40 cm.

i) Calculate how many tiles Julie needs to cover the floor.

The tiles are for sale at The Tile Warehouse at £49.99 for a pack of 25.
Tiles Direct also sell the same tiles on a special offer price of £18.99 for a pack of 10.

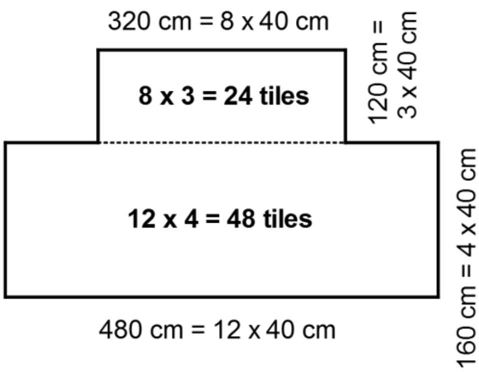
ii) Julie compares the prices and reasons as follows:

" The Tile Warehouse's offer is £49.99 - let's call it £50, and 50 divided by 25 is 2, so it's £2 per tile. Tiles Direct's price of £18.99 – call it £19, and one tenth of £19 is £1.90, which is 10p less per tile. Tiles Direct is the better choice."

a) Explain why Julie's reasoning might not be fully correct.

b) Calculate, for each shop, the price Julie would have to pay for tiling the floor, showing all working.

i) The floor area can be divided up into two rectangles, a larger one of (480 × 160) cm and a smaller one of (320 × 120) cm. (Note: 1 m = 100 cm)



Since all the lengths in centimetres are whole-number multiples of 40 cm, we can divide each length by 40 to obtain the number of tiles for each rectangular area.

Since $\frac{480}{40}$ = 12 and $\frac{160}{4}$ = 4, the number of tiles needed to cover the larger rectangle is 12 × 4 = 48.

Also, as $\frac{320}{40}$ = 8 and $\frac{120}{40}$ = 3, the number of tiles needed to cover the smaller rectangle is 8 × 3 = 24.
Hence Julie needs 48 + 24, or **72**, tiles in total to cover the floor.

ii) a) Julie's reasoning is quite correct when it comes to calculating the prices per tile, but the issue here is that the number of tiles used, namely 72, is neither a multiple of 10 nor of 25, and so there would inevitably be some tiles left over no matter which shop she chose. Julie did not realise that this wastage would be greater with the Tiles Direct deal, thus cancelling out the lower price per tile

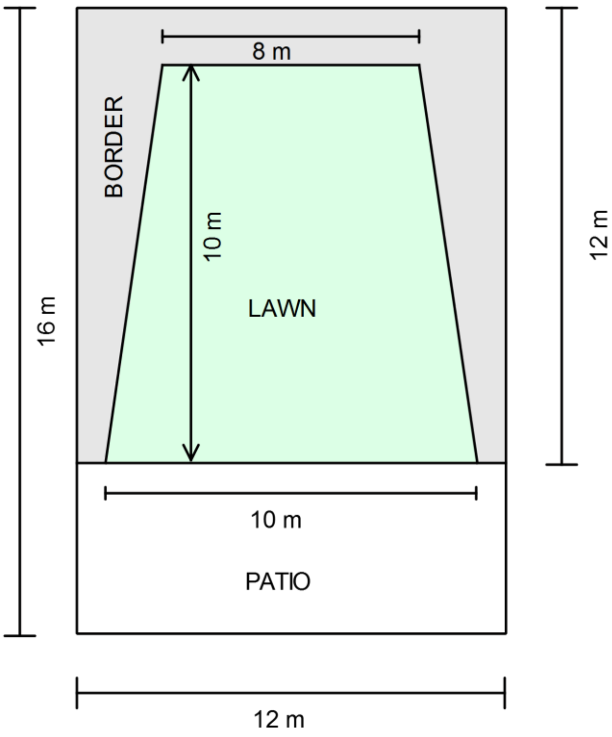
ii) b) If Julie were to go to Tiles Direct and buy in packs of 10, she would need to buy 8 packs, totalling 80 tiles, as 7 packs, or 70 tiles, would not be enough. The tiles would therefore cost Julie (8 × £18.99) , or **£151.92**, at Tiles Direct, and she would have 8 tiles left over.

If Julie were to go to The Tile Warehouse and buy in packs of 25, she would need to buy 3 packs, totalling 75 tiles. Here the tiles would cost Julie (3 × £49.99) , or **£149.97**, at The Tile Warehouse, leaving 3 tiles to spare.

The Tile Warehouse is cheaper, since buying in packs of 25 results in fewer tiles being left over even though the price per tile is higher.

There is an interesting aside here. Julie could save more money if she were to split her purchase by buying 1 pack of 25 tiles from The Tile Warehouse and 5 packs of 10 tiles from Tiles Direct.
She would then end up paying (1 × £49.99) + (5 × £18.99) = **£144.94** for 75 tiles with 3 tiles to spare.

Example (11): The plan of Sam's garden (not to scale) is shown on the right. The lawn is in the shape of a trapezium, and both the patio and the garden as a whole are rectangular.



Sam's maintenance tasks for the year are to clean the patio, spread fresh topsoil on all the borders, and to treat and feed the lawn once in spring.

His materials cost as follows :

Patio cleaner : £8.99 a bottle, treating 25 m^2

Topsoil : £59 for a bulk bag covering 20 m^2

Lawn food and treatment : available in two sizes, £3.99 for 60 m^2 , £2.49 for 30 m^2

The garden centre also offers a discount of 10% for purchases over £200 in total.

Calculate the total cost, after discount if appropriate, of all of Sam's gardening materials for the year.

The area of the patio is $12 \times (16 - 12)$ m^2 = 48m^2. The number of bottles of patio cleaner he needs is $\frac{48}{25}$ = 1.92, or 2 to the next integer above, for a total of £17.98.

The area of the lawn is ½(8 + 10) × 10 m^2 = 90 m^2. One large pack at £3.99 (for 60 m^2) plus one small one at £2.49 (for 30 m^2) will therefore be sufficient.

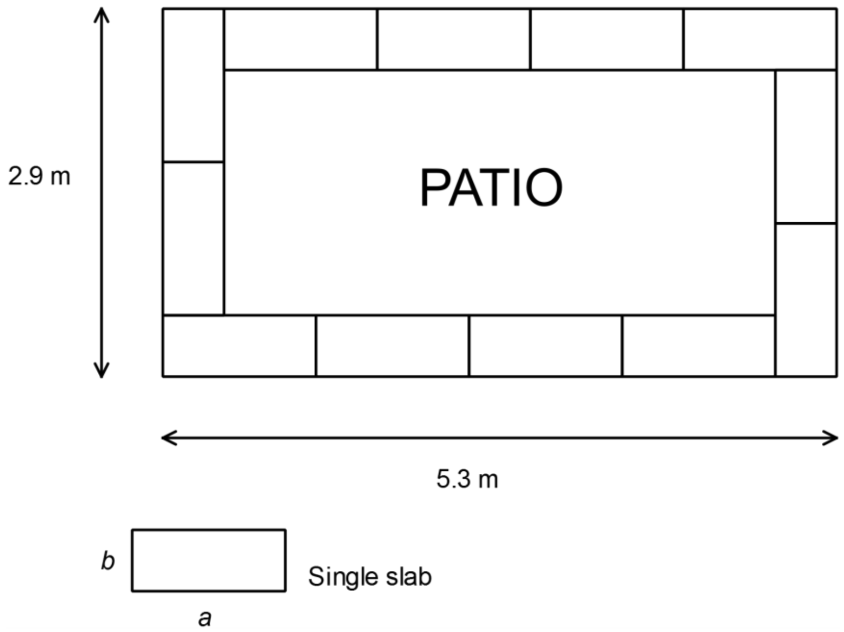
The border and the lawn have a combined area of 12 × 12 – 144 m^2, so we subtract the 90 m^2 area of the lawn to obtain the border's area of 54 m^2.
Sam needs $\frac{54}{20}$ = 2.7 bulk bags of topsoil, or 3 to the next integer above. These three bags of topsoil will cost him £177.

His total cost before discount is £201.46, so his purchase qualifies for a further 10% off, giving him a final bill of £181.31.

His final cost breakdown is :

2 bottles patio cleaner @ £8.99 =	£17.98
1 pack lawn food for 60m^2 @ £3.99 =	£3.99
1 pack lawn food for 30m^2 @ £2.49 =	£2.49
3 bulk bags topsoil @ £59 =	£177.00
TOTAL =	£201.46
Less discount of 10% =	-£20.15
FINAL TOTAL	**£181.31**

Example (12): Rakesh wants to border his rectangular patio with slabs all round, as illustrated on the plan below (not to scale), along with a single rectangular slab.



i) Find the lengths of the long and short sides of a single slab.
ii) Hence calculate the area of the patio.

i) The long side of the bordered patio is 5.3 m in length, which is equivalent to four long sides '*a*' plus one short side '*b*' of a single slab. The length of the short side is 2.9 m, or two long sides '*a*' and one short side '*b*'.

From this information, we can set up simultaneous equations.

$4a + b = 5.3$ *A*
$2a + b = 2.9$ *B*

By subtracting equation *B* from equation *A* we can eliminate *b* :

$4a + b = 5.3$ *A*
$2a + b = 2.9$ *B*

$2a = 2.4$ *A-B* $\therefore$ $\boldsymbol{a = 1.2}$

The long side of the slab is 1.2 m long, so by substituting 1.2 for *a* in the first equation we have $2.4 + b = 2.9$, so $b = 0.5$.

The slabs therefore measure **1.2 m by 0.5 m.**

ii) The short sides of the slabs have a length of 0.5 m, so we subtract twice that length, or 1 m, from both the length and height of the bordered patio.

The patio without the border therefore measures 4.3 m by 1.9 m, giving an area of **8.17m^2**.

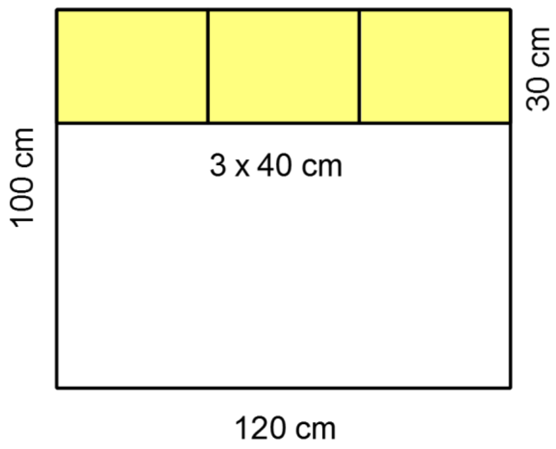
Example (13): A warehouse stores flat packs of carpet tiles measuring 40 cm $\times$ 30 cm. The packs are stored in layers on a pallet measuring 120 cm $\times$ 100 cm.

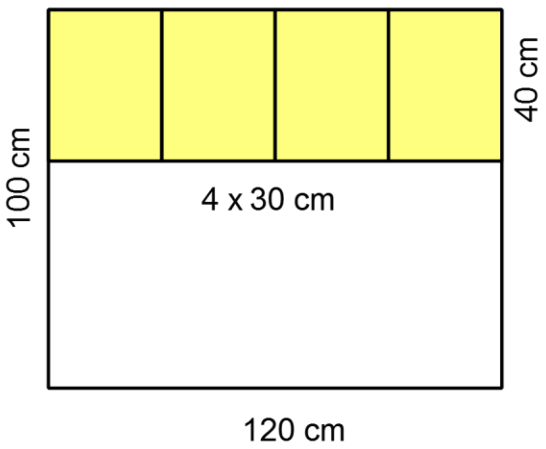
Show that ten such packs can completely cover the pallet to form a layer. You may use a diagram.

When we look at the dimensions of the pallet and the packs of tiles, we can see that both 30 and 40 are factors of 120, but that neither is a factor of 100.

This means that a long side of the pallet can have rectangular rows, but a short side cannot.

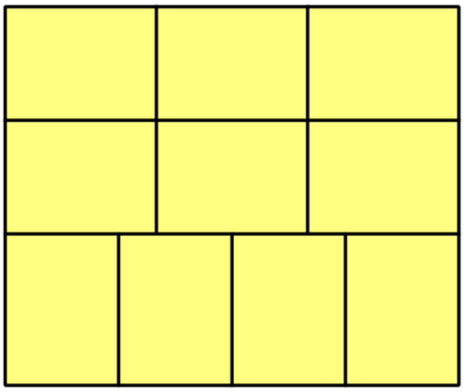
We can stack the cases to form rows on the pallet base in two distinct ways :





In the first case, we can place three packs to form a row measuring 120 cm by 30 cm; in the second case, the row measures 120 cm by 40 cm.

We need to find some combination of multiples of 30 and multiples of 40 that can add to 100 in order to find the required layer pattern. By trial and error, we find that $100 = (2 \times 30) + (1 \times 40)$, as the diagram on the right shows.

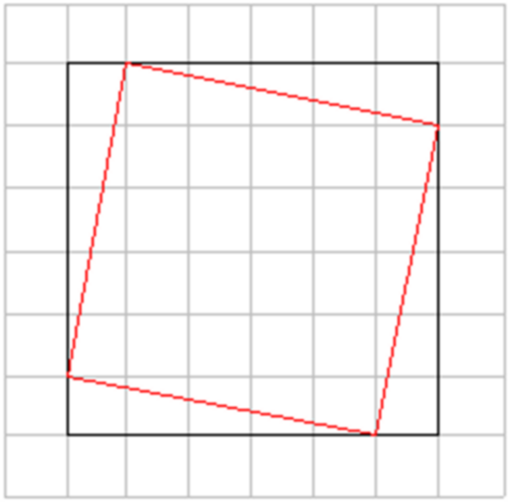


There are 3 packs in the first row from the top, 3 in the second and 4 in the third, making 10 packs per row in total.

Sometimes it is possible to find the area of a shape using differing methods.

Example (14): Find the area of the tilted square shown in the diagram on the right. (Do not simply count squares).

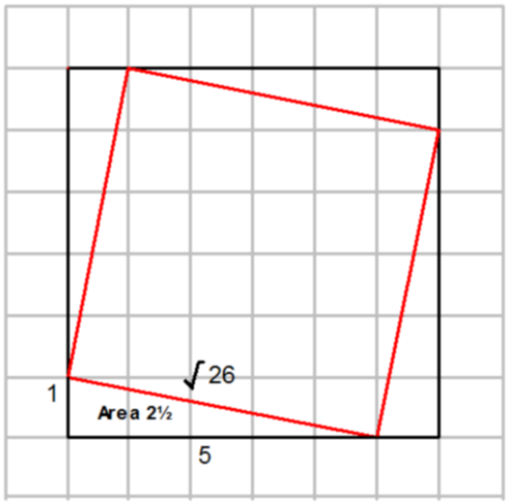
There are two possible methods here.



Method (1): We can see that the tilted square is enclosed in a larger 6 × 6 square, and that there are 4 identical right-angled triangles making up the difference.

The height and base of each triangle are 1 unit and 5 units, so the area of each is ½ × 5 × 1 or 2½ square units.

The large square has an area of 36 square units, and the 4 triangles have a combined area of 10 square units, therefore the tilted square has an area of (36 – 10) or 26 square units.



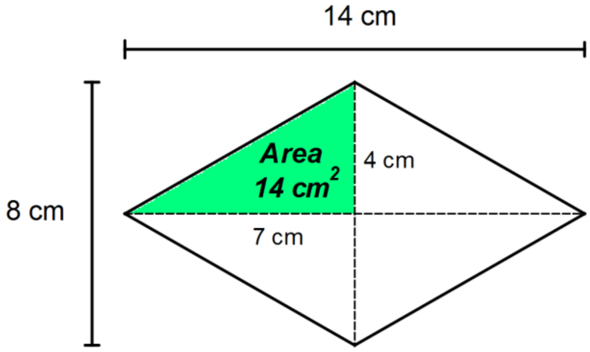
Method (2): We can see that the side of the tilted square makes up the hypotenuse of a right-angled triangle whose other two sides are 1 and 5 units.

Using Pythagoras' Theorem, the area of the tilted square is the square of the hypotenuse.
The square on the hypotenuse is $1^2 + 5^2 = 1 + 25 = 26$ square units.

(The length of the hypotenuse is √26, but as we are squaring it, the area is 26 square units.)

In certain cases, the area of a plane figure can be determined from the lengths of the diagonals alone.

Example (15): Find the area of a rhombus whose diagonals are 8 cm and 14 cm long.



(Remember that the diagonals of a rhombus bisect each other at right angles.)

Although a rhombus is a type of parallelogram, the base × height formula cannot be used here since we are not given either.

The rhombus can be broken up into four right-angled triangles, each of which has a base of 7 cm and a height of 4 cm (i.e. half the diagonal length).

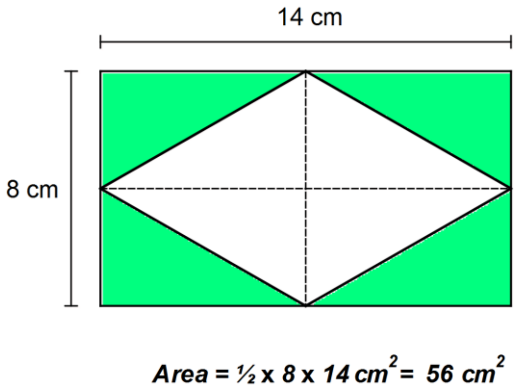
The area of a triangle is ½ (base × height), so here one small triangle has an area of 14 cm^2, and thus the entire rhombus has an area of **56 cm^2.**

This is also half the product of the diagonals, as can be shown below :

We can enclose the rhombus in a rectangle whose sides are parallel to the diagonals of the rhombus.

The sides of the rectangle are equal in length to the diagonals of the rhombus.

The unshaded triangles making up the rhombus have the same combined area as the shaded ones completing the rectangle.



The method of halving the product of the diagonals to find the area of a rhombus can also be used to find the area of a kite.