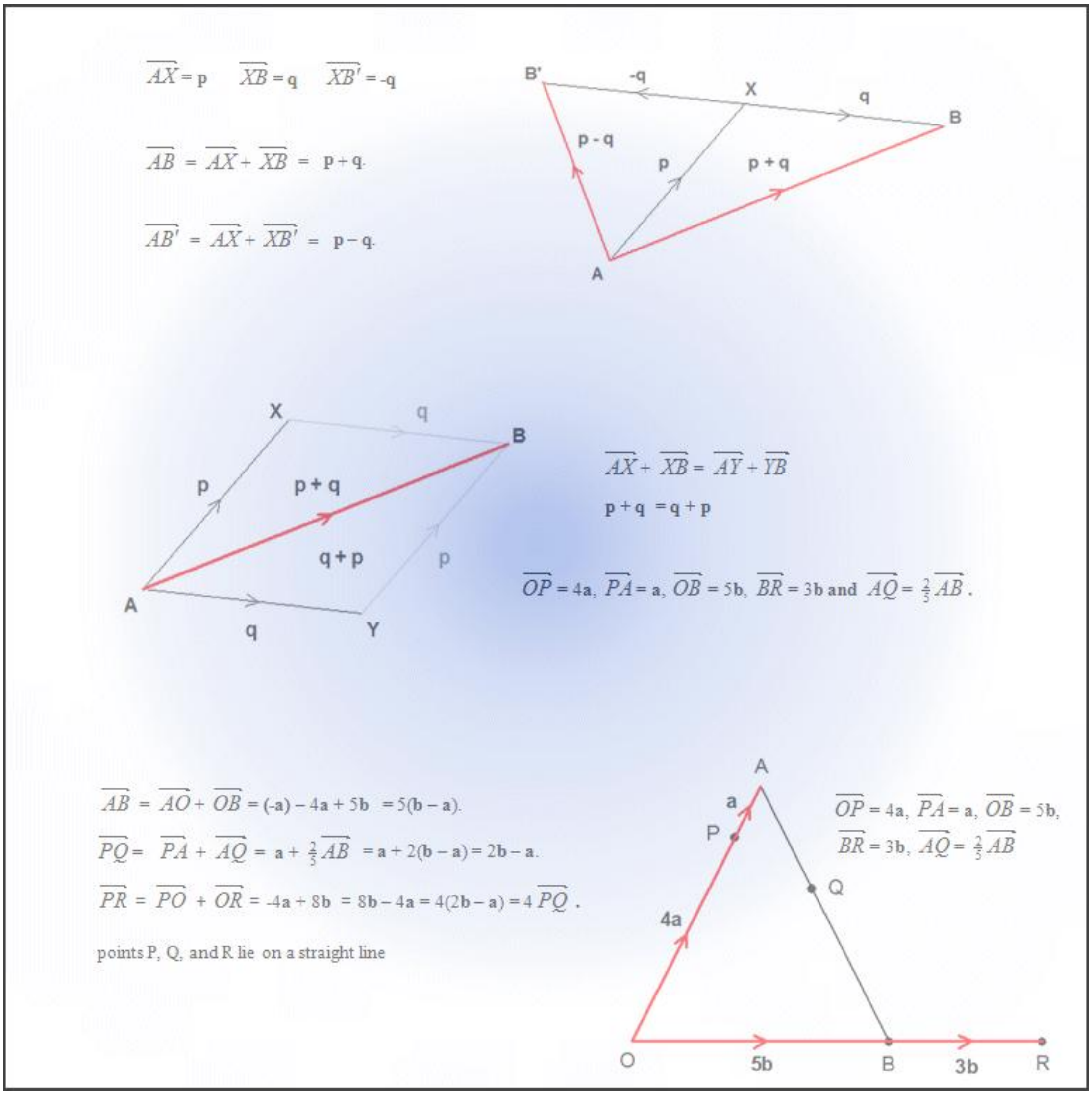
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

VECTORS



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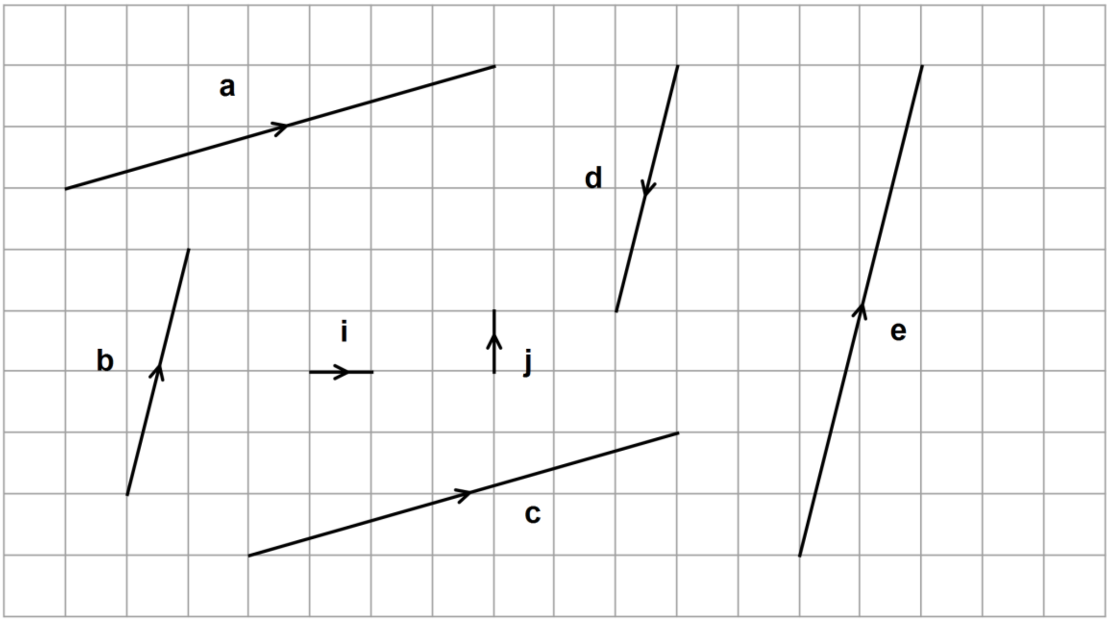
VECTORS

Vectors are used in mathematics to illustrate quantities that have size (magnitude) and direction.
Quantities like mass and length have magnitude only, and are called **scalars**.
Velocity and force, on the other hand, have direction as well as size and can be expressed as vectors.

There are various ways of denoting vectors: typed documents use boldface, but written work uses underlining. Thus **a** and $\underline{a}$ are the same vector.



Example (1): The diagram above shows a collection of vectors in the plane.

Describe the relationships between the following vector pairs :
i) **a** and **c** ; ii) **b** and **d** ; iii) **b** and **e ;** iv) **i** and **j**.

i) Vectors **a** and **c** are equal here; hence **a** = **c**.
Two vectors are equal if they have the same size and the same direction.

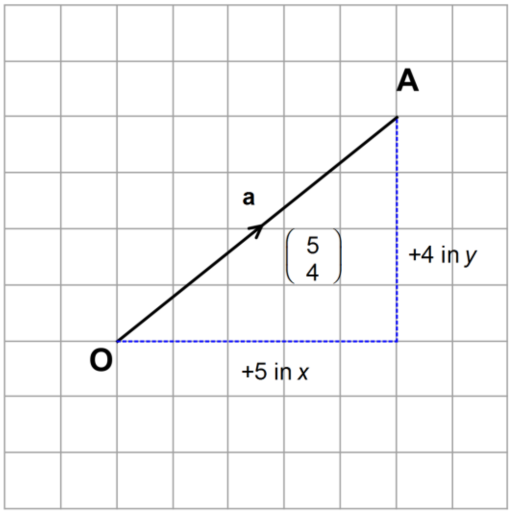
The fact that **a** and **c** have different start and end points is irrelevant.

ii) Vectors **b** and **d** have the same size, but opposite directions, therefore **d** = **-b**.
Two vectors are inverses of each other if they have the same size, but opposite directions.

iii) Vector **e** is exactly twice as long as vector **b**, so **e** = 2**b**.
The 2 is what is known as a **scalar** multiplier.
(A scalar multiplier of -1 signifies an inverse vector.)

iv) Vectors **i** and **j** are perpendicular to each other.

We have come across vectors in the section on "Transformations", when describing a translation.



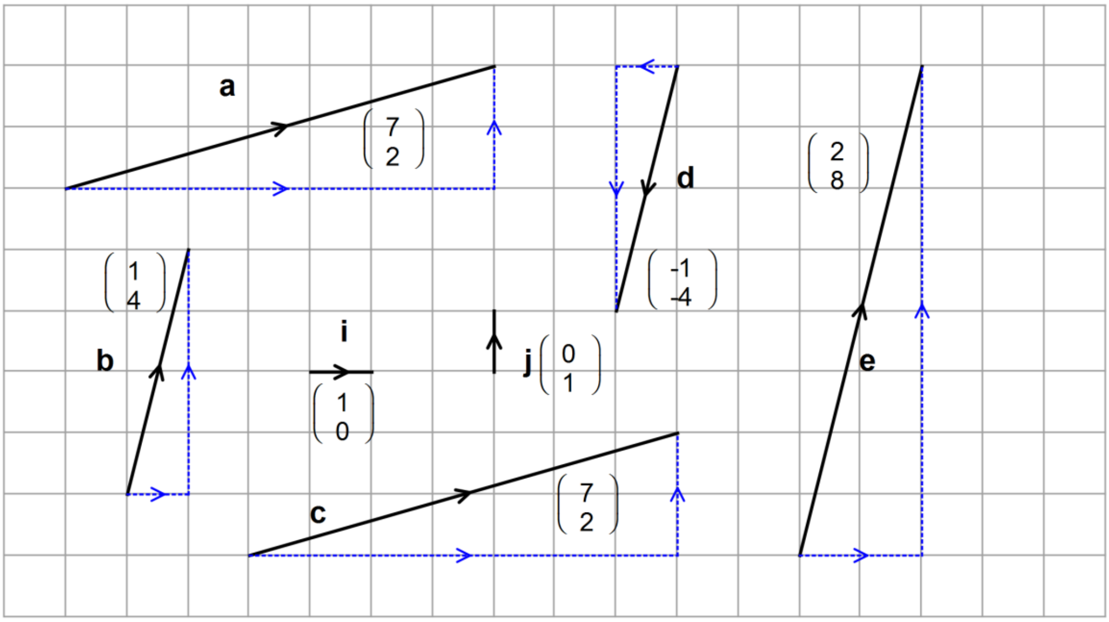
In the diagram on the right, the point O is transformed to point A by a translation of +5 units in the x-direction and +4 units in the y-direction.

We can denote this translation as the vector

$\mathbf{a} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, termed **column notation**.

Also, written work uses underlining where typing uses boldface, so we print **a** but write $\underline{a}$.

Example (2): Express the six vectors in the last example in column notation.



A movement in the direction of vector **a** corresponds to 7 units horizontally and 2 units vertically, as does that in the direction of vector **c**, given that the two vectors are equal .

Hence **a** = **c** = $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$

For vector **b**, the values are 1 horizontally and 4 vertically. Vector **d** is the inverse of vector **b**, so the movement is -1 unit horizontally and -4 units vertically.

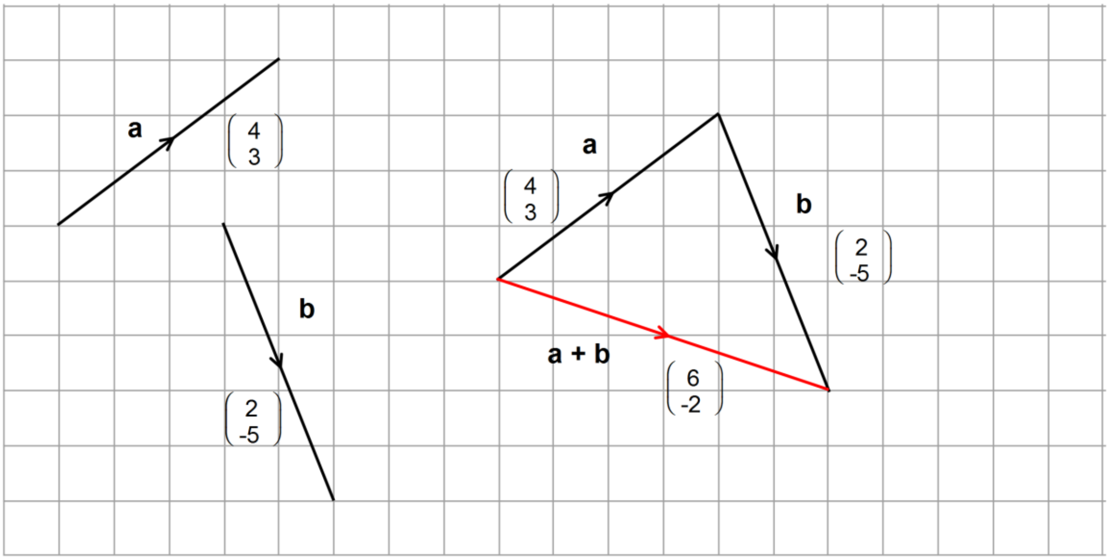
Hence **b** = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and **d** = - $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ = $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Vector **e** is twice vector **b** , so **e** = $2\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ = $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

Finally the vector **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Addition of vectors.

Example (3):



To add two vectors, join them "nose to tail" as in the diagram.

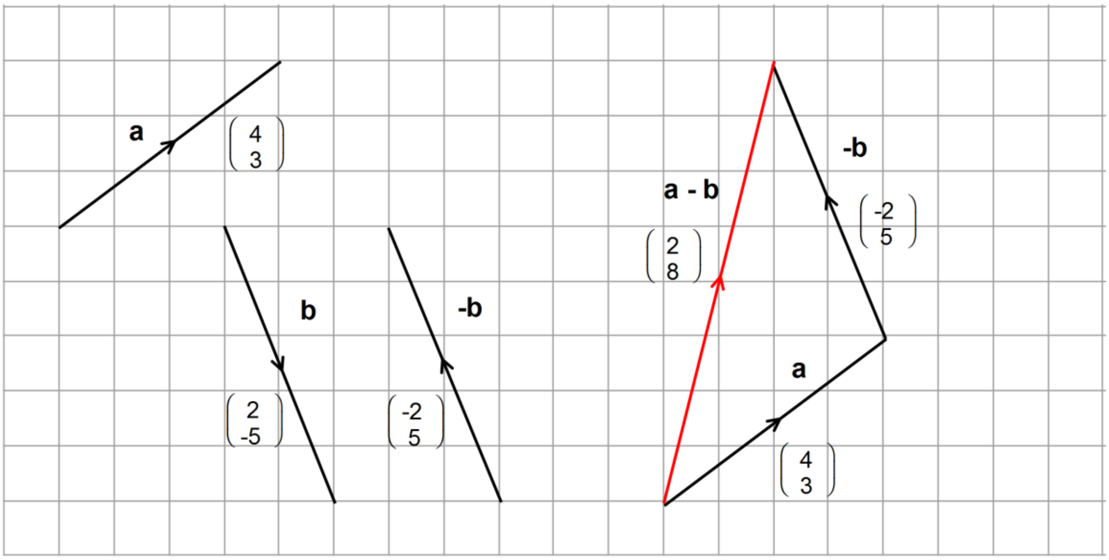
In column notation, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and **a+b =** $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

This result could also have been obtained without drawing the diagram.

a+b = $\begin{pmatrix} 4+2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Subtraction of vectors.

Example (4):



Subtracting vector **b** from **a** is identical to adding the inverse of vector **b** to **a**.

This time, we join **–b** to **a** "nose to tail".

In column notation, **a** = $\begin{pmatrix}4\\3\end{pmatrix}$, **b** = $\begin{pmatrix}2\\-5\end{pmatrix}$, **-b** = $\begin{pmatrix}-2\\5\end{pmatrix}$ and **a-b** = **a** + **(-b)** = $\begin{pmatrix}2\\8\end{pmatrix}$.

This result could again have been obtained without drawing the diagram.

a-b = $\begin{pmatrix}4-2\\3+5\end{pmatrix} = \begin{pmatrix}2\\8\end{pmatrix}$.

Addition or subtraction of vectors in column form is very easy - just add or subtract the components !

Another special case is **a-a** = $\begin{pmatrix}4-4\\3-3\end{pmatrix}$, where for example, **a** = $\begin{pmatrix}4\\3\end{pmatrix}$.

The result here is the **zero vector**, **0**. This is not the same as the number 0, which is a scalar.

Standard Unit Vectors.

In Example (2), we came across two special vectors in the two-dimensional *x*-*y* plane :

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are termed the **standard unit vectors.**

Vector **i** is parallel to the *x*-axis and vector **j** is parallel to the *y*-axis.

All two-dimensional vectors can also be expressed as combinations of **i** and **j**.
(This is also known as component form.)

Thus the vector $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ from Examples (3) and (4) can be expressed as $4\mathbf{i} + 3\mathbf{j}$,

and vector $\mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ as $2\mathbf{i} - 5\mathbf{j}$.

Example (5): Let vector $\mathbf{r} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{s} = \mathbf{i} + 4\mathbf{j}$.

Express the following in column form:

i) $\mathbf{r} + 3\mathbf{s}$; ii) $2\mathbf{s} - \mathbf{r}$; iii) $\mathbf{r} + \mathbf{i} - \mathbf{j}$.

i) $\mathbf{r} + 3\mathbf{s} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+3 \\ -1+12 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$

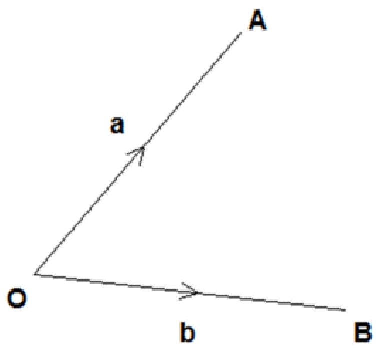
ii) $2\mathbf{s} - \mathbf{r} = 2\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 8+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

iii) $\mathbf{r} + \mathbf{i} - \mathbf{j} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

Alternative Notation ("Two-letter")

Another way of denoting vectors is by stating their end points and writing an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.



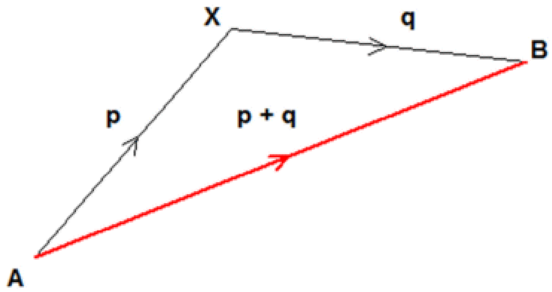
Therefore $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The direction of the arrow is important here;
the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$ although it has the same magnitude.

Hence $\overrightarrow{AO} = -\overrightarrow{OA} = -\mathbf{a}$.

The Triangle Law.

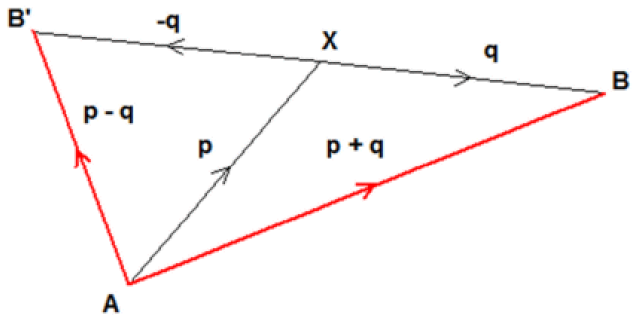
(Recall) To add two vectors, apply the first, and then the second.



Thus $\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$.

Here $\overrightarrow{AX} = \mathbf{p}$ and $\overrightarrow{XB} = \mathbf{q}$.

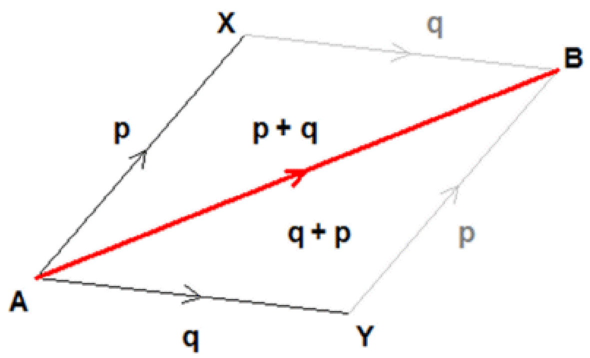
(Recall) Subtracting a vector is the same as adding its inverse, i.e. the parallel vector of the same magnitude but in the opposite direction.



Here, $\overrightarrow{XB'} = -\mathbf{q}$.

Thus $\overrightarrow{AB'} = \overrightarrow{AX} + \overrightarrow{XB'} = \mathbf{p} - \mathbf{q}$.

The Parallelogram Law.



Going from *A* to *B* via *Y* gives the same result as going from *A* to *B* via *X*.

Therefore $\overrightarrow{AX} + \overrightarrow{XB} = \overrightarrow{AY} + \overrightarrow{YB}$.

In other words, **p + q** = **q + p**.

Notice as well that $\overrightarrow{AX} = \overrightarrow{YB}$ = **p** and $\overrightarrow{AY} = \overrightarrow{XB}$ = **q**.

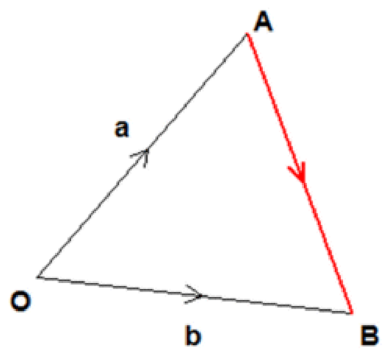
Since the opposite pairs of sides of any parallelogram are equal and parallel, they can always be represented by the same vector provided their directions are equal, thus $\overrightarrow{BY}$ = **-p** here.

Geometrical Applications.

Many problems and theorems in geometry can be analysed using vectors.

When asked to find an unknown vector between two points, just work it out as an alternative route made up of known segments, as per the parallelogram law.

Example (6): Express the vector $\overrightarrow{AB}$ in terms of **a** and **b**.



We want to go from *A* to *B* directly, but we do not have the vector for it.

We therefore go via *O*, as in $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now $\overrightarrow{AO}$ is the same as **a** but in the reverse direction, whilst $\overrightarrow{OB}$ = **b**.

Hence $\overrightarrow{AB}$ = **-a** + **b** or **b – a**.

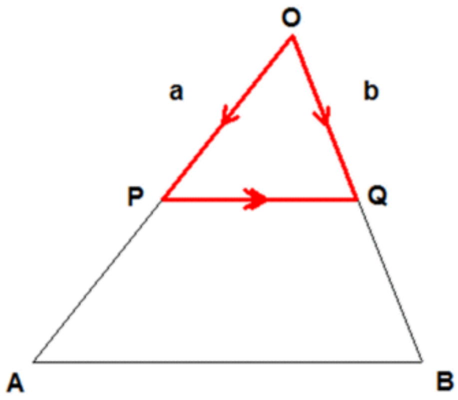
Example (6a): The coordinates of points *A* and *B* are (2, 4) and (6, 1) respectively. Find $\overrightarrow{AB}$ in column notation given that *O* is the origin.

Since *O* is the origin, vector **a** = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and **b** = $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Hence $\overrightarrow{AB}$ = **b – a** = $\begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Example (7): In the triangle *OAB*, point *P* is the midpoint of *OA* and point *Q* is the midpoint of *OB*.

Show that *PQ* is parallel to *AB*, and also half its length.



$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ = **-a** + **b** or **b – a**.

We are also told that

$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$, thus $\overrightarrow{OA} = 2\,\overrightarrow{OP}$ = 2**a**.

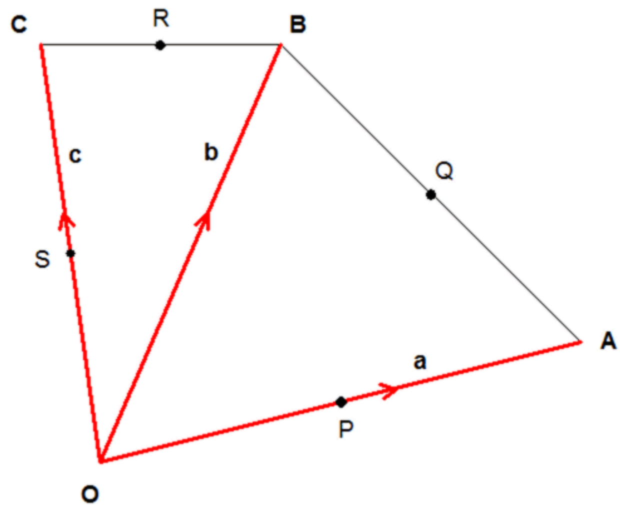
$\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OB}$, thus $\overrightarrow{OB} = 2\,\overrightarrow{OQ}$ = 2**b**.

Finally, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = -2**a** + 2**b** = 2(**b – a**) = $2\overrightarrow{PQ}$.

$\therefore$ *PQ* is parallel to *AB*, and half its length.
(Two vectors are parallel if either can be expressed as a scalar multiple of the other).

Example (8): *OABC* is a quadrilateral. $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b** and $\overrightarrow{OC}$ = **c.**

Points *P*, *Q*, *R* and *S* are the midpoints of *OA*, *AB*, *BC* and *OC* respectively.



i) Find the following vectors in terms of **a**, **b** and **c**: $\overrightarrow{OP}$, $\overrightarrow{AB}$, $\overrightarrow{AQ}$, $\overrightarrow{PQ}$ **and** $\overrightarrow{SR}$.
ii) Prove that *PQ* is parallel to *SR*.
iii) What type of quadrilateral is *PQRS* ?

i) *P* is the midpoint of *OA*, so $\overrightarrow{OP} = \frac{1}{2}$ **a.**

($\overrightarrow{PA}$ **is also** $\frac{1}{2}$ **a**).

By going via *O*, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = **b – a.**

Since *Q* is the midpoint of *AB*, $\overrightarrow{AQ} = \frac{1}{2}$ **(b – a).**

By going via *A*, $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \frac{1}{2}\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{b}.$

To find $\overrightarrow{SR}$, we must find $\overrightarrow{CB}$ first; it is (via *O*) **b** – **c**. Now $\overrightarrow{CR} = \frac{1}{2}(\mathbf{b} - \mathbf{c})$.

Hence $\overrightarrow{SR} = \overrightarrow{SC} + \overrightarrow{CR} = \frac{1}{2}\mathbf{c} + \frac{1}{2}(\mathbf{b} - \mathbf{c}) = \frac{1}{2}\mathbf{b}.$

ii) The vectors $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are equal, so *PQ* is parallel to *SR* and also equal in length.

iii) Because *PQ* is parallel to *SR*, the quadrilateral *PQRS* must be at least a trapezium. However , *PQ* and *SR* are also equal, so *PQRS* is a parallelogram (sides equal and parallel).

Note : We can prove that *PS* = *QR*, and that *PQRS* is a parallelogram, as follows :

By going via *O*, $\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$ = **c – a.**

We can find $\overrightarrow{QR}$ by going via P and S;

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR} = -\frac{1}{2}\mathbf{b} + \mathbf{c} - \mathbf{a} + \frac{1}{2}\mathbf{b} = \mathbf{c} - \mathbf{a}.$$

The vectors $\overrightarrow{PS}$ and $\overrightarrow{QR}$ are equal, so *PQ* is equal and parallel to *SR*.

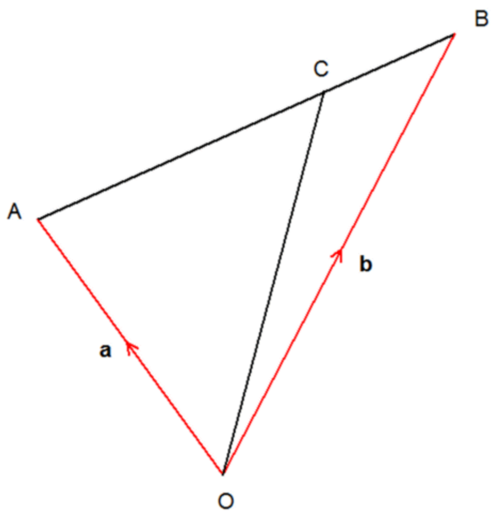
Example (9):

In the diagram, $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**.

Point C is on the line AB so that

$\overrightarrow{AC} = k\overrightarrow{AB}$. where $0 < k < 1$,

and $\overrightarrow{OC}$ = s**a** + t**b** where s and t are scalar multipliers.



i) Find s and t in terms of k.

ii) We are then told that point C is three-fifths along AB.
Using the result from i), find $\overrightarrow{OC}$.

i) Firstly, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = **b** – **a.**

Hence $\overrightarrow{AC} = k\overrightarrow{AB}$ = k(**b** – **a**).

Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, or

a + k(**b** – **a**) = **a** + k**b** – k**a**

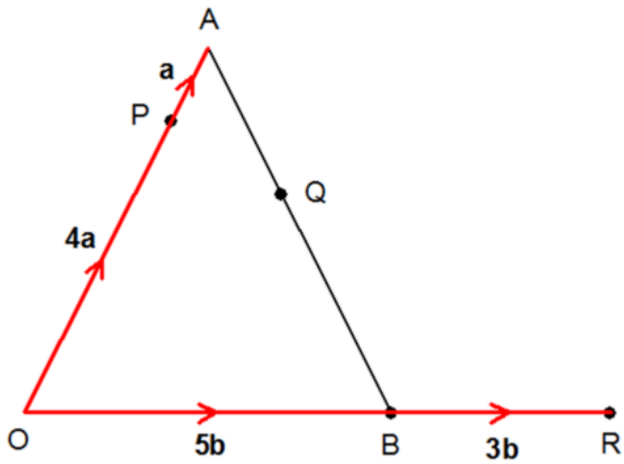
= $(1 – k)$**a** + k**b.**

Hence $s = 1-k$ and $t = k$.

ii) Given that C is three-fifths of the distance along AB, $k = \frac{3}{5}$.

Hence in this case $\overrightarrow{OC} = \frac{2}{5}$ **a** + $\frac{3}{5}$ **b.**

Example (10): In the diagram, $\overrightarrow{OP} = 4\mathbf{a}$, $\overrightarrow{PA} = \mathbf{a}$, $\overrightarrow{OB} = 5\mathbf{b}$, $\overrightarrow{BR} = 3\mathbf{b}$ **and** $\overrightarrow{AQ} = \frac{2}{5}\overrightarrow{AB}$ **.**



i) Find $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of **a** and **b**.

ii) Show clearly that points *P*, *Q*, and *R* lie on a straight line.

i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = (\mathbf{-a}) - 4\mathbf{a} + 5\mathbf{b}$

$= 5(\mathbf{b} - \mathbf{a})$.

$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \mathbf{a} + \frac{2}{5}\overrightarrow{AB}$

$= \mathbf{a} + 2(\mathbf{b} - \mathbf{a}) = 2\mathbf{b} - \mathbf{a}$.

ii) We want to show that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are scalar multiples of each other.

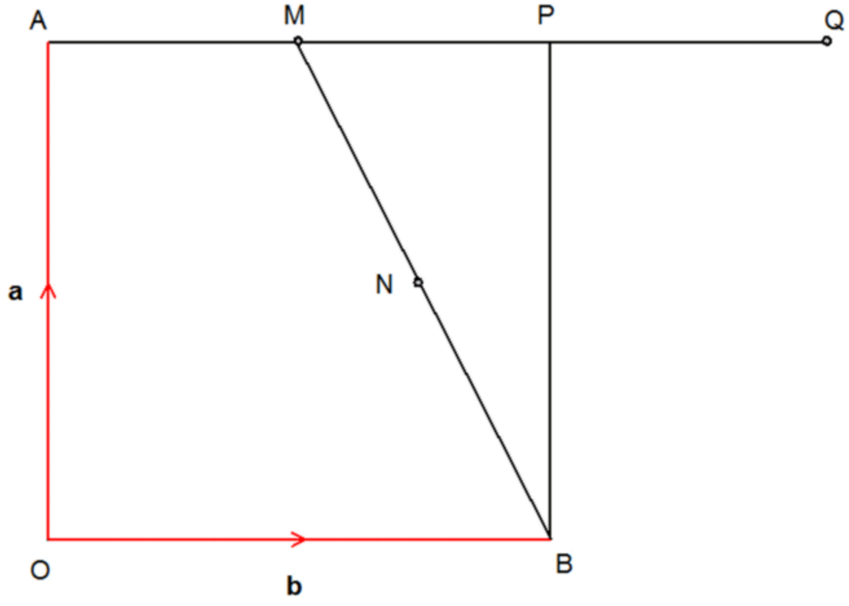
Now $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -4\mathbf{a} + 8\mathbf{b} = 8\mathbf{b} - 4\mathbf{a} = 4(2\mathbf{b} - \mathbf{a}) = 4\overrightarrow{PQ}$ **.**

Because $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are scalar multiples of each other and contain the point P in common, the points *P*, *Q*, and *R* lie on a straight line.

Point *Q* is one quarter of the way between *P* and *R*.

Example (11): The diagram shows a square *OAPB*.
M and *N* are the midpoints of *AP* and *BM* respectively.
The side *AP* is extended to point *Q* where *AQ* = 1½ *AP*.

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



Write the following vectors in terms of **a** and **b** , giving your answers in the simplest form.

i) $\overrightarrow{OQ}$ ii) $\overrightarrow{BM}$ iii) $\overrightarrow{BN}$ iv) $\overrightarrow{ON}$

v) What can be deduced about points *O*, *N* and *Q* ? Justify your answer.

Start with the obvious: $\overrightarrow{AP} = \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{BP} = \overrightarrow{OA} = \mathbf{a}$
since the opposite sides of a square are equal in length and parallel.

i) $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \mathbf{a} + \frac{3}{2}\mathbf{b}$.

ii) $\overrightarrow{BM} = \overrightarrow{BP} + \overrightarrow{PM} = \mathbf{a} - \frac{1}{2}\mathbf{b}$.

iii) $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b}$.

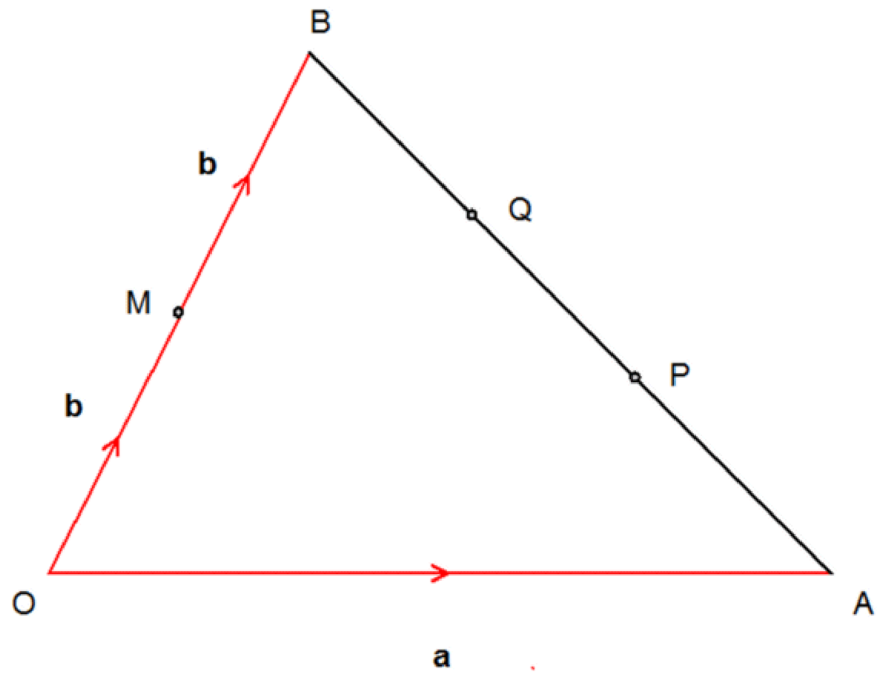
iv) $\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN} = \mathbf{b} + \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$.

v) The results from i) and iv) show that $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OQ}$.

Because $\overrightarrow{ON}$ and $\overrightarrow{OQ}$ are scalar multiples of each other and contain the point O in common, the points *O*, *N* and *Q* are therefore collinear.

In addition, *N* is the midpoint of *OQ*.

Example (12):



OAB is a triangle where *M* is the midpoint of *OB*.
P and *Q* are points on *AB* such that *AP* = *PQ* = *QB*.

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$.

Find expressions for the following in terms of **a** and **b**:

i) $\overrightarrow{BA}$; ii) $\overrightarrow{MQ}$; iii) $\overrightarrow{OP}$

iv) What can you deduce about the quadrilateral *OMQP* ? Justify your answer.

i) $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -2\mathbf{b} + \mathbf{a}$

$= \mathbf{a} - 2\mathbf{b}$.

ii) $\overrightarrow{MQ} = \overrightarrow{MB} + \overrightarrow{BQ} = \mathbf{b} + \frac{1}{3}\overrightarrow{BA} = \mathbf{b} + \frac{1}{3}(\mathbf{a} - 2\mathbf{b}) = \mathbf{b} + \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

$= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

iii) $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = 2\mathbf{b} + \frac{2}{3}\overrightarrow{BA} = 2\mathbf{b} + \frac{2}{3}\mathbf{a} - \frac{4}{3}\mathbf{b}$

$= \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$.

iv) The quadrilateral *OMQP* is a trapezium because $\overrightarrow{OP} = 2\,\overrightarrow{MQ}$.

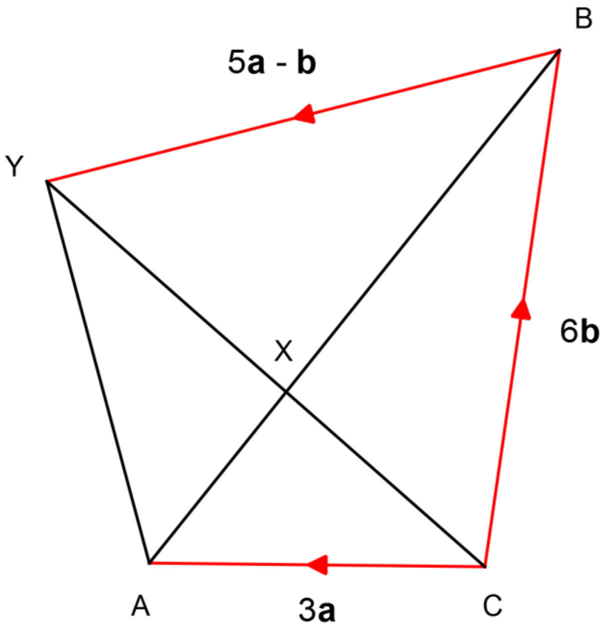
If one vector is a scalar multiple of another, then the two vectors are parallel.

Example (13):

ACBY is a quadrilateral, with the diagonals *AB* and *CY* intersecting at point *X*.
The point *X* divides the line *AB* in the ratio 1 : 2.

$\overrightarrow{CA} = 3\mathbf{a}$, $\overrightarrow{CB} = 6\mathbf{b}$ and $\overrightarrow{BY} = 5\mathbf{a} - \mathbf{b}$.

Prove that *X* divides the line *CY* in the ratio 2 : 3 .



$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = -3\mathbf{a} + 6\mathbf{b}$, or $6\mathbf{b} - 3\mathbf{a}$.

Point *X* divides *AB* in the ratio 1 : 2, so it lies one-third of the way along *AB.*

Hence $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AB} = 2\mathbf{b} - \mathbf{a}.$

We then find vectors $\overrightarrow{CX}$ and $\overrightarrow{CY}$:

$\overrightarrow{CX} = \overrightarrow{CA} + \overrightarrow{AX} = 3\mathbf{a} + 2\mathbf{b} - \mathbf{a} = 2\mathbf{a} + 2\mathbf{b}$.

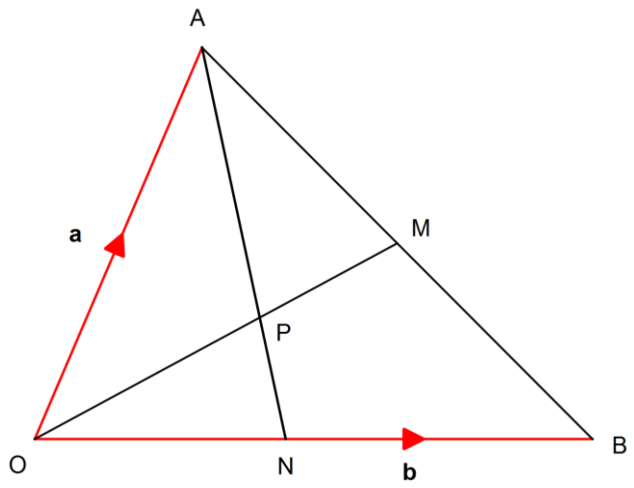
$\overrightarrow{CY} = \overrightarrow{CB} + \overrightarrow{BY} = 6\mathbf{b} + 5\mathbf{a} - \mathbf{b} = 5\mathbf{a} + 5\mathbf{b}$.

The length of *CX* is evidently two-fifths that of *CY*, so point *X* does indeed divide the diagonal *CY* in the ratio 2 : 3.

Example (13a):

OAB is a triangle where *M* is the midpoint of *AB*.
$\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**.

The point *P* divides the line *OM* in the ratio 3 : 2.
The line *AN* also passes through point *P*.
Find the ratio *ON* : *NB* in its simplest form.



The point *N* lies on *OB*, so we can say that
$\overrightarrow{ON} = s\mathbf{b}$ where *s* is a constant and $0 < s < 1$.

We can also say that $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$.

The trickiest part is to find a vector equation for $\overrightarrow{AN}$, but we can begin with
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.

Since *M* is the midpoint of *AB*, we have $\overrightarrow{AM} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$ and
$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

Next, we find $\overrightarrow{OP}$ and $\overrightarrow{AP}$.

Since *P* divides *OM* in the ratio 3 : 2, $\overrightarrow{OP} = \frac{3}{5}\overrightarrow{OM} = \frac{3}{10}\mathbf{a} + \frac{3}{10}\mathbf{b}$.
Next, $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\mathbf{a} + \frac{3}{10}\mathbf{a} + \frac{3}{10}\mathbf{b} = \frac{3}{10}\mathbf{b} - \frac{7}{10}\mathbf{a}$.

Since *A*, *P* and *N* lie on a straight line, we can also say that $\overrightarrow{AN}$ is a scalar multiple of $\overrightarrow{AP}$,
or $\overrightarrow{AN} = k\left(\frac{3}{10}\mathbf{b} - \frac{7}{10}\mathbf{a}\right)$ where *k* is another constant.

We can get rid of the fractions by bringing out a factor of $\frac{1}{10}$ outside the brackets to obtain
$\overrightarrow{AN} = \frac{1}{10}k(3\mathbf{b} - 7\mathbf{a})$, and use a new constant *t* to replace $\frac{1}{10}k$. Hence $\overrightarrow{AN} = t(3\mathbf{b} - 7\mathbf{a})$,

Also, since $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$, we now have $\overrightarrow{ON} = \mathbf{a} + t(3\mathbf{b} - 7\mathbf{a}) = (1-7t)\,\mathbf{a} + (3t)\,\mathbf{b}$.

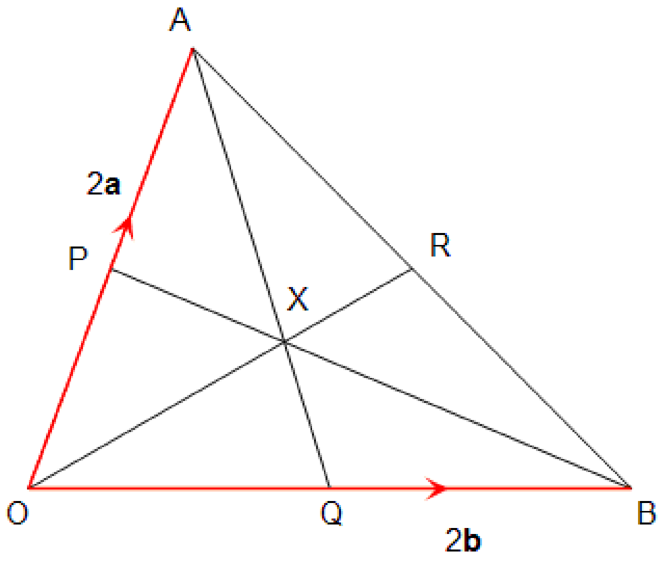
We now have two vector equations for $\overrightarrow{ON}$:
$\overrightarrow{ON} = s\mathbf{b}$ and $\overrightarrow{ON} = (1-7t)\,\mathbf{a} + (3t)\,\mathbf{b}$.

The final stage is to equate the **a-** and **b-** components - in other words, to compare them.

As $\overrightarrow{ON}$ has an **a**-component of zero, $1-7t = 0$, and so $t = \frac{1}{7}$.
Equating the **b**-components, we have $s = 3t$, hence $s = \frac{3}{7}$.

Hence $\overrightarrow{ON} = \frac{3}{7}\overrightarrow{OB}$ and $\overrightarrow{NB} = \frac{4}{7}\overrightarrow{OB}$, with *N* dividing *OB* in the ratio **3 : 4.**

Example (14): *OAB* is a triangle where *P*, *Q* and *R* are the midpoints of *OA*, *OB* and *AB* respectively, and *X* is the point at which all three intersect.



$\overrightarrow{OA} = 2\mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$.

As a matter of interest, the lines *AQ*, *BP* and *OR* are known as the **medians** of the triangle, and point *X* is the **centroid** and also the triangle's centre of gravity..

Find expressions for the following in terms of **a** and **b**:

i) $\overrightarrow{AB}$; ii) $\overrightarrow{OR}$; iii) $\overrightarrow{AQ}$; iv) $\overrightarrow{BP}$

The ratios *OX* : *OR* , *AX* : *AQ* and *BX* : *BP* are all equal to *k* : 1 where *k* is a fractional constant.

v) Express $\overrightarrow{OX}$ in terms of a) $\overrightarrow{OR}$; b) **a** and $\overrightarrow{AQ}$; b) **b** and $\overrightarrow{BP}$.

vi) Hence solve the vector equations in v) and thus find the value of *k*.

i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -2\mathbf{a} + 2\mathbf{b} = 2\mathbf{b} - 2\mathbf{a}$.

ii) $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR} = 2\mathbf{a} + \frac{1}{2}\overrightarrow{AB} = 2\mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{a} + \mathbf{b}$.

iii) $\overrightarrow{AQ} = \overrightarrow{AO} + \overrightarrow{OQ} = -2\mathbf{a} + \mathbf{b} = \mathbf{b} - 2\mathbf{a}$.

iv) $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP} = -2\mathbf{b} + \mathbf{a} = \mathbf{a} - 2\mathbf{b}$.

v) a) $\overrightarrow{OX} = k\overrightarrow{OR} = k\mathbf{a} + k\mathbf{b}$.

v) b) $\overrightarrow{OX} = 2\mathbf{a} + k\overrightarrow{AQ} = 2\mathbf{a} + k(\mathbf{b} - 2\mathbf{a}) = 2\mathbf{a} + k\mathbf{b} - 2k\mathbf{a} = (2-2k)\mathbf{a} + k\mathbf{b}$.

v) c) $\overrightarrow{OX} = 2\mathbf{b} + k\overrightarrow{BP} = 2\mathbf{b} + k(\mathbf{a} - 2\mathbf{b}) = 2\mathbf{b} + k\mathbf{a} - 2k\mathbf{b} = (2-2k)\mathbf{b} + k\mathbf{a}$.

vi) Equating the results in v), we have $k = 2 - 2k \rightarrow 3k = 2$
and thus $k = \frac{2}{3}$.

Hence the centroid *X* is two-thirds of the way along all three medians *AQ*, *OR* and *BP*.

A few extras . (Seen on some exam papers)

The Magnitude of a Vector.

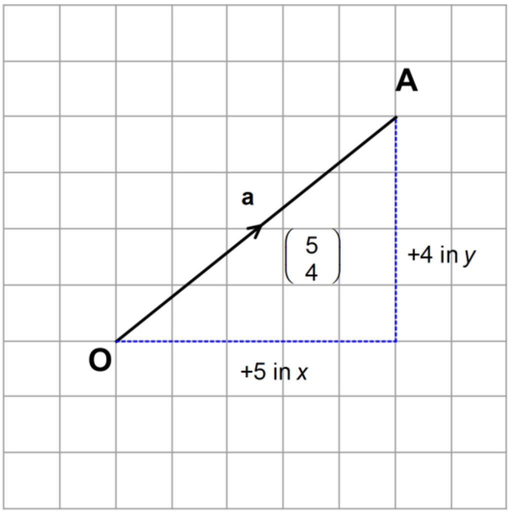
An important property of a vector is its **magnitude**, and it can be determined very easily by Pythagoras.

Since the vector **a** from Example (1) can be visualised as the hypotenuse of a right-angled triangle with a base of 5 units and a height of 4 units, its magnitude is simply

$\sqrt{5^2 + 4^2} = \sqrt{41}$ units.

In general, the magnitude of any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\sqrt{a^2 + b^2}$.

(Recall the method used to find the distance between two given points in "Straight Line Graphs".)



Example (15): Find the magnitudes of the following vectors:

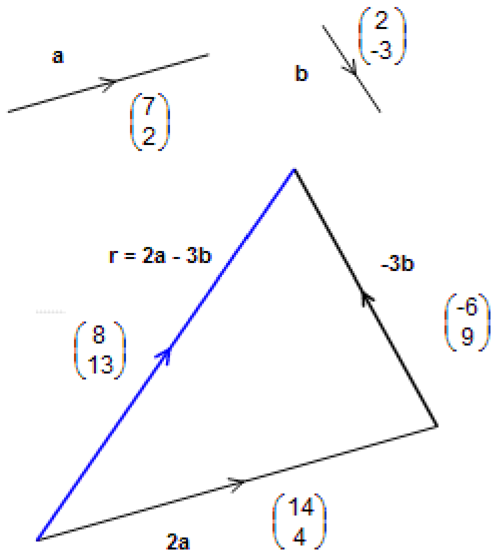
i) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$; ii) $\begin{pmatrix} 0.28 \\ 0.96 \end{pmatrix}$

i) The magnitude of the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\sqrt{3^2 + 4^2} = 5$ units.

ii) The vector $\begin{pmatrix} 0.28 \\ 0.96 \end{pmatrix}$ has a magnitude of $\sqrt{0.28^2 + 0.96^2} = 1$ unit.

Simultaneous Vector Equations.

The diagram on the right shows how scalar multiples of vectors **a** and **b** can be combined to form vector **r = 2a – 3b.**



r = 2a – 3b = $2\begin{pmatrix}7\\2\end{pmatrix} - 3\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}8\\13\end{pmatrix}$

Question: How can we express **r** in terms of **a** and **b** without using a diagram ?

In other words, we want to express **r** in the form **r** = *s***a** + *t***b** where *s* and *t* are constants to be determined.

Example (16):

We therefore set up a vector equation as follows:

r = *s*a + *t*b $\rightarrow$ $s\begin{pmatrix}7\\2\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}8\\13\end{pmatrix}$

From this vector equation, we can set up a pair of linear simultaneous equations by reading the column entries from left to right;

$7s + 2t = 8$ A
$2s - 3t = 13$ B

$21s + 6t = 24$ $3A$
$4s - 6t = 26$ $2B$

$25s = 50$ $3A + 2B$ $\therefore$ $\boldsymbol{s = 2}$

Substituting in equation *A* gives $14 + 2t = 8$, so $t = -3$.

Hence **r** = $2\begin{pmatrix}7\\2\end{pmatrix} - 3\begin{pmatrix}2\\-3\end{pmatrix}$.